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The electromagnetic energy of a point charge

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1. INTRODUCTION

There are many reasons for preferring the point model of the electron, in which the field equations of empty space hold all the way up to the centre of the electron, to the Lorentz model, in which the charge is distributed over a small sphere. The point model is not without difficulties, however, and two have attracted special attention. The first is that the field becomes infinite at the charge, so that the Lorentz equations of motion cannot be applied directly; the second is that the ordinary expression leads to an infinite electromagnetic energy in the neighbourhood of the charge. As these difficulties occur both in classical and in quantum electrodynamics it seems reasonable to look for their solution, first in the classical theory, and then try to translate it into the quantum theory. A recent paper by Dirac (1938) has satisfactorily removed the first difficulty from the classical theory. The present paper shows how the second can be removed also. The translation of these methods to quantum theory has not yet been accomplished. Some papers by Wentzel (1933, 1934) have also dealt with this subject, both from the classical and the quantum standpoint, but they do not seem to be altogether without difficulties, and the method is rather complicated to use in actual problems.

2. The electromagnetic energy

The ordinary expression for the electromagnetic energy in a region of space is

$$\frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{H}^2)\,dV,$$

where **E**, **H** are the electric and magnetic vectors, and dV is the element of volume. The classical derivation of this is briefly as follows: one calculates the rate at which the electromagnetic forces, $\rho\mathbf{E}+\mathbf{j}\times\mathbf{H}$ per unit volume (ρ, $\mathbf{j}$ being the charge and current density), do mechanical work on the charge distributions in the region. This is $c\int\mathbf{j}\,\mathbf{E}\,dV$, which can be transformed by means of the Maxwell equations to

$$-\frac{1}{8\pi}\frac{\partial}{\partial t}\int(\mathbf{E}^2+\mathbf{H}^2)\,dV-\frac{c}{4\pi}\int[\mathbf{E}\times\mathbf{H}].d\mathbf{S}$$

the second integral being taken over the boundary of the region. Thus the rate of change of the sum of the mechanical energy and $\int(\mathbf{E}^2+\mathbf{H}^2)\,dV/8\pi$ is given by the surface integral $c\int[\mathbf{E}\times\mathbf{H}].d\mathbf{S}/4\pi$; this can be interpreted as meaning the conservation of energy if $\int(\mathbf{E}^2+\mathbf{H}^2)\,dV/8\pi$ is the electromagnetic energy in the region and $\int c[\mathbf{E}\times\mathbf{H}].\,d\mathbf{S}/4\pi$ the rate of flow of energy out of the region across the boundary. This derivation is strictly valid only if ρ is everywhere finite (but not necessarily continuous), and therefore does not apply to point charges. Furthermore it gives only the rate of change of the energy, leaving the energy itself indeterminate to the extent of an additive constant.

It is reasonable to assume that the concentration of the charge into a point will not alter the expression for the energy in a region of space, apart from an additive constant, if this region is free from charges. (A convenient formulation of this is as follows: if two electromagnetic fields differ only in a region containing no charges, the difference of the electromagnetic energies associated with them is the integral of the difference of their energy densities $U\,(=[\mathbf{E}^2+\mathbf{H}^2]/8\pi)$ over that region.)

It is possible to write down a finite expression for the energy, consistent with this principle. Let it be assumed for convenience that there is only one charge in the field; all that follows can be easily extended to many charges. Then let the charge be surrounded by a small closed surface, whose diameter is smaller than any physically important dimension and will eventually be made to tend to zero, and let the integral of the energy density U be formed over the whole of space outside the surface. As the diameter tends to zero the integral tends to infinity. If, however, a quantity can be found, depending only on the surface and the variables describing the charge, such that its sum with the integral tends to a finite limit, then this sum will satisfy the above requirement. This will be so if a vector field **K**, depending only on the variables of the charge, can be found such that $U' = U - \operatorname{div} \mathbf{K}$ is integrable in the neighbourhood of the charge (i.e. does not go to infinity more rapidly than r^{-2}). A possible expression for the electromagnetic energy is then

$$W_{\text{el}} = \lim \left\{ \int U\,dV + \int \mathbf{K} \,.\, d\mathbf{S} \right\}, \tag{2·1}$$

the volume integral being taken over the region of space outside the surface, and the surface integral being taken with the outward normal. Equation 2·1 expresses the central idea of this paper. Loosely speaking, the first term is the usual infinite energy and the second term is a negatively infinite energy arising from the charge itself.

If **K** falls off at infinity more rapidly than R^{-2}, this can also be written as an integral over the whole of space:

$$W_{\text{el}} = \int (U - \operatorname{div} \mathbf{K})\,dV = \int U'\,dV. \tag{2·2}$$

Although the physical significance is hidden in this formulation, it is often more convenient to work with it than with equation 2·1.

The *total* energy is of course the sum of the electromagnetic energy and some energy associated with the charge itself, which may be called the mechanical energy. The distinction between electromagnetic and mechanical energy is not sharp, for a different vector **K** will in general give a different value of the electromagnetic energy, which must be compensated in the mechanical energy to keep the total unaltered.

For this procedure to be physically correct two conditions must be satisfied: the conservation of energy must follow from the field equations and the equations of motion for a point charge, and it must be relativistically invariant. Momentum and energy must therefore be treated on an equal footing.

3. Relativistic formulation

Relativistic vector and tensor notation will be used throughout. The velocity of light will be taken as unit. Greek indices take the values 0, 1, 2, 3; Roman indices take the values 1, 2, 3. The co-ordinates of a point in space-time are written x^μ; x^0 refers to the time co-ordinate, x^1, x^2, x^3 to the space co-ordinates. Indices are raised and lowered by means of the metric tensor

$g_{\mu\nu}$ ($g_{00} = 1, g_{11} = g_{22} = g_{33} = -1, g_{\mu\nu} = 0$ if $\mu \neq \nu$). The electromagnetic field is given by the antisymmetric tensor $F_{\mu\nu}$ ($E_1 = F_{01}$, $H_1 = F_{32}$, etc.). The total energy and momentum are components of a vector p^σ, which can be separated into an electromagnetic and a mechanical part,

$$p^\sigma = p^\sigma_{\text{el}} + p^\sigma_{\text{mech}},$$

although, as already noted, the division is not absolute.

The ordinary expression for p^σ_{el} is given by the space integral of the component $T^{0\sigma}$ of the energy tensor:

$$p^\sigma_{\text{el}} = \int T^{0\sigma}\, dx^1 dx^2 dx^3,$$

with
$$T^{\nu\sigma} = \frac{1}{4\pi}\{F^{\nu\alpha}F_\alpha^{\ \sigma} + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\nu\sigma}\}. \tag{3·1}$$

($T^{00} = U$). $T^{\nu\sigma}$ is symmetric and satisfies the conservation law

$$\frac{\partial}{\partial x^\nu} T^{\nu\sigma} = 0. \tag{3·2}$$

In the relativistic theory, the components of **K**, together with those of the similar (three-dimensional) vectors associated with the components of momentum, can be seen to form part of a tensor of third rank $K^{\mu\nu\sigma}$, antisymmetric in μ and ν,

$$K^{\mu\nu\sigma} = -K^{\nu\mu\sigma}. \tag{3·3}$$

Those of **K** are in fact K^{a00} ($a = 1, 2, 3$), and those of the vector associated with the component p^σ_{el} are $K^{a0\sigma}$. The (three-dimensional) divergence of these vectors must differ from $T^{0\sigma}$ by an integrable amount. Because of the antisymmetry, the three-dimensional divergence is the same as $\partial K^{\mu 0\sigma}/\partial x^\mu$. Thus $T'^{\nu\sigma}$, defined analogously to U' by

$$T'^{\nu\sigma} = T^{\nu\sigma} - \frac{\partial}{\partial x^\mu} K^{\mu\nu\sigma}, \tag{3·4}$$

must be integrable. $T'^{\nu\sigma}$ also satisfies a conservation law

$$\frac{\partial}{\partial x^\nu} T'^{\nu\sigma} = 0 \tag{3·5}$$

at all points not on the world-line of the charge. This follows from 3·2 and the antisymmetry of $K^{\mu\nu\sigma}$.

At first sight it would not seem necessary to require that $T'^{\nu\sigma}$ should be symmetric; but if these methods are also applied to discuss angular momentum, it will be convenient to have it so. It will appear later that $K^{\mu\nu\sigma}$ can be so chosen that $\partial K^{\mu\nu\sigma}/\partial x^\mu$ is symmetric, although $K^{\mu\nu\sigma}$ itself is not symmetric in ν and σ. It is therefore natural to add this restriction.

The relativistic extension of equation 2·2 is

$$p^\sigma_{\text{el}} = \int T'^{0\sigma}\, dx^1 dx^2 dx^3 \tag{3·6}$$

This is a more convenient form to take than the extension of 2·1, for it can be discussed with less mathematical apparatus. It is of course strictly valid only if $K^{\mu\nu\sigma}$ falls off more rapidly than R^{-2} at infinity, but since it is only its behaviour near the charge that is important and at large distances $K^{\mu\nu\sigma}$ may be made to behave as we please, this is unimportant.

Let $p^\sigma_{\text{el}}(1)$ and $p^\sigma_{\text{el}}(2)$ denote the values at two times t_1 and t_2. These are given by integrals over two three-dimensional sections of space-time, $x^0 = t_1$ and $x^0 = t_2$. Let the world-line of the charge be surrounded by a tube of small diameter (a section $x^0 =$ const. will cut the tube in a closed surface surrounding the charge); the diameter of the tube will ultimately tend to zero. In the region of space-time between the sections $x^0 = t_1$ and $x^0 = t_2$, and outside the tube, 3·5 holds exactly. From this it follows, by the theorem of Gauss and Green, that the integral of the "normal component" of $T'^{\nu\sigma}$ over the boundary of the region is zero; i.e.

$$\int_1 T'^{\nu\sigma}\, dS_\nu - \int_2 T'^{\nu\sigma}\, dS_\nu + \int_{\text{tube}} T'^{\nu\sigma}\, dS_\nu = 0, \tag{3·7}$$

where $\int_1$, $\int_2$ are taken over the part of the sections $x^0 = t_1$ and $x^0 = t_2$ outside the tube, and dS_ν is the directed "surface element"; in the first integral it has been taken in the inward direction to correspond with the positive direction of time (it is assumed that $t_2 > t_1$), and away from the world-line on the tube. As the diameter of the tube tends to zero, the first two integrals tend to $p^\sigma_{\text{el}}(1)$, $p^\sigma_{\text{el}}(2)$ respectively. The integral over the tube therefore tends to a definite limit, irrespective of the shape of the tube. This limit must be expressible as a line integral along the world-line. If τ denotes the proper time along the world-line, measured from some arbitrary zero, it may be written

$$\lim \int_{\text{tube}} T'^{\nu\sigma}\, dS_\nu = -\int_1^2 R^\sigma\, d\tau. \tag{3·8}$$

Equation 3·7 therefore gives

$$p^\sigma_{\text{el}}(2) - p^\sigma_{\text{el}}(1) = -\int_1^2 R^\sigma\, d\tau. \tag{3·9}$$

Energy and momentum will be conserved if p^{σ}_{mech} satisfies the equation of motion

$$\dot{p}^{\sigma}_{\text{mech}} = R^{\sigma}. \tag{3·10}$$

(A dot means differentiation with respect to the proper time.)

If a different $K^{\mu\nu\sigma}$ is taken, differing from the first by $\Delta K^{\mu\nu\sigma}$ ($\partial\Delta K^{\mu\nu\sigma}/\partial x^{\mu}$ being integrable), the new p^{σ}_{el} will differ from the old by an amount

$$\Delta p^{\sigma}_{\text{el}} = \lim \int \Delta K^{a0\sigma}\, d\sigma_a,$$

where $d\sigma_a$ stand for the components of the surface element of the surface round the charge, in the three-dimensional space $x^0 = \text{const}$. This limit exists, for $\partial\Delta K^{a0\sigma}/\partial x^a$ is integrable. Equation 3·9 shows that R^{σ} is changed by an amount

$$\Delta R^{\sigma} = -\Delta\dot{p}^{\sigma}_{\text{el}}.$$

The change in p^{σ}_{mech} compensates the change in p^{σ}_{el}:

$$\Delta p^{\sigma}_{\text{mech}} = -\Delta p^{\sigma}_{\text{el}}.$$

The equation of motion now reads

$$\dot{p}^{\sigma}_{\text{mech}} + \Delta\dot{p}^{\sigma}_{\text{mech}} = R^{\sigma} + \Delta R^{\sigma},$$

which is of course identical with 3·10.

Equation 3·7 still holds if the sections $x^0 = t_1$, $x^0 = t_2$ are replaced by any two time-like sections, not necessarily parallel to one another. The vectors $p^{\sigma}_{\text{el}}(1)$, $p^{\sigma}_{\text{el}}(2)$ in 3·9 are now the electromagnetic energy and momentum vectors defined in two different Lorentz frames of reference (frames in which the two sections respectively are sections of constant time). Combined with the (integrated) equation of motion 3·10, this gives

$$p^{\sigma}(1) = p^{\sigma}(2),$$

which now expresses, not only the conservation of energy and momentum, but also the fact that the definition of p^{σ} is independent of the co-ordinate system.

4. Finite energy and momentum—equations of motion

It must now be shown that a tensor $K^{\mu\nu\sigma}$ exists. We shall use the following notation:

x^μ are the co-ordinates of a point in space, at which the fields, etc., are being evaluated.

z^μ are the co-ordinates of the point where the world-line of the charge is cut by the *past* branch of the light-cone starting from the point x^μ; the point z^μ will be called the retarded point.

v^μ is the four-velocity $\dot{z}^\mu$, evaluated at the retarded point.

$\dot{v}^\mu$, $\ddot{v}^\mu$ are the derivatives of the velocity with respect to τ, at the retarded point.

s^μ is the null-vector joining the retarded point to the variable point $(s^\mu = x^\mu - z^\mu)$.

s is the invariant "retarded distance" $s^\mu v_\mu$.

$\kappa = s^\mu \dot{v}_\mu$.

The inner product of two vectors will be written in bracket notation: $A^\mu B_\mu = (A, B)$.

The actual field will be written as the sum of the retarded field of the charge and the "incoming" external field

$$F = F_{\text{ret}} + F_{\text{in}}. \tag{4·1}$$

The retarded field is given by

$$F^{\mu\nu}_{\text{ret}} = e\left\{\frac{(1-\kappa)}{s^3}(s^\mu v^\nu - s^\nu v^\mu) + \frac{1}{s^2}(s^\mu \dot{v}^\nu - s^\nu \dot{v}^\mu)\right\}. \tag{4·2}$$

In the neighbourhood of the charge, s tends to zero, and it is convenient to classify the orders of infinity of the various terms with the help of the powers of s; s, s^μ, κ are $O(s)$, and v^μ, $\dot{v}^\mu$, $\ddot{v}^\mu$ are $O(1)$. The retarded field is $O(s^{-2})$.

The energy tensor $T^{\nu\sigma}$ is quadratic in $F_{\mu\nu}$, and can therefore be written

$$T = T_{\text{ret}} + T_{\text{mix}} + T_{\text{in}}, \tag{4·3}$$

where T_{ret}, T_{in} are the tensors formed from F_{ret}, F_{in} according to 3·1, and T_{mix} is the cross-term. T_{ret} is $O(s^{-4})$. T_{mix} and T_{in} are $O(s^{-2})$ and $O(1)$ respectively, and are therefore integrable. Thus $K^{\mu\nu\sigma}$ must be such that the terms of order s^{-4} and s^{-3} in both $T^{\nu\sigma}_{\text{ret}}$ and $\partial K^{\mu\nu\sigma}/\partial x^\mu$ are equal.

$T^{\nu\sigma}_{\text{ret}}$ is given by

$$T^{\nu\sigma}_{\text{ret}} = \frac{e^2}{4\pi}\left\{-\frac{(1-\kappa)^2 s^\nu s^\sigma}{s^6} + \frac{(1-\kappa)(v^\nu s^\sigma + v^\sigma s^\nu)}{s^5} + \frac{(\dot{v}^\nu s^\sigma + s^\nu \dot{v}^\sigma) - (\dot{v}, \dot{v})\, s^\nu s^\sigma - \frac{1}{2}g^{\nu\sigma}}{s^4}\right\}. \tag{4·4}$$

In the appendix it is shown that the divergence of the tensor

$$K^{\mu\nu\sigma} = \frac{e^2}{16\pi}\left\{\frac{9\kappa(s^\mu v^\nu - s^\nu v^\mu)\, s^\sigma}{s^5} + \frac{(v^\nu g^{\mu\sigma} - v^\mu g^{\nu\sigma})}{s^3} + \frac{(1+2\kappa)\,(s^\mu g^{\nu\sigma} - s^\nu g^{\mu\sigma}) + 3(v^\mu s^\nu - v^\nu s^\mu)\, v^\sigma + 3(\dot v^\mu s^\nu - \dot v^\nu s^\mu)\, s^\sigma}{s^4}\right\} \quad (4\cdot5)$$

is $$\frac{\partial}{\partial x^\mu}K^{\mu\nu\sigma} = \frac{e^2}{4\pi}\left\{\frac{(8\kappa^2+2\kappa-1)\, s^\nu s^\sigma}{s^6} + \frac{(\dot v^\nu s^\sigma + s^\nu \dot v^\sigma) - \frac{1}{2}g^{\nu\sigma}}{s^4} + \frac{(1-\kappa)\,(s^\nu v^\sigma + v^\nu s^\sigma) - 2(\ddot v, s)\, s^\nu s^\sigma}{s^5}\right\}, \quad (4\cdot6)$$

which differs from 4·4 only by terms of order s^{-2}. Defining a $T'^{\nu\sigma}_{\text{ret}}$ analogous to $T'^{\nu\sigma}$,

$$T'^{\nu\sigma}_{\text{ret}} = T^{\nu\sigma}_{\text{ret}} - \frac{\partial}{\partial x^\mu}K^{\mu\nu\sigma}, \quad (4\cdot7)$$

this gives $$T'^{\nu\sigma}_{\text{ret}} = \frac{e^2}{4\pi}\left\{-\frac{9\kappa^2}{s^6} + \frac{2(\ddot v, s)}{s^5} - \frac{(\dot v, \dot v)}{s^4}\right\} s^\nu s^\sigma. \quad (4\cdot8)$$

To ensure that the amount of energy radiated away from the charge during its past history be finite, let it be assumed that previous to some remote time it was moving uniformly in a straight line. $K^{\mu\nu\sigma}$ is then of order R^{-3} at infinity, for at sufficiently large distances $\dot v^\mu$ and κ vanish; equation 3·6 is therefore valid. The expression for the electromagnetic energy is then

$$p^\sigma_{\text{el}} = \int (T'^{0\sigma}_{\text{ret}} + T^{0\sigma}_{\text{mix}} + T^{0\sigma}_{\text{in}})\, dx^1 dx^2 dx^3. \quad (4\cdot9)$$

This is finite. For a charge in uniform motion, with no external field, it is zero; this is in agreement with the result of Wentzel (1934). The electromagnetic energy is not necessarily positive.

To determine R^σ one must evaluate the integral of $T'^{\nu\sigma}$ along a world-tube of infinitesimal diameter. Only T'_{ret} and T_{mix} will contribute anything, for the integral of T_{in} will tend to zero. The only important terms in T_{mix} are

$$T^{\nu\sigma}_{\text{mix}} \sim \frac{1}{4\pi s^3}\{(s^\nu v^\tau - s^\tau v^\nu)\, F_\tau{}^\sigma + F^\nu{}_\tau (s^\tau v^\sigma - s^\sigma v^\tau) + \tfrac{1}{2}(s^\alpha v^\beta - s^\beta v^\alpha)\, F_{\alpha\beta} g^{\nu\sigma}\}.$$

To evaluate the contribution from an element $\delta\tau$ of the world-line it is simplest to take a co-ordinate system in which the components of v^μ at the element are (1, 0, 0, 0), and a tube of spherical section. The contribution is then

$$R^{\sigma}\delta\tau = -\int T'^{\alpha\sigma} d\sigma_{\alpha}\,\delta\tau,$$

i.e.

$$R^{\sigma} = -\int T'^{\alpha\sigma} d\sigma_{\alpha}, \tag{4·10}$$

where $d\sigma_a$ has the same meaning as on p. 394.

Let x, y, z be the co-ordinates of a point on the sphere and ϵ its radius; then to the required order of approximation

$$s^0 = s = \epsilon,\ s^1 = x,\ s^2 = y,\ s^3 = z \quad (x^2+y^2+z^2 = \epsilon^2).$$

The integration can be carried out separately for R^0, R^a; it is quite straightforward if one remembers that in the particular co-ordinate system chosen $\dot{v}^0 = 0$, $\ddot{v}^0 = -(\dot{v}, \dot{v})$ (these follow from $(v, \dot{v}) = 0$, $(v, \ddot{v}) + (\dot{v}, \dot{v}) = 0$). The result is

$$R^0 = 0, \quad R^a = \tfrac{2}{3}e^2\ddot{v}^a + eF^{a0}_{\text{in}}.$$

In a general co-ordinate system this reads

$$R^{\sigma} = \tfrac{2}{3}e^2\{\ddot{v}^{\sigma} + (\dot{v}, \dot{v})\, v^{\sigma}\} + eF^{\sigma\nu}_{\text{in}} v_{\nu}. \tag{4·11}$$

R^{σ} satisfies the equation

$$v_{\sigma}R^{\sigma} = 0; \tag{4·12}$$

p^{σ}_{mech} must therefore satisfy the condition

$$v_{\sigma}\dot{p}^{\sigma}_{\text{mech}} = 0. \tag{4·13}$$

This is most simply achieved if p^{σ}_{mech} is a constant multiple of v^{σ}.

$$p^{\sigma}_{\text{mech}} = mv^{\sigma}. \tag{4·14}$$

We have seen that the electromagnetic part of p^{σ} vanishes for a charge in uniform motion; its total energy and momentum is then therefore mv^{σ}. The constant m is therefore the actual inertial mass of the charge and its own field.

The equation of motion is

$$m\dot{v}^{\sigma} - \tfrac{2}{3}e^2\{\ddot{v}^{\sigma} + (\dot{v}, \dot{v})\, v^{\sigma}\} = eF^{\sigma\nu}_{\text{in}} v_{\nu}. \tag{4·15}$$

This is the equation given by Dirac (1938, equation 24).

The tensor 4·5 is not the only one that can be built up from the retarded variables to satisfy the requirements. To it could be added

$$\frac{\lambda e^2}{16\pi}\frac{\kappa(s^{\mu}v^{\nu} - s^{\nu}v^{\mu})\, s^{\sigma}}{s^5},$$

where λ is a numerical factor. The divergence of this is

$$\frac{\lambda e^2}{16\pi}\left\{\frac{4\kappa^2}{s^6}-\frac{(\ddot{v},s)}{s^5}\right\}s^\nu s^\sigma,$$

which is also of order s^{-2}. It is easily found that this changes p^σ_{el} by an amount $-\frac{1}{3}\lambda e^2\dot{v}^\sigma$. The corresponding mechanical vector is $mv^\sigma+\frac{1}{3}\lambda e^2\dot{v}^\sigma$. The tensor 4·5 was chosen to satisfy 4·12, so that the mechanical part of p^σ could be taken to be mv^σ, without any additional terms.

5. Symmetry between retarded and advanced variables

Up to this point the theory has been developed in an unsymmetrical way as regards the advanced and retarded fields. It is possible to formulate it quite symmetrically if one forms a tensor like 4·5 in which the advanced variables occur instead of the retarded. Let us call these two K_{ret} and K_{adv}. In the appendix it is shown that near the world-line $K_{ret}-K_{adv}$ is $O(s^{-2})$, and that, further, its integral over a small closed surface tends to zero. It is therefore immaterial which of the two is used in defining p^σ_{el}, and a symmetrical presentation is achieved by using

$$K_{sym}=\tfrac{1}{2}(K_{ret}+K_{adv}). \tag{5·1}$$

Let the field $F^{\mu\nu}$ be written in the form

$$F=\tfrac{1}{2}(F_{ret}+F_{adv})+f, \tag{5·2}$$

and let $T^{\nu\sigma}_{sym}$ denote the energy tensor formed from $\frac{1}{2}(F_{ret}+F_{adv})$ according to 3·1, $t^{\nu\sigma}$ that from f, and $t^{\nu\sigma}_{mix}$ the cross-term. We define $T'^{\nu\sigma}_{sym}$ by

$$T'^{\nu\sigma}_{sym}=T^{\nu\sigma}_{sym}-\frac{\partial}{\partial x^\mu}K^{\mu\nu\sigma}_{sym}.$$

In evaluating R^σ by integrating

$$\int(T'^{\nu\sigma}_{sym}+t^{\nu\sigma}_{mix}+t^{\nu\sigma})\,dS_\nu$$

over a world-tube, it is easily seen from the symmetry that the contribution from $T'^{\nu\sigma}_{sym}$ is zero, and the only contribution comes from $t^{\sigma\nu}_{mix}$, namely

$$R^\sigma=ef^{\sigma\nu}v_\nu.$$

This gives the symmetrical form of the equation of motion (Dirac, 1938, equation 22),

$$m\dot{v}^\sigma=ef^{\sigma\nu}v_\nu. \tag{5·3}$$

I am greatly indebted to Professor Dirac for letting me use the manuscript of his paper.

Appendix

The rules for differentiating the retarded variables are easily obtained by differentiating

$$s^\nu = x^\nu - z^\nu,$$

giving

$$\frac{\partial s^\nu}{\partial x^\mu} = \delta^\nu_\mu - v^\nu \frac{\partial \tau}{\partial x^\mu}.$$

Now s^ν is a null-vector; hence $\quad s_\nu \dfrac{\partial s^\nu}{\partial x^\mu} = 0.$

I.e.

$$\frac{\partial \tau}{\partial x^\mu} = \frac{s_\mu}{s},$$

$$\frac{\partial s^\nu}{\partial x^\mu} = \delta^\nu_\mu - \frac{v^\nu s_\mu}{s}.$$

The derivatives of v^ν, $\dot{v}^\nu$ and s are

$$\frac{\partial v^\nu}{\partial x^\mu} = \frac{\dot{v}^\nu s_\mu}{s}, \quad \frac{\partial \dot{v}^\nu}{\partial x^\mu} = \frac{\ddot{v}^\nu s_\mu}{s},$$

$$\frac{\partial s}{\partial x^\mu} = v_\mu - \frac{(1-\kappa)\, s_\mu}{s}.$$

With these rules it is easy to derive the following results:

$$\frac{\partial}{\partial x^\mu}\left[\frac{\kappa(s^\mu v^\nu - s^\nu v^\mu)\, s^\sigma}{s^5}\right] = \frac{4\kappa^2 s^\nu s^\sigma}{s^6} - \frac{(\ddot{v}, s)\, s^\nu s^\sigma}{s^5},$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{s^\mu g^{\nu\sigma} - s^\nu g^{\mu\sigma}}{s^4}\right] = \frac{-4(1-\kappa)\, s^\nu s^\sigma}{s^6} + \frac{v^\nu s^\sigma + 4 s^\nu v^\sigma}{s^5} - \frac{2g^{\nu\sigma}}{s^4},$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{\kappa(s^\mu g^{\nu\sigma} - s^\nu g^{\mu\sigma})}{s^4}\right] = \frac{-4\kappa(1-\kappa)\, s^\nu s^\sigma}{s^6} + \frac{\kappa(v^\nu s^\sigma + 4 s^\nu v^\sigma) - (\ddot{v}, s)\, s^\nu s^\sigma}{s^5} - \frac{\kappa g^{\nu\sigma} + s^\nu \dot{v}^\sigma}{s^4},$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{(v^\mu s^\nu - v^\nu s^\mu)\, v^\sigma}{s^4}\right] = \frac{-3\kappa s^\nu v^\sigma}{s^5} + \frac{v^\nu v^\sigma + s^\nu \dot{v}^\sigma}{s^4},$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{(\dot{v}^\mu s^\nu - \dot{v}^\nu s^\mu)\, s^\sigma}{s^4}\right] = \frac{4\kappa(1-\kappa)\, s^\nu s^\sigma}{s^6} + \frac{(\ddot{v}, s)\, s^\nu s^\sigma - \kappa(v^\nu s^\sigma + s^\nu v^\sigma)}{s^5} + \frac{\dot{v}^\nu s^\sigma + s^\nu \dot{v}^\sigma}{s^4},$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{v^\nu g^{\mu\sigma} - v^\mu g^{\nu\sigma}}{s^3}\right] = \frac{3(1-\kappa)\, v^\nu s^\sigma}{s^5} + \frac{2\kappa g^{\nu\sigma} - 3 v^\nu v^\sigma + \dot{v}^\nu s^\sigma}{s^4}.$$

Multiplying these by 9, 1, 2, 3, 3, 1 respectively, adding and multiplying the sum by $e^2/16\pi$, we find

$$\frac{e^2}{16\pi}\frac{\partial}{\partial x^\mu}\left\{\frac{9\kappa(s^\mu v^\nu - s^\nu v^\mu)\,s^\sigma}{s^5} + \frac{(v^\nu g^{\mu\sigma} - v^\mu g^{\nu\sigma})}{s^3} + \frac{(1+2\kappa)\,(s^\mu g^{\nu\sigma} - s^\nu g^{\mu\sigma}) + 3(v^\mu s^\nu - v^\nu s^\mu)\,v^\sigma + 3(\dot v^\mu s^\nu - \dot v^\nu s^\mu)\,s^\sigma}{s^4}\right\}$$

$$= \frac{e^2}{4\pi}\left\{\frac{(8\kappa^2+2\kappa-1)\,s^\nu s^\sigma}{s^6} + \frac{(\dot v^\nu s^\sigma + \dot s^\nu v^\sigma) - \frac{1}{2}g^{\nu\sigma}}{s^4} + \frac{(1-\kappa)\,(s^\nu v^\sigma + v^\nu s^\sigma) - 2(\ddot v, s)\,s^\nu s^\sigma}{s^5}\right\},$$

which is the result quoted in 4·6.

To compare K_{ret} and K_{adv} near the world-line let us express them in terms of new variables. Let y^μ be the point on the world-line such that $\gamma^\mu = x^\mu - y^\mu$ is orthogonal to the velocity λ^μ, taken at y^μ. Let $(\gamma,\gamma) = -r^2$, and let τ be the difference in the proper times at z^μ and y^μ. Then to the required order of small quantities

$$\begin{aligned}
s^\mu &= \gamma^\mu + \tau\lambda^\mu - \tfrac{1}{2}\tau^2\dot\lambda^\mu + O(r^3),\\
\tau &= r\{1+\tfrac{1}{2}(\gamma,\dot\lambda)\} + O(r^3),\\
v^\mu &= \lambda^\mu - r\dot\lambda^\mu + O(r^2),\\
\dot v^\mu &= \dot\lambda^\mu + O(r),\\
s &= r\{1-\tfrac{1}{2}(\gamma,\dot\lambda)\} + O(r^2),\\
\kappa &= (\gamma,\dot\lambda) + O(r^2).
\end{aligned}$$

Calculating $K^{\mu\nu\sigma}_{\text{ret}}$ to the order r^{-2} we find

$$K^{\mu\nu\sigma}_{\text{ret}} = \frac{e^2}{16\pi}\left\{\frac{9(\gamma,\dot\lambda)\,\gamma^\mu\lambda^\nu\gamma^\sigma}{r^5} + \frac{\{1+4(\gamma,\dot\lambda)\}\,\gamma^\mu g^{\nu\sigma} + 3\{1-(\gamma,\dot\lambda)\}\,\lambda^\mu\gamma^\nu\lambda^\sigma + 3\dot\lambda^\mu\gamma^\nu\gamma^\sigma}{r^4} + \frac{3(\gamma,\dot\lambda)\,\lambda^\mu g^{\nu\sigma} + 3\gamma^\mu\lambda^\nu\dot\lambda^\sigma + 3\dot\lambda^\mu\lambda^\nu\gamma^\sigma}{r^3} + \frac{\dot\lambda^\mu g^{\nu\sigma} + 3\dot\lambda^\mu\lambda^\nu\lambda^\sigma}{2r^2}\right\} - \ldots + O(r^{-1}),$$

where the minus sign at the end indicates that the similar terms with μ and ν interchanged must be subtracted.

The corresponding result for K_{adv} is obtained by reversing the sign of λ and leaving that of $\dot\lambda$ unchanged. The difference of the two is

$$K^{\mu\nu\sigma}_{\text{ret}} - K^{\mu\nu\sigma}_{\text{adv}} = \frac{3e^2}{8\pi}\left\{\frac{3(\gamma,\dot\lambda)\,\gamma^\mu\lambda^\nu\gamma^\sigma}{r^5} + \frac{(\gamma,\dot\lambda)\,\lambda^\mu g^{\nu\sigma} + \gamma^\mu\lambda^\nu\dot\lambda^\sigma + \dot\lambda^\mu\lambda^\nu\gamma^\sigma}{r^3}\right\} - \ldots + O(r^{-1}),$$

which is $O(r^{-2})$.

To calculate $\int(K^{a0\sigma}_{\text{ret}} - K^{a0\sigma}_{\text{adv}})\,d\sigma_a$ it is best to take a co-ordinate system in which λ has components (1, 0, 0, 0) and to integrate over a sphere. The components of $d\sigma_a$ are then $\epsilon\gamma^a\,d\Omega$, where $d\Omega$ is the element of solid angle on the sphere, γ^a the co-ordinates of a point on the sphere, and ϵ the radius. For $\sigma = 0$ the integrand vanishes; for $\sigma = b$ the integral is

$$\int\left(-3\frac{\dot{\lambda}^a\gamma^a\gamma^b}{\epsilon^2}+\dot{\lambda}^b\right)d\Omega+O(\epsilon)=O(\epsilon).$$

Both tensors therefore give the same result for the energy and momentum.

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The porous diaphragm method of measuring diffusion velocity, and the velocity of diffusion of potassium chloride in water

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The velocity of diffusion of substances in dilute solution is a very informative property, but one of which it has been possible in the past to make but little use, on account of the very low accuracy of all but the most extended and laborious experiments, as is well seen in the collection of data for KCl in water from various authors plotted by McBain and Dawson (1935). The chief experimental difficulty has been the elimination of convection which will be produced to some extent by any temperature fluctuations or vibration however small, and particularly by the process of dividing up the diffusing solution when this is necessary for analysis. Convection is least serious when concentrated solutions of a heavy solute are examined, because the large density gradient has a stabilizing influence: it is also less serious the more rapidly the solute diffuses. Unfortunately, however, there is as