Physical Significance of the Poynting Vector in Static Fields

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Even in static fields, where there is no observable energy flow, Poynting vector momentum must be considered to avoid an apparent violation of the angular-momentum law. This often-neglected aspect of the Poynting vector, is illustrated in an easily calculated example. Two other simple and rigorously solvable pedagogical examples illustrate the role of the Poynting vector in defining the energy flow in static fields.

It is conventional to use the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$, where $\mathbf{H}$ is defined by $\mathbf{H}=(\mathbf{B}-\mathfrak{M})/\mu_0$, to represent rates of flow of energy and momentum in electromagnetic waves. This vector, however, is seldom applied to situations where the electromagnetic fields are static. A permanent magnet placed in a static electric field usually results in nonzero values for $\mathbf{S}$ that indicate energy flowing in closed paths with zero divergence which cannot be observed. Since $\mathbf{S}$ is usually defined only in terms of its divergence, an infinite number of functions such as $\mathbf{S}'=\mathbf{E}\times\mathbf{H}+\mathbf{f}$, where $\nabla\cdot\mathbf{f}=0$, satisfies this definition.

Not surprisingly, doubts have often been expressed concerning the physical significance of such divergence-free energy flows.[1] However, the angular momentum implied by the conventional Poynting vector cannot be ignored. Specifically, the addition of a function $\mathbf{f}$, $\nabla\cdot\mathbf{f}=0$, to the Poynting vector will, in fact, change the physical significance of the vector unless the net angular momentum implied by $\mathbf{f}$ is constant with time (regardless of what physical experiment is performed). The *changes* in net angular momentum implied by the conventional Poynting vector are essential to the conservation of angular momentum. As Feynman[2] concludes concerning such static fields, "There really is a momentum flow. It is needed to maintain conservation of angular momentum in the whole world."

Our purpose is to underline the operational significance of this statement, using a gedanken classical experiment in which the relevant field energies and momenta are confined to a finite space where the energies, momenta, and fields can be exactly calculated.

Imagine a solid sphere of radius a, consisting of a ferromagnetic material with a uniform magnetization parallel to the z axis. For simplicity, consider the sphere to be a nonconductor covered with a thin conducting film, so that any current flow is confined to the surface, as illustrated in Fig. 1. Let this be surrounded by a spherical shell of nonmagnetic metal having the inner radius b. Assume both spheres to have negligible resistivity. If charges of $+Q$ and $-Q$ are placed on the inner and outer spheres, respectively, an electric field confined to the space between spheres will result. This electric field combined with the magnetic field from the inner sphere will produce values of $\mathbf{E}\times\mathbf{H}$ that circle the z axis.

Assume the two spheres to be free to rotate without friction about the z axis. Start with the spheres at rest and uncharged, so that the system initially has no angular momentum.

Next, slowly charge the two spheres, bringing equal but opposite charges along a coaxial cable whose axes coincide with the positive z axis as

[1] As an example, one reviewer of this paper has called our attention to an article published in this Journal by one of the authors [E. M. Pugh, Am. J. Phys. **32**, 879–883 (1964)] where such a caveat was inserted and worded too strongly (p. 882) in response to comments by another reviewer. In the earlier text [E. M. Pugh and E. W. Pugh, *Principles of Electricity and Magnetism* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1960)] the Poynting vector is used for static fields without restriction.

[2] R. P. Feynman, *The Feynman Lectures on Physics* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1964), Vol. 2, 27, 11.

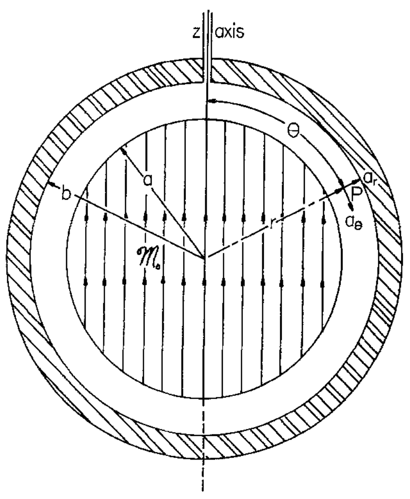

FIG. 1. Nonconducting sphere surrounded by two conducting spherical shells of radii a and b. The inner sphere is solid and magnetized.

shown in Fig. 1. The resulting electric currents in the sphere interact with the magnetic field ($\mathbf{F}=\mathbf{J}\times\mathbf{B}$) producing opposite but *not* equal angular momenta in the two spheres. The combined angular momentum of the two spheres is not zero. Thus, a system originally with no angular momentum and with no angular momentum being introduced is given a net mechanical angular momentum.

First let the magnitude of the charging current be computed. Assume a positive charge flows on to the inner sphere and a negative charge flows on to the outer sphere, the magnitude of each being Q. Further assume the charges flow onto each sphere so slowly that they can always be considered as uniformly distributed over each spherical surface. The magnitudes of these uniform charges are $q=q(t)$, where

$$\int_0^\infty q(t)dt=Q.$$

The current past a point P on the outer sphere can be derived from the rate of change of the negative charge below the zone of P. This charge is $-q(1+\cos\theta)/2$ and its rate of change is $-\{(1+\cos\theta)/2\}dq/dt$. Hence the total current $\mathbf{J}$ flowing past the zone of P is given by

$$\mathbf{J}=-\mathbf{a}_\theta[\tfrac{1}{2}(1+\cos\theta)](dq/dt)$$

and the rotational force per unit zone width at P is given by

$$\mathbf{J}\times\mathbf{B}=\mathbf{J}\times(\mathfrak{M}_0a^3/3b^3)(2\mathbf{a}_r\cos\theta+\mathbf{a}_\theta\sin\theta)$$
$$=\tfrac{1}{2}(1+\cos\theta)(dq/dt)(\mathfrak{M}_0a^3/3b^3)\mathbf{a}_\phi 2\cos\theta.$$

The force on the zone width $bd\theta$ at P is

$$d\mathbf{F}=\mathbf{a}_\phi(\mathfrak{M}_0a^3/3b^2)(1+\cos\theta)\cos\theta d\theta(dq/dt),$$

and the resulting torque about the axis is

$$dL=(\mathfrak{M}_0a^3/3b)(1+\cos\theta)\sin\theta\cos\theta d\theta(dq/dt).$$

Now the angular impulse delivered to the zone of the outer sphere between θ and $\theta+d\theta$ is

$$dI=\int_0^\infty dLdt,$$

or

$$dI=\frac{\mathfrak{M}_0a^3}{3b}(1+\cos\theta)\cos\theta\sin\theta d\theta\int_0^\infty\frac{dq}{dt}dt,$$

$$dI=(Q\mathfrak{M}_0a^3/3b)(1+\cos\theta)\cos\theta\sin\theta d\theta$$

and the total angular impulse delivered to the outer sphere is

$$I_b=\int_0^\pi dI,$$

which gives

$$I_b=\frac{Q\mathfrak{M}_0a^3}{3b}\int_0^\pi(1+\cos\theta)\cos\theta\sin\theta d\theta,$$

$$I_b=(2/9)(Q\mathfrak{M}_0a^3/b).$$

The same type of analysis shows that the total angular impulse delivered to the inner sphere is $I_a=-(2/9)(Q\mathfrak{M}_0a^3/a)$. Thus the net impulse is

$$I=I_a+I_b=-(2/9)(Q\mathfrak{M}_0a^3)[(1/a)-(1/b)],$$

which must give the net mechanical angular momentum of the system if the spheres rotate without friction.

Thus, a system having no angular momentum has been taken and, without adding any angular momentum, has been transformed into a system containing a net mechanical angular momentum. This appears to violate the law of conservation of angular momentum. However, there is a Poynting-vector flux circulating about the z axis in the existing static fields. Let us investigate its angular momentum. The magnetic field due to the uniformly magnetized sphere is

$$\mathbf{H}=(\mathfrak{M}_0 a^3/3\mu_0 r^3)(2\mathbf{a}_r\cos\theta+\mathbf{a}_\theta\sin\theta)$$

from $r=a$ to $r=b$. The electric field exists only between the spheres. Its value is $\mathbf{E}=\mathbf{a}_r Q/4\pi\epsilon_0 r^2$. The Poynting-vector momentum density is

$$\mathbf{p}=\mathbf{S}/C^2=\mathbf{a}_\phi(\mathfrak{M}_0 Qa^3\sin\theta/12\pi\mu_0\epsilon_0 C^2 r^5)$$

or

$$\mathbf{a}_\phi(\mathfrak{M}_0 Qa^3\sin\theta/12\pi r^5),$$

since $C^2\mu_0\epsilon_0=1$. The total angular momentum of the Poynting vector is given by

$$P=\int pr\sin\theta d\tau$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_a^b\frac{\mathfrak{M}_0 Qa^3}{12\pi r^4}r^2\sin^3\theta d\phi d\theta dr$$

$$P=(2/9)(2\mathfrak{M}_0 Qa^3)[(1/a)-(1/b)].$$

As expected, the inclusion of the Poynting vector in the preceding "experiment" provides a rigorous accounting for the change in mechanical angular momentum. Obviously, the same results will apply in the limit where $b\rightarrow\infty$. Thus, for example, the earth might be thought of as a charged magnetic sphere, which would have a significant circulating Poynting-vector flux.

In the preceding calculations, we have dealt only with the ϕ component of the Poynting vector (which remains after the charging of the spheres is complete). During the actual charging process, there is also a small θ component which results from the interaction of the static electric field with the magnetic field produced by the charging current itself (this field was ignored in the previous calculation). As guaranteed by the divergence derivation of the Poynting vector, this θ component is exactly the size required to account for the spatial distribution of the electrostatic energy accumulated in the field between the two spheres.

The existence of such a simple configuration, where angular momentum can be conserved only through the Poynting vector, should help dispel doubts about its reality. Although, the magnitude of the effect is too small to be easily observed, it is sufficiently large that modern experimental techniques, cleverly applied, might be able to detect or even measure the effect.

The foregoing analysis is all that is required to demonstrate our basic point concerning the "physical reality" of momentum carried by the Poynting vector. There are, however, a couple of details of the way momentum is absorbed by the field in the "experiment" which may be puzzling to some, and may merit some comment.

Although the net integrated torque experienced by the spheres is exactly right to account for the momentum in the field, the θ distribution of the torque does not correspond to the resultant θ distribution of momentum in the field. In particular, the torque on the spheres is largest in the upper hemispheres where the charging current is largest; but the resulting field momentum is symmetrical about the equator. If one imagines a modification of the experiment so that separate θ zones of the spheres are free to rotate independently, one would find that the mechanical angular momentum on a zone-by-zone basis would not balance the angular momentum of the field on a corresponding zone-by-zone basis. One wonders how the symmetry of the Poynting vector about the equator is established when the initial interaction of the spheres with the field is concentrated in the upper hemisphere. Apparently, angular momentum for Poynting vector is transported through the field from the upper to the lower hemisphere. The simple Poynting vector, of course, has nothing to say about this momentum transport since no net flow of energy is involved. Presumably a more complete analysis, using the full Maxwell stress energy tensor, would be required to account for this momentum transport.

Secondly, it is interesting to note that in this experiment, as in any situation where an electrostatic field is superimposed on an existing magnetic field, the addition of a small amount of electrostatic energy can result in a polarization of the momentum state for a much larger quantity of energy present due to the initial magnetic field. Consider an arbitrary point in the field. At any time in the charging process the energy density is given by

$$U=\tfrac{1}{2}\epsilon_0 E^2+\tfrac{1}{2}\mu_0 H^2$$

while the momentum density is given by

$$P=(\epsilon_0\mu_0)^{\frac{1}{2}}|\mathbf{E}\times\mathbf{H}|.$$

The value of U is quadratic with E while P is linear with E. At the beginning of the charging process, $\partial U/\partial E=0$ but $\partial P/\partial E$ is finite. Thus the initial change in angular momentum must result primarily from a change in the momentum state of energy *already* in the field. It cannot be attributed exclusively to the momentum of the energy added.

There are other interesting and informative problems in connection with static fields near dc circuits. For example, consider a small storage battery centrally placed between two large parallel disks. If the disks are metallic, circular, and of large diameter, the electric field in the air space between them will be perpendicular to their surfaces and equal to the potential difference divided by the distance between the plates. For simplicity consider the plates of the battery also to be disks, but of much smaller diameter, with centers on the axis of the larger disks. This configuration is connected to external circuits through superconducting wires attached to the centers of the large metallic disks.

(1) When the battery is being charged, the vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ between the disks but outside the battery will be directed radially inward toward the battery. The integrated flux of $\mathbf{S}$ will be the same at different radial distances and will be equal to the power flowing into the battery to appear as chemical and heat energy.

When the same battery is discharging, the electric current and the magnetic field are reversed, so the Poynting flux is directed outward. The total Poynting-vector flux then equals the rate at which chemical energy is being converted to electrical energy minus the energy appearing as heat within the battery.[3] A complete circuit might involve one such battery charging another such battery of lower voltage. Obviously, the Poynting flux emerging from the first finds its way into the second battery, but it is not easy to follow this flux in greater detail.

(2) The difficulty in following the Poynting flux from one battery to the other can be eliminated by placing these same batteries between concentric spheres instead of between parallel disks. Let the z axis pass vertically through the center of these spheres. Center the discharging battery on the z axis, and connect it between the top two points where this axis passes through the surfaces of the two spheres. Similarly, connect the battery being charged between the bottom two points where the z axis passes through the spherical surfaces. If the resistance of the spheres can be neglected and the positive terminals of both batteries are connected to the inner sphere, the electric field in the space between the two spheres will be given by

$$\mathbf{E}=\mathbf{a}_r Vab/(b-a)r^2,$$

where $\mathbf{a}_r$ is a unit vector parallel to an outward drawn radius, V is the potential difference between the spheres, r is the radial distance from the center to the field point P, and a and b are the radii of the inner and outer surfaces, respectively. If I is the total current, then at P

$$\mathbf{H}=-\mathbf{a}_\phi I/(2\pi r\sin\theta)$$

and

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mathbf{a}_\theta[VI/(2\pi r^3\sin\theta)]ab/(b-a).$$

The flux of $\mathbf{S}$ past any zone at $\theta=\theta$ is then

$$\int_a^b \mathbf{S}_\theta 2\pi r\sin\theta dr=VI,$$

which is just equal to the power flowing from the battery at the top to that at the bottom. These problems have been found to be good illustrative problems for students.

[3] If the source of this flux were traced in detail inside the battery, it would be found to originate at the interfaces between the battery plates and the electrolyte, where chemical energy is converted to electrical energy and where quantum effects force the electrons to flow in a direction counter to the prevailing electric field.