Dwight-Bewley paradox

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Indexing terms: Electromagnetic field theory, Dwight-Bewley paradox

Abstract: The concept of motional induced EMF is implicitly based on a reference frame relative to which the magnetic field pattern is unchanging or stationary, and uses the relative velocity v between the frame and the conductor in the expression $B\ v\ l$. For ease of conception, the velocity may be taken as that between the field pattern itself and the conductor. Where there are multiple field patterns in relative motion, no 'velocity' can be ascribed to the resultant field pattern relative to any object, fixed or moving. Therefore the EMF induced in a conductor situated in such a field cannot be expressed as a single motional EMF, but must be found by superposition of EMFs induced by the individual fluxes. From the impossiblity of applying the $B\ v\ l$ rule to the resultant flux, Bewley concludes that the EMF in the conductor is not motional but variational. It is shown here, mathematically from a consideration of reference frames, and physically from the fact of electromechanical energy conversion taking place, that Bewley's assertion is invalid and that the EMF is in fact motional, not variational. The $B\ v\ l$ rule is then generalised for application to such situations.

1 Introduction

Among the many 'paradoxes' described and solved by Bewley in his classic little book 'Flux linkages and electromagnetic induction' [1], one due to Dwight, called 'conductor in zero resultant magnetic field' is described below.

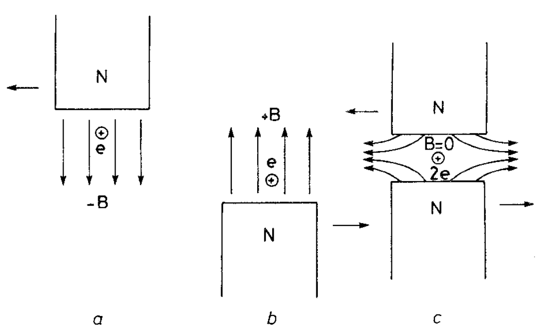

Fig. 1 *Superposition*

IEE Proceedings online no. 19960429

Paper first received 30th October 1995 and in revised form 4th March 1996

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In Fig. 1*a*, a north pole above a conductor is shown moving to the left and, taking the flux density to be 'cutting' the conductor, one arrives at an EMF e directed into the paper. In Fig. 1*b*, the north pole is below the conductor and moving to the right, and the induced EMF is again e and directed into the paper. Now suppose that two north poles, one above the conductor and moving to the left, and the other below the conductor and moving to the right, are used as shown in Fig. 1*c*. What is the EMF? Here the flux density is zero, and yet by the principle of superposition, the EMF should be 2 e!

2 Bewley's solution to the paradox

After presenting this as 'an interesting example of how a misconception may exist about (flux) cutting action', Bewley goes on to solve the paradox. According to him, 'the fallacy in the above reasoning is that the EMFs induced in the conductor are variational and not motional (cutting action) components at all'. He then proceeds to prove this on the following lines.

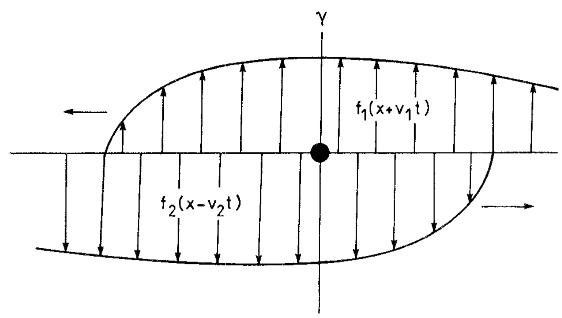

Fig. 2 *Travelling waves of flux density*

If the reference frame is taken on the conductor, the field is varying rather than moving. Even though no closed circuit is involved, the EMF induced in the conductor can be calculated as follows. Let the flux density distributions due to the two oppositely moving poles be expressed as travelling waves, so that

$$B_y = f_1(x + v_1 t) + f_2(x - v_2 t) \qquad (1)$$

Here, both the flux density space distributions and their velocities may be different, as indicated in Fig. 2. In vector form,

$$\mathbf{B} = \mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z$$

with $B_x = B_z = 0$.

The differential form of Maxwell's equation derived from Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (2)$$

which, when expanded, becomes

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} = 0 \quad (3a)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$
$$= -v_1 f_1'(x + v_1 t) + v_2 f_2'(x - v_2 t) \quad (3b)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial E_y} = -\frac{\partial B_z}{\partial t} = 0 \quad (3c)$$

The only nonzero part of these equations is, from eqn. 3*b*,

$$\frac{\partial E_z}{\partial x} = v_1 f_1'(x + v_1 t) - v_2 f_2'(x - v_2 t) \quad (4)$$

The solution to this differential equation, compatible with the implied boundary conditions, is

$$E_z = v_1 f_1(x + v_1 t) - v_2 f_2(x - v_2 t) \quad (5)$$

But this is the same resuft as is obtained by applying the 'cutting rule' separately to each component of flux density wave and superimposing the results.

In the particular situation described, three conditions apply:

(i) $x = 0$ (6)

(ii) $v_1 = v_2 = v(\text{say})$ (7)

(iii) The flux density at the conductor is zero which, together with eqns. 6 and 7, when imposed on eqn. 1, gives

$$B_y = f_1(vt) + f_2(-vt) = 0$$

or

$$f_1(vt) = -f_2(-vt) \quad (8)$$

Substituting eqns. 6–8 in eqn. 5,

$$E = vf_1(vt) + vf_1(vt) = 2vf_1(vt) \quad (9)$$

or twice the EMF for either wave alone.

If the flux densities are constant in space,

$$f_1(vt) = B \quad \text{and eqn.9 becomes}$$
$$E = 2Bv$$

or

$$e = 2Bvl \quad (10)$$

The superposition of EMFs calculated on the basis of *relative* motion works in this case, whereas, if the flux densities are superimposed first, there is no possibility of using the cutting rule.

Bewley concludes that 'the whole idea of cutting action is basically wrong in this case, and the induction is actually a pure example of variational induction'.

3 Bewley's reasoning

Consider a coil on the armature of an alternator. If the reference frame is fixed to the rotor, there is no flux variation with respect to the frame, but the coil has a veleocity relative to it, and the induced EMF in the coil is $B\ v\ l$, i.e. motional. On the other hand, if the reference frame is fixed to the coil, there is no relative velocity between the frame and the coil, both of which merely experience a flux variation, and the EMF appears as variational.

Consider next the induced EMF in the secondary winding of a transformer. Since both the winding and the flux are stationary in space, practical considerations dictate that the reference frame be also stationary, fixed to the winding and core. No velocity is involved and the coil experiences only a flux variation. The EMF is purely variational and cannot be expressed as motional. Hence it is clear that, by a change of reference frame, a motional EMF can always be expressed as a variational EMF, but a variational EMF can never be expressed as a motional EMF.

Any velocity used in $B\ v\ l$ must be with respect to the reference frame chosen [1]. The frame on which Bewley has expressed the flux density, eqn. 1, is fixed to the conductor itself, making the relative velocity between the frame and the conductor zero. The conductor merely experiences a flux variation, and naturally the EMF appears as variational. Furthermore, since the EMF cannot be expressed as motional EMF, i.e. as a single $B\ v\ l$, using the *resultant* B, Bewley concludes that it must be variational.

4 Is Bewley's conclusion valid?

First, Bewley's conclusion contradicts physical facts, for there is motion, and if it is stopped there will be no EMF. Secondly, if either magnet is moved alone, the induced EMF is obviously motional. If the components are motional EMFs, must not the resultant, *a fortiori*, be a motional EMF?

But the ultimate criterion is whether the situation involves electromechanical power conversion, or merely electric power transfer between circuits. If a circuit with a motional EMF is closed there will be power conversion from mechanical to electrical, i.e. generator action. If a circuit with a purely variational EMF is closed, there will only be electric power transfer to this circuit from another circuit, i.e. transformer action.

To apply this criterion to the present situation, let the conductor form part of a closed circuit. (If practical or conceptual difficulties are encountered in visualising this condition, alternatively let the configuration of Fig. 1*c* represent the development of one half of a rotating machine. The other half will have a pair of south poles, one on either side of the 'airgap', with another conductor placed symetrically between them. Let the two conductors be connected at one end of the machine so as to form a single-turn coil. The inner and outer magnetic systems will have to be coupled in such a way that, driven by a prime mover, they will not only run at the same speed but in opposite directions, but also each pair of like poles is always symmetrical about the conductors. The turn can easily be made part of a closed circuit).

It is obvious that there will be electric power dissipation in the circuit, and this power can only come from the mechanical agent moving the magnets (or the prime mover), which means that there is electromechanical power conversion. Therefore the EMF is purely *motional* and not *variational* at all!

Why then is it not possible to express it as $B\ v\ l$ using the resultant B?

5 Solution to the paradox

A clue can be had from the following. In the situation described, the conductor is stationary and the fields are moving. Is it possible to reverse their conditions, i.e. keep the fields stationary and move the conductor, and still get the same effect? Obviously not.

More formally, the issue is one of reference frames. To obtain motional EMF, the reference frame should be such that, with respect to it, the field pattern remains unchanging. Such a frame is obviously the source of the field itself, which may be a current-carrying coil or a magnet. In the case of the alternator, the rotor obviously constitutes such a frame. But, in the Dwight–Bewley situation, there is no such reference frame, as there is not one, but two separate magnets or unchanging field patterns in relative motion with respect to each other. To which of them should the reference frame be fixed? This is the crux of the matter.

Let us now view the problem mathematically and try to apply the $B\ v\ l$ rule. The questions to be answered concern the values of B and v.

Bewley says that if the flux densities are superimposed, the resultant at the conductor becomes zero and it is not possible to apply the $B\ v\ l$ rule. But then, in that case, neither is it possible to apply the $\partial \mathbf{B}/\partial t$ rule, since $\mathbf{B}$ is permanently zero. However, for his own derivation, Bewley tacitly sidesteps this problem by starting with a general expression for the total flux density at any point distant x from the origin (and imposing the condition $B_y = 0$ only at the end). Let us try the same procedure. Eqn. 1 is reproduced below, for convenience in algebraic form:

$$B = B_1 + B_2 = f_1(x + v_1 t) + f_2(x - v_2 t) \qquad (11)$$

Since Bewley implies that the $B\ v\ l$ rule might be applicable to points where $B \neq 0$, consider a point $x \neq 0$. At such a point, B will have a finite nonzero value under Dwight's conditions. Now, what is the velocity of this B relative to the conductor? The answer is that no single 'total' or resultant velocity can be ascribed to this B as a whole. So we find that even where there is a finite resultant B, we are unable to apply the $B\ v\ l$ rule owing to the nonexistence of a resultant v.

We looked for the velocity of B only because we know that the conductor is stationary. Now, what can 'velocity of B' possibly mean? It can only mean the velocity of the field source relative to the conductor. If so, what is the 'resultant field source'? Again, no single resultant source can be identified as the two component sources are in relative motion. This is what the problem of reference frames amounts to in practical terms.

With the real problem thus identified, both physically and mathematically, the solution becomes apparent. It is to recognise that each component flux pattern and its velocity are both integral to their respective source. This means that each B and its v are inseparable and no attempt should be made to combine the two flux densities or the two velocities separately. The fundamental or elemental quantity that can be combined or superimposed is the *electric field* $E = B\ v$ rather than the flux density B. The induced electric field in the conductor should therefore be found from

$$E = \sum (B_n v_n) \qquad (12)$$

$$= B_1 v_1 + B_2(-v_2)$$
$$= f_1(x + v_1 t)v_1 + f_2(x - v_2 t)(-v_2) \qquad (13)$$

This naturally takes us directly to eqn. 5 of Bewley's derivation, and the rest of the steps are simply those following it there — eqns. 6–10.

In vector notation, the electric field in a conductor due to flux cutting is given by

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \qquad (14)$$

where v is the velocity of the conductor relative to the field. Again this equation needs to be generalised as

$$\mathbf{E} = \sum (\mathbf{v_n} \times \mathbf{B_n}) \qquad (15)$$

Since Dwight's EMF is derivable by eqn. 12 or eqn. 15, it is motional, not variational.

6 Bewley's contribution

Bewley's conclusion may be erroneous, but his derivation is of great significance. Other writers [2–4] obtain eqn. 14 for 'flux-cutting electric field' from the Lorentz formula (for force on a current-carrying conductor situated across a magnetic field) which is treated as fundamental. Bewley has shown that flux-cutting EMF in a *straight conductor* is directly derivable from the relevant Maxwell equation. He has also demonstrated that in situations where the applicability of other theories is doubtful, a recourse to Maxwell's equations can lead to the correct quantitative result.

7 Conclusion

The EMF induced in a conductor situated across multiple magnetic field patterns in relative motion cannot be found as a single resultant $B\ v\ l$ because no resultant v exists, but this fact does not make the EMF variational because physically it is motional. The elemental quantity that can be combined to obtain a resultant in such situations has been shown to be the electric field $E = Bv$. The law of motional EMF has accordingly been generalised for application to such situations.

8 Acknowledgment

The author learnt the essentials of electromagnetism from the late Prof. G.W. Carter of Leeds University and from his classic book 'The electromagnetic field in its engineering aspects' [5].

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