It will be seen above that the heavy barium crown glasses are those which show the greatest variation in homogeneity in the same melting.

My thanks are due to Sir David Gill for his great kindness and help in arranging the paper.

Electrical Vibrations on a Thin Anchor-Ring.

By Lord RAYLEIGH, O.M., F.R.S.

(Received May 24,—Read June 27, 1912.)

Although much attention has been bestowed upon the interesting subjec of electric oscillations, there are comparatively few examples in which definite mathematical solutions have been gained. These problems are much simplified when conductors are supposed to be perfect, but even then the difficulties usually remain formidable. Apart from cases where the propagation may be regarded as being in one dimension,* we have Sir J. Thomson's solutions for electrical vibrations upon a conducting sphere or cylinder.† But these vibrations have so little persistence as hardly to deserve their name. A more instructive example is afforded by a conductor in the form of a circular ring, whose circular section is supposed small. There is then in the neighbourhood of the conductor a considerable store of energy which is more or less entrapped, and so allows of vibrations of reasonable persistence. This problem was very ably treated by Pocklington‡ in 1897, but with deficient explanations.§ Moreover, Pocklington limits his detailed conclusions to one particular mode of free vibration. I think I shall be doing a service in calling attention to this investigation, and in exhibiting the result for the radiation of vibrations in the higher modes. But I do not attempt a complete re-statement of the argument.

Pocklington starts from Hertz's formulæ for an elementary vibrator at the origin of co-ordinates ξ, η, ζ,

$$P = \frac{d^2\Pi}{d\xi\, d\zeta}, \qquad Q = \frac{d^2\Pi}{d\eta\, d\zeta}, \qquad R = \frac{d^2\Pi}{d\zeta^2} + \alpha^2\Pi, \tag{1}$$

where

$$\Pi = e^{i\alpha\rho}e^{ipt}/\rho, \tag{2}$$

in which P, Q, R denote the components of electromotive intensity, $2\pi/p$ is

* 'Phil. Mag.,' 1897, vol. 43, p. 125; 1897, vol. 44, p. 199; 'Scientific Papers,' vol. 4, pp. 276, 327.

† 'Recent Researches,' 1893, §§ 301, 312.

‡ 'Camb. Proceedings,' 1897, vol. 9, p. 324.

§ Compare W. McF. Orr, 'Phil. Mag.,' 1903, vol. 6, p. 667.

the period of the disturbance, and $2\pi/\alpha$ the wave-length corresponding in free æther to this period. At a great distance ρ from the source, we have from (1)

$$\mathrm{P, Q, R} = \frac{\alpha^2 e^{i\alpha\rho}}{\rho}\left(-\frac{\xi\zeta}{\rho^2}, -\frac{\eta\zeta}{\rho^2}, 1-\frac{\zeta^2}{\rho^2}\right). \tag{3}$$

The resultant is perpendicular to ρ, and in the plane containing ρ and ζ. Its magnitude is

$$-\frac{\alpha^2 e^{i\alpha\rho}}{\rho}\sin\chi, \tag{4}$$

where χ is the angle between ρ and ζ.

The required solution is obtained by a distribution of elementary vibrators of this kind along the circular axis of the ring, the axis of the vibrator being everywhere tangential to the axis of the ring and the coefficient of intensity proportional to $\cos m\phi'$ where m is an integer and ϕ' defines a point upon the axis. The calculation proceeds in terms of semi-polar co-ordinates z, ϖ, ϕ, the axis of symmetry being that of z, and the origin being at the centre of the circular axis. The radius of the circular axis is a, and the radius of the circular section is ϵ, ϵ being very small relatively to a. The condition to be satisfied is that at every point of the surface of the ring, where $(\varpi-a)^2+z^2 = \epsilon^2$, the tangential component of (P, Q, R), shall vanish. It is not satisfied absolutely by the above specification; but Pocklington shows that to the order of approximation required the specification suffices, provided α be suitably chosen. The equation determining α expresses the evanescence of that tangential component which is parallel to the circular axis, and it takes the form

$$\int_0^\pi d\phi\,\Pi_0 \cos m\phi\,(m^2-\alpha^2 a^2\cos\phi) = 0, \tag{5}$$

where

$$\Pi_0 = \frac{e^{i\alpha[\epsilon^2+4\varpi a\sin^2\frac{1}{2}\phi]}}{\sqrt{[\epsilon^2+4\varpi a\sin^2\frac{1}{2}\phi]}}. \tag{6}$$

In (5) we are to retain the large term, arising in the integral when ϕ is small, and the finite term, but we may reject *small* quantities. Thus Pocklington finds

$$\int_0^\pi \frac{(a^2\alpha^2\cos\phi-m^2)\cos m\phi\,d\phi}{\sqrt{\{\epsilon^2+4a^2\sin^2\frac{1}{2}\phi\}}} + \int_0^\pi \frac{(e^{2i\alpha a\sin\frac{1}{2}\phi}-1)(a^2\alpha^2\cos\phi-m^2)\cos m\phi\,d\phi}{2a\sin\frac{1}{2}\phi} = 0, \tag{7}$$

the condition being to this order of approximation the same at all points of a cross-section.

The first integral in (7) may be evaluated for any (integral) value of m. Writing $\frac{1}{2}\phi = \psi$, we have

$$\int_0^{\frac{1}{2}\pi} \frac{(a^2\alpha^2 \cos 2\psi - m^2) \cos 2m\psi \, d\psi}{a\sqrt{\{\epsilon^2/4a^2 + \sin^2\psi\}}}. \tag{8}$$

The large part of the integral arises from small values of ψ. We divide the range of integration into two parts, the first from 0 to ψ where ψ, though small, is large compared with $\epsilon/2a$, and the second from ψ to $\frac{1}{2}\pi$. For the first part we may replace $\cos 2\psi$, $\cos 2m\psi$ by unity, and $\sin^2\psi$ by ψ^2. We thus obtain

$$\frac{\alpha^2 a^2 - m^2}{a} \log \{\psi + \sqrt{(\epsilon^2/4a^2 + \psi^2)}\}_0^{\psi} = \frac{\alpha^2 a^2 - m^2}{a} (\log 4a/\epsilon + \log \psi). \tag{9}$$

Thus to a first approximation $\alpha a = \pm m$. In the second part of the range of integration we may neglect $\epsilon^2/4a^2$ in comparison with $\sin^2\psi$, thus obtaining

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{(a^2\alpha^2 \cos 2\psi - m^2) \cos 2m\psi \, d\psi}{a \sin \psi}. \tag{10}$$

The numerator may be expressed as a sum of terms such as $\cos^{2n}\psi$, and for each of these the integral may be evaluated by taking $\cos\psi = z$, in virtue of

$$\int \frac{z^{2n}\,dz}{z^2-1} = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots + \frac{z^{2n-1}}{2n-1} + \tfrac{1}{2} \log \frac{1-z}{1+z}.$$

Accordingly

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^{2n}\psi \, d\psi}{\sin\psi} = -\cos\psi - \frac{\cos^3\psi}{3} - \ldots\ldots - \frac{\cos^{2n-1}\psi}{2n-1} - \log \tan \tfrac{1}{2}\psi$$

$$= -1 - \tfrac{1}{3} - \ldots\ldots - \frac{1}{2n-1} - \log \tfrac{1}{2}\psi, \tag{11}$$

when small quantities are neglected. For example,

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^2\psi \, d\psi}{\sin\psi} = -1 - \log \tfrac{1}{2}\psi, \qquad \int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^4\psi \, d\psi}{\sin\psi} = -\tfrac{4}{3} - \log \tfrac{1}{2}\psi.$$

The sum of the coefficients in the series of terms (analogous to $\cos^{2n}\psi$) which represents the numerator of (10) is necessarily $a^2\alpha^2 - m^2$, since this is the value of the numerator itself when $\psi = 0$. The particular value of ψ chosen for the division of the range of integration thus disappears from the sum of (9) and (10), as of course it ought to do.

When $m = 1$, corresponding to the gravest mode of vibration specially considered by Pocklington, the numerator in (10) is

$$4a^2\alpha^2 \cos^4\psi - (4a^2\alpha^2 + 2) \cos^2\psi + a^2\alpha^2 + 1,$$

and the value of the integral is accordingly

$$\frac{1}{a}\left[2-\frac{4a^2\alpha^2}{3}-(a^2\alpha^2-1)\log\tfrac{1}{2}\psi\right].$$

To this is to be added from (9)

$$\frac{a^2\alpha^2-1}{a}\left[\log\frac{4a}{\epsilon}+\log\psi\right],$$

making altogether for the value of (8)

$$\frac{1}{a}\left[(a^2\alpha^2-1)\log\frac{8a}{\epsilon}+2-\frac{4a^2\alpha^2}{3}\right]. \tag{12}$$

The second integral in (7) contributes only finite terms, but it is important as determining the imaginary part of α and thus the rate of dissipation. We may write it

$$\frac{m^2}{2a}\int_0^{\frac{1}{2}\pi}d\psi\,\frac{e^{ix\sin\psi}-1}{\sin\psi}\{\cos(2m+2)\psi+\cos(2m-2)\psi-2\cos 2m\psi\}, \tag{13}$$

where $x^2 = 4a^2\alpha^2 = 4m^2$ approximately.

Pocklington shows that the imaginary part of (13) can be expressed by means of Bessel's functions. We may take

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}d\psi\cos 2n\psi\, e^{ix\sin\psi} = J_{2n}(x)+iK_{2n}(x), \tag{14}*$$

whence
$$\int_0^{\frac{1}{2}\pi}d\psi\cos 2n\psi\,\frac{e^{ix\sin\psi}-1}{\sin\psi} = \frac{i\pi}{2}\int_0^x\{J_{2n}(x)+iK_{2n}(x)\}\,dx. \tag{15}$$

Accordingly, (13) may be replaced by

$$\frac{im^2\pi}{4a}\int_0^x dx\,\{J_{2m+2}(x)-2J_{2m}(x)+J_{2m-2}(x)+i(K_{2m+2}-2K_{2m}+K_{2m-2})\}. \tag{16}$$

Now† $J_{2m+2}-2J_{2m}+J_{2m-2} = 4J''_{2m}$,

so that
$$\int_0^x dx\,\{J_{2m+2}-2J_{2m}+J_{2m-2}\} = 4J'_{2m} = 2J_{2m-1}-2J_{2m+1}. \tag{17}$$

The imaginary part of (13) is thus simply

$$\frac{im^2\pi}{2a}\{J_{2m-1}(x)-J_{2m+1}(x)\}. \tag{18}$$

A corresponding theory for the K functions does not appear to have been developed

When $m = 1$, our equation becomes

$$\left(\frac{x^2}{4}-1\right)\log\frac{8a}{\epsilon} = -\frac{i\pi}{2}\{J_1(x)-J_3(x)\}+\frac{x^2}{3}-2$$
$$-\int_0^{\frac{1}{2}\pi}d\psi\,\frac{\cos(x\sin\psi)-1}{2\sin\psi}(1-2\cos 2\psi+\cos 4\psi), \tag{19}$$

* Compare 'Theory of Sound,' § 302.

† Gray and Mathews, 'Bessel's Functions,' p. 13.

and on the right we may replace x by its first approximate value. Referring to (2) we see that the negative sign must be chosen for α and x, so that $x = -2$. The imaginary term on the right is thus

$$\frac{i\pi}{2}\{J_1(2)-J_3(2)\} = 0{\cdot}70336i.$$

For the real term Pocklington calculates 0·485, so that, L being written for $\log(8\,a/\epsilon)$,

$$-\alpha = \frac{1}{a}\{1+(0{\cdot}243+0{\cdot}352i)/L\}. \qquad (20)$$

"Hence the period of the oscillation is equal to the time required for a free wave to traverse a distance equal to the circumference of the circle multiplied by $1-0{\cdot}243/L$, and the ratio of the amplitudes of consecutive vibrations is $1 : e^{-2{\cdot}21/L}$ or $1-2{\cdot}21/L$."

For the general value of m (19) is replaced by

$$(a^2\alpha^2-m^2)\,L = \frac{im^2\pi}{2}\{J_{2m-1}(2m)-J_{2m+1}(2m)\}+R, \qquad (21)$$

where R is a real finite number, and finally

$$-\alpha = \frac{m}{a}\left[1+\frac{R}{2m^2L}+\frac{i\pi}{4L}\{J_{2m-1}(2m)-J_{2m+1}(2m)\}\right]. \qquad (22)$$

The ratio of the amplitudes of successive vibrations is thus

$$1 : 1-\pi^2\{J_{2m-1}(2m)-J_{2m+1}(2m)\}/2L, \qquad (23)$$

in which the values of $J_{2m-1}(2m) - J_{2m+1}(2m)$ can be taken from the tables (see Gray and Mathews). We have as far as m equal to 12:—

m.	$J_{2m-1}(2m)-J_{2m+1}(2m)$.	m.	$J_{2m-1}(2m)-J_{2m+1}(2m)$.
1	0·448	7	0·136
2	0·298	8	0·125
3	0·232	9	0·116
4	0·194	10	0·108
5	0·169	11	0·102
6	0·150	12	0·096

It appears that the damping during a *single vibration* diminishes as m increases, viz., the greater the number of subdivisions of the circumference.

An approximate expression for the tabulated quantity when m is large may be at once derived from a formula due to Nicholson,* who shows that when n and z are large and nearly equal, $J_n(z)$ is related to Airy's integral. In fact,

$$J_n(z) = \frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}}\int_0^\infty \cos\left\{w^3+(n-z)\left(\frac{6}{z}\right)^{\frac{1}{3}}w\right\}dw$$

$$= \frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}}\left[\frac{\Gamma(\frac{1}{3})}{2\sqrt{3}} - (n-z)\left(\frac{6}{z}\right)^{\frac{1}{3}}\frac{\Gamma(\frac{2}{3})}{2\sqrt{3}}\right], \qquad (24)$$

* 'Phil. Mag.,' 1908, vol. 16, pp. 276, 277.

so that
$$J_{2m-1}(2m)-J_{2m+1}(2m)=\left(\frac{3}{m}\right)^{\frac{2}{3}}\frac{\Gamma(\frac{2}{3})}{\pi\sqrt{3}}. \tag{25}$$

If we apply this formula to $m = 10$. we get 0·111 as compared with the tabular 0·108.*

It follows from (25) that the damping in each vibration diminishes without limit as m increases. On the other hand, the damping in a *given time* varies as $m^{\frac{1}{3}}$ and increases indefinitely, if slowly, with m.

We proceed to examine more in detail the character at a great distance of the vibration radiated from the ring. For this purpose we choose axes of x and y in the plane of the ring, and the coördinates (x, y, z) of any point may also be expressed as $r \sin\theta\cos\phi$, $r\sin\theta\sin\phi$, $r\cos\theta$. The contribution of an element $a\,d\phi'$ at ϕ' is given by (4). The direction cosines of this element are $\sin\phi'$, $-\cos\phi'$, 0; and those of the disturbance due to it are taken to be l, m, n. The direction of this disturbance is perpendicular to r and in the plane containing r and the element of arc $a d\phi'$. The first condition gives $lx+my+nz=0$, and the second gives

$$l\,.\,z\cos\phi'+m\,.\,z\sin\phi'-n(x\cos\phi'+y\sin\phi')=0;$$

so that
$$\frac{l}{(z^2+y^2)\sin\phi'+xy\cos\phi'}=\frac{-m}{(z^2+x^2)\cos\phi'+xy\sin\phi')}=\frac{n}{zy\cos\phi'-zx\sin\phi'}. \tag{26}$$

The sum of the squares of the denominators in (26) is

$$r^2\{z^2-(y\sin\phi'+x\cos\phi')^2\},$$

Also in (4)
$$\sin^2\chi=1-\frac{(x\sin\phi'-y\cos\phi')^2}{r^2}=\frac{z^2+(x\cos\phi'+y\sin\phi')^2}{r^2}; \tag{27}$$

and thus
$$\left.\begin{aligned} r^2\,.\,l\sin\chi &= (z^2+y^2)\sin\phi'+xy\cos\phi',\\ -r^2\,.\,m\sin\chi &= (z^2+x^2)\cos\phi'+xy\sin\phi',\\ r^2\,.\,n\sin\chi &= zy\cos\phi'-zx\sin\phi'.\end{aligned}\right\} \tag{28}$$

To these quantities the components P, Q, R due to the element $a d\phi'$ are proportional.

Before we can proceed to an integration there are two other factors to be regarded. The first relates to the intensity of the source situated at $a d\phi'$. To represent this we must introduce $\cos m\phi'$. Again, there is the question of phase. In $e^{ia\rho}$ we have

$$\rho=r-a\sin\theta\cos(\phi'-\phi);$$

and in the denominator of (4) we may neglect the difference between ρ and r.

* $\log_{10}\Gamma(\frac{2}{3}) = 0{\cdot}13166$.

Thus, as the components due to $ad\phi'$, we have

$$\mathrm{P} = -\frac{\alpha^2 a e^{i\alpha r}}{r} d\phi' e^{-i\alpha a \sin\theta \cos(\phi'-\phi)} \cos m\phi' \frac{(z^2+y^2)\sin\phi' + xy\cos\phi'}{r^2}, \tag{29}$$

with similar expressions for Q and R corresponding to the right-hand members of (28). The integrals to be considered may be temporarily denoted by S, C, where

$$\mathrm{S}, \mathrm{C} = \int_{-\pi}^{+\pi} d\phi' \cos m\phi' e^{-i\zeta\cos(\phi'-\phi)} (\sin\phi', \cos\phi), \tag{30}$$

ζ being written for $\alpha a \sin\theta$. Here

$$\mathrm{S} = \tfrac{1}{2}\int_{-\pi}^{+\pi} d\phi' e^{-i\zeta\cos(\phi'-\phi)} \{\sin(m+1)\phi' - \sin(m-1)\phi'\},$$

and in this, if we write ψ for $\phi'-\phi$,

$$\sin(m+1)\phi' = \sin(m+1)\psi\cos(m+1)\phi + \cos(m+1)\psi\sin(m+1)\phi.$$

We thus find

$$\mathrm{S} = \Theta_{m+1}\sin(m+1)\phi - \Theta_{m-1}\sin(m-1)\phi, \tag{31}$$

where

$$\Theta_n = \int_0^\pi d\psi \cos n\psi e^{-i\zeta\cos\psi}. \tag{32}$$

In like manner,

$$\mathrm{C} = \Theta_{m+1}\cos(m+1)\phi + \Theta_{m-1}\cos(m-1)\phi. \tag{33}$$

Now

$$\Theta_n = \int_0^\pi d\psi \cos n\psi \{\cos(\zeta\cos\psi) - i\sin(\zeta\cos\phi)\}.$$

When n is even, the imaginary part vanishes, and

$$\Theta_n = \frac{\pi \mathrm{J}_n(\zeta)}{\cos\frac{1}{2}n\pi}. \tag{34}$$

On the other hand, when n is odd, the real part vanishes, and

$$\Theta_n = -\frac{i\pi \mathrm{J}_n(\zeta)}{\sin\frac{1}{2}n\pi}. \tag{35}$$

Thus, when m is even, $m+1$ and $m-1$ are both odd and S and C are both pure imaginaries. But when m is odd, S and C are both real.

As functions of direction we may take P, Q, R to be proportional to

$$\mathrm{S}\frac{z^2+y^2}{r^2} + \mathrm{C}\frac{xy}{r^2}, \quad -\mathrm{C}\frac{z^2+x^2}{r^2} - \mathrm{S}\frac{xy}{r^2}, \quad \mathrm{C}\frac{zy}{r^2} - \mathrm{S}\frac{zx}{r^2}.$$

Whether m be odd or even, the three components are in the same phase. On the same scale the intensity of disturbance, represented by $\mathrm{P}^2+\mathrm{Q}^2+\mathrm{R}^2$, is in terms of θ, ϕ

$$\cos^2\theta\,(\mathrm{S}^2+\mathrm{C}^2) + \sin^2\theta\,(\mathrm{C}\cos\phi + \mathrm{S}\sin\phi)^2, \tag{36}$$

an expression whose sign should be changed when m is even. Introducing the values of C and S in terms of Θ from (31), (33), we find that $P^2+Q^2+R^2$ is proportional to

$$\cos^2\theta\,\{\Theta_{m+1}{}^2+\Theta_{m-1}{}^2+2\Theta_{m+1}\Theta_{m-1}\cos 2m\phi\}+\sin^2\theta\cos^2 m\phi\,\{\Theta_{m+1}+\Theta_{m-1}\}^2. \tag{37}$$

From this it appears that for directions lying in the plane of the ring ($\cos\theta=0$) the radiation vanishes with $\cos m\phi$. The expression (37) may also be written

$$\Theta_{m+1}{}^2+\Theta_{m-1}{}^2+2\Theta_{m+1}\Theta_{m-1}\cos 2m\phi-\tfrac{1}{2}\sin^2\theta\,(\Theta_{m+1}-\Theta_{m-1})^2(1-\cos 2m\phi), \tag{38}$$

or, in terms of J's, by (34), (35),

$$\pi^2\,[J_{m+1}{}^2+J_{m-1}{}^2-2J_{m+1}J_{m-1}\cos 2m\phi-\tfrac{1}{2}\sin^2\theta\,(J_{m+1}+J_{m-1})^2(1-\cos 2m\phi)], \tag{39}$$

and this whether m be odd or even. The argument of the J's is $\alpha a\sin\theta$.

Along the axis of symmetry ($\theta=0$) the expression (39) should be independent of ϕ. That this is so is verified when we remember that $J_n(0)$ vanishes except $n=0$. The expression (39) thus vanishes altogether with θ unless $m=1$, when it reduces to π^2 simply.* In the neighbourhood of the axis the intensity is of the order θ^{2m-2}.

In the plane of the ring ($\sin\theta=1$) the general expression reduces to

$$\pi^2\,(J_{m+1}-J_{m-1})^2\cos^2 m\phi, \quad \text{or} \quad 4\pi^2 J_m'^2\cos^2 m\phi. \tag{40}$$

It is of interest to consider also the *mean* value of (39) reckoned over angular space. The mean with respect to ϕ is evidently

$$\pi^2\,[J_{m+1}{}^2+J_{m-1}{}^2+\tfrac{1}{2}\sin^2\theta\,(J_{m+1}+J_{m-1})^2]. \tag{41}$$

By a known formula in Bessel's functions

$$\{J_{m+1}(\zeta)+J_{m-1}(\zeta)\}^2=\frac{4m^2}{\zeta^2}J_m{}^2(\zeta). \tag{42}$$

For the present purpose

$$\zeta^2=a^2\alpha^2\sin^2\theta=m^2\sin^2\theta\,;$$

and (41) becomes

$$\pi^2\,[J_{m+1}{}^2(\zeta)+J_{m-1}{}^2(\zeta)-2J_m{}^2(\zeta)]. \tag{43}$$

To obtain the mean over angular space we have to multiply this by $\sin\theta\,d\theta$, and integrate from 0 to $\frac{1}{2}\pi$. For this purpose we require

$$\int_0^{\frac{1}{2}\pi} J_n{}^2(m\sin\theta)\sin\theta\,d\theta, \tag{44}$$

an integral which does not seem to have been evaluated.

* [*June* 20.—Reciprocally, plane waves, travelling parallel to the axis of symmetry and incident upon the ring, excite none of the higher modes of vibration.]

By a known expansion* we have

$$J_0(2m\sin\theta\sin\tfrac{1}{2}\beta) = J_0{}^2(m\sin\theta)+2J_1{}^2(m\sin\theta)\cos\beta+2J_2{}^2(m\sin\theta)\cos 2\beta + \ldots\ldots,$$

whence

$$\int_0^{\frac{1}{2}\pi} J_0(2m\sin\theta\sin\tfrac{1}{2}\beta)\sin\theta\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} J_0{}^2(m\sin\theta)\sin\theta\,d\theta + 2\cos\beta\int_0^{\frac{1}{2}\pi} J_1{}^2(m\sin\theta)\sin\theta\,d\theta+\ldots\ldots$$
$$+2\cos n\beta\int_0^{\frac{1}{2}\pi} J_n{}^2(m\sin\theta)\sin\theta\,d\theta. \qquad (45)$$

Now† for the integral on the left

$$\int_0^{\frac{1}{2}\pi} J_0(2m\sin\theta\sin\tfrac{1}{2}\beta)\sin\theta\,d\theta = \frac{\sin(2m\sin\frac{1}{2}\beta)}{2m\sin\frac{1}{2}\beta};$$

and thus

$$\int_0^{\frac{1}{2}\pi} J_n{}^2(m\sin\theta)\sin\theta\,d\theta = \frac{1}{2\pi m}\int_0^{\pi} d\beta\cos n\beta\,\frac{\sin(2m\sin\frac{1}{2}\beta)}{2m\sin\frac{1}{2}\beta}$$
$$= \frac{1}{\pi m}\int_0^{\frac{1}{2}\pi} d\psi\cos 2n\psi\,\frac{\sin(2m\sin\psi)}{\sin\psi} = \frac{1}{2m}\int_0^{2m} J_{2n}(x)\,dx, \qquad (46)$$

as in (15). Thus the mean value of (43) is

$$\frac{\pi^2}{2m}\int_0^{2m} dx\,\{J_{2m+2}(x)+J_{2m-2}(x)-2J_{2m}(x)\} = \frac{2\pi^2}{m}J_{2m}'(2m)$$
$$= \frac{\pi^2}{m}\{J_{2m-1}(2m)-J_{2m+1}(2m)\}, \qquad (47)$$

as before.

In order to express fully the mean value of $P^2+Q^2+R^2$ at distance r, we have to introduce additional factors from (29). If $\alpha = -\alpha_1 - i\alpha_2$, $e^{i\alpha r} = e^{-i\alpha_1 r}e^{\alpha_2 r}$, and these factors may be taken to be $\alpha^4 a^2 e^{2\alpha_2 r}/r^2$. The occurrence of the factor $e^{2\alpha_2 r}$, where α_2 is positive, has a strange appearance; but, as Lamb has shown,‡ it is to be expected in such cases as the present, where the vibrations to be found at any time at a greater distance correspond to an earlier vibration at the nucleus.

The calculations just effected afford an independent estimate of the dissipation. The rate at which energy is propagated outwards away from the sphere of great radius r, is

$$-\frac{dE}{dt} = V\,.\,4\pi r^2\,.\,\frac{\alpha^4 a^2 e^{2\alpha_2 r}}{r^2}\,\frac{\pi^2}{m}\{J_{2m-1}-J_{2m+1}\}, \qquad (48)$$

* Gray and Mathews, p. 28.

† 'Enc. Brit.,' "Wave Theory of Light," Equation (43), 1888; 'Scientific Papers,' vol. 3, p. 98.

‡ 'Proc. Math. Soc.,' 1900, vol. 32, p. 208.

or, since τ (the period) $= 2\pi a/m\mathrm{V}$, the loss of energy in one complete vibration is given by

$$-\frac{d\mathrm{E}\,.\,\tau}{dt} = \frac{8\pi^4\alpha^4 a^3 e^{2\alpha_2 r}}{m^2}\{\mathrm{J}_{2m-1}-\mathrm{J}_{2m+1}\}. \tag{49}$$

With this we have to compare the total energy to be found within the sphere. The occurrence of the factor $e^{2\alpha_2 r}$ is a complication from which we may emancipate ourselves by choosing r great in comparison with a, but still small enough to justify the omission of $e^{2\alpha_2 r}$, conditions which are reconcilable when ϵ is sufficiently small. The mean value of $\mathrm{P}^2+\mathrm{Q}^2+\mathrm{R}^2$ at a small distance ρ from the circular axis is $2m^2/a^2\rho^2$. This is to be multiplied by $2\pi a\,.\,2\pi\rho\,d\rho$, and integrated from ϵ to a value of ρ comparable with a, which need not be further specified. Thus

$$\mathrm{E} = \frac{8m^2\pi^2}{a}\int\frac{d\rho}{\rho} = -\frac{8m^2\pi^2}{a}\log\epsilon; \tag{50}$$

and

$$-\frac{d\mathrm{E}\,.\,\tau}{\mathrm{E}\,dt} = \frac{\pi^2\{\mathrm{J}_{2m-1}(2m)-\mathrm{J}_{2m+1}(2m)\}}{-\log\epsilon}, \tag{51}$$

in agreement with (23).

On the Apparent Change in Weight during Chemical Reaction.

By J. J. Manley, Hon. M.A. Oxon., Daubeny Curator, Magdalen College, Oxford.

(Communicated by Prof. J. H. Poynting, F.R.S. Received May 29,—Read June 27, 1912.)

(Abstract.)

In this communication are considered some of the causes which give rise to apparent changes in the total mass of chemically reacting substances.

A short review of the work of and the conclusions drawn by the late Prof. Landolt is first given, and then we deal with the conditions under which one of Landolt's final experiments was repeated by us. Next our balance and reaction vessels are briefly described, together with the preliminary treatment to which the latter were subjected prior to their being weighed. Following this, our first plan for weighing is outlined and then illustrated with the aid of data given in the form of tables and otherwise; then, by means of an actual example, the method adopted for calculating final results from our two series of preliminary experiments is explained.