ON THE ELECTROMAGNETIC FIELD FROM A VERTICAL HALF-WAVE AERIAL ABOVE A PLANE EARTH

By *P. Rjasin*

The paper is divided into two parts. In the first part the calculation of the electromagnetic field from a radiating half-wave aerial in a homogenous non-conducting medium is carried out based on the representation of the field from the Hertzian vector of an elementary dipole by a zero order Bessel function. This calculation leads exactly to the same expressions which had been obtained previously by the author * when solving the same problem in a different way.

In the second part starting from the results of part 1 the vertical component from the electrical vector of a half-wave aerial above a finitely conducting plane earth is calculated. It is also shown that the field from the vertical component of the electrical vector (E_z *) of the aerial is identical with the field from the Hertzian vector of two elementary dipoles located at the top and the base of the aerial, the electrical constants of the surrounding medium remaining the same in both cases.

Introduction

As is well known from the literature on the subject, the up till now published investigations of propagation of electromagnetic waves radiated by a source located close to a finitely conducting earth are usually based on the assumption of infinitesimal dimensions of the radiator (elementary dipole).** Such idealization of the real radiators (aerials) possessing a finite length which is compared to the wavelength is quite admissible when treating the problem of the electromagnetic field in the distant zone where it practically leads to quite satisfactory results. It is however entirely unsuitable for the determination of the structure of the electromagnetic field close to the aerial at a distance compared to its length. In the field of radio engineering there exists a number of special problems which require a minute knowledge of the field structure (particularly from the viewpoint of phase relations) just in the immediate neighbourhood of the aerial.

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* P. A. Rjasin, Calculation of the radiation from a straight aerial in the immediate neighbourhood. — J. Techn. Phys. (Russian), **VII**, 646. No. 6, 1937.

** A. Sommerfeld, Ann. d. Phys., **28**, 1909 and **81**, 1926; H. Herschelmann, JdTT, **5**, 1911; H. Weyl, Ann. d. Phys., **60**, 1919; B. van der Pol, JdTT, **37**, 1931; B. van der Pol and K. Niessen, Ann. d. Phys., **10**, 1931 etc.; V. Fock, Ann. d. Phys., **18**, 1933.

In the present paper expressions for the electromagnetic field from a radiator of finite length (a half-wave vibrator) close to a finitely conducting earth are developed which are also valid for the immediate neighbourhood including the surface of the radiator.

1. Calculation of the field from a half-wave aerial in homogenous space

Exact formulae for a perfectly thin radiating half-wave aerial have been derived by the author in a previous work * by the help of operations on elementary functions. Here will be shown at first how the same formulae are obtained when starting with the well known representation of the field from a Hertzian vector of an elementary dipole by a zero order Bessel function: **

$$\frac{e^{jk_1R}}{R}=\int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2-k_1^2}}\, e^{\pm\sqrt{\lambda^2-k_1^2}(z-\tau)}\,. \tag{1}$$

Although for the case discussed in the previous paper the here described method leads to the already known results, it still seems expedient to carry out the calculation as this method can be successfully used for the calculation of the field from an aerial above a finitely conducting earth, as will be shown in the second part of the present paper.

Starting from formula (1) we obtain in our case:

$$d\Pi^* = -\frac{\cos\left(\frac{2\pi}{\lambda_0}\tau\right)}{k_1}\cdot d\tau\int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2\cdot k_1^2}}\cdot e^{\sqrt{\lambda^2-k_1^2}(z-\tau)} \tag{1a}$$

under the following conditions:

A. In the present work as well as in the previous one, the current in the elementary dipole is given by:

$$I_0 = I\cos\left(\frac{2\pi}{\lambda_0}\tau\right)\sin\omega t$$

where $\cos\left(\frac{2\pi}{\lambda_0}\tau\right)$ takes into account the sinusoidal distribution of the current amplitudes along the aerial and $\sin\omega t$ — the harmonical variation of current with time. I_0 is the current amplitude at the antinode. For such distribution of the current along the aerial the moment of the elementary dipole dp is expressed by:

* Loc. cit.

** See for inst. F. Frank and R. Mises, Differential equations of Mathematical Physics (Russian), part II, 1937, p. 941, f-la (11).

$$dp = I_0 \cos\left(\frac{2\pi}{\lambda_0}\tau\right) d\tau \int \sin \omega t\, dt = -\frac{I_0}{\omega} \cos\left(\frac{2\pi}{\lambda_0}\tau\right) \cos \omega t\, d\tau$$

hence the Hertzian vector in the complex form becomes:

$$d\Pi^* = -\frac{I_0}{\omega} \cos\left(\frac{2\pi}{\lambda_0}\tau\right) d\tau\, \frac{e^{jk_1 R}}{R}.$$

If we express now Π in *CGSM* units (instead of *CGSE*) which will be used further and take $I_0 = 1$, then

$$d\Pi^* = -\frac{1}{k_1} \cos\left(\frac{2\pi}{\lambda_0}\tau\right) d\tau\, \frac{e^{jk_1 R}}{R} \tag{1b}$$

where

$$k_1 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}.$$

B. The expression for $\frac{e^{jk \cdot R}}{R}$ in the form (1) refers to an elementary dipole raised at a height τ above the origin of coordinates, the sign of the exponent in the function before the root being taken as + for $z - t < 0$ and − for $z - t > 0$. We assume for definitiveness that $z - t < 0$ in all points of the aerial (Fig. 1). This will not affect the generality since for a different orientation of the point of observation and of the aerial the same result is obtained. For the given current distribution the following expression for the Hertzian vector referring to the whole aerial is obtained:

$$\Pi^* = -\frac{1}{k_1} \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}} \int_{\frac{-\lambda_0}{4}}^{\frac{+\lambda_0}{4}} e^{\sqrt{\lambda^2 - k_1^2}\,(z-\tau)} \cdot \cos\left(\frac{2\pi}{\lambda_0}\tau\right) d\tau \tag{II}$$

where λ_0 is the wavelength.

Here

$$\int_{\frac{-\lambda_0}{4}}^{\frac{+\lambda_0}{4}} e^{\sqrt{\lambda^2 - k_1^2}(z-\tau)} \cdot \cos\left(\frac{2\pi}{\lambda_0}\tau\right) dt =$$

$$= \frac{1}{\lambda^2} \left| e^{\sqrt{\lambda^2 - k_1^2}\,(z-\tau)} \right| - \sqrt{\lambda^2 - k_1^2} \cos\left(\frac{2\pi}{\lambda_0}\tau\right) + k \sin\left(\frac{2\pi}{\lambda_0}\tau\right) \Bigg|_{\tau = -\frac{\lambda_0}{4}}^{\tau + \frac{\lambda}{4}} =$$

$$= \frac{k_1}{\lambda_2} \left[e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} + e^{\sqrt{\lambda^2 - k_1^2}\left(z + \frac{\lambda_0}{4}\right)} \right], \tag{IIa}$$

so that

$$\Pi^* = -\int_0^\infty \frac{d\lambda J_0(\lambda r)}{\lambda\sqrt{\lambda^2-k_1^2}}\left[e^{\sqrt{\lambda^2-k_1^2}\left(z-\frac{\lambda_0}{4}\right)}+e^{\sqrt{\lambda^2-k_1^2}\left(z+\frac{\lambda_0}{4}\right)}\right]. \qquad \text{(IIb)}$$

In order to prove the validity of the following mathematical operations which will be used starting from the integral (11a) we shall make the following remark. Although the denominator of the integrand for $\lambda \to 0$ tends to zero according to the first order of λ its numerator tends to zero according to the second order of λ, i. e. λ^2 and indeed expanding the bracketed factor in powers of λ we obtain:

$$\left| e^{\sqrt{\lambda^2-k_1^2}\left(z-\frac{\lambda_0}{4}\right)}+e^{\sqrt{\lambda^2-k_1^2}\left(z+\frac{\lambda_0}{4}\right)}\right|_{\lambda\to 0} =$$

$$= 2\left| e^{\sqrt{\lambda^2-k_1^2}\cdot z}\cdot\cos\left(\frac{\pi}{2}\sqrt{1-\frac{\lambda^2}{k_1^2}}\right)\right|_{\lambda\to 0} =$$

$$= 2e^{jk_1 z}\left|\frac{\pi\lambda^2}{4k_1^2} + \text{members of higher powers of } \lambda^2\right|_{\lambda\to 0} \to 0,$$

so that for $\lambda = 0$ the integrand remains finite. When approaching the upper limit, i. e. $\lambda \to \infty$ the integrand will surely tend to zero more rapidly than $\frac{1}{\lambda^2}$, so that convergence of the integral along the integration portion running to infinity is secured.

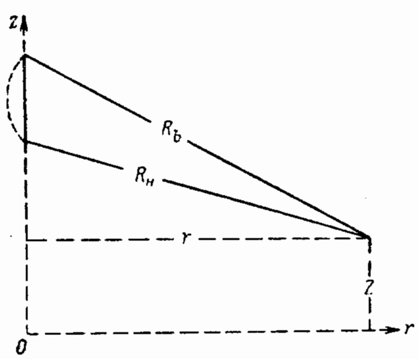

Fig. 1.

Finally the fact that over the integration interval $\lambda = k_1$ the integrand becomes infinity does not lead to the divergence of the integral either, as for $\lambda \to k_1$ the integrand tends to infinity as:

$$\frac{\sqrt{2}J_0(k_1 r)}{k^{\frac{3}{2}}}\left|\frac{1}{(\lambda-k_1)^{\frac{1}{2}}}\right|_{\lambda\to k_1}$$

Hence from the Cauchy test we conclude that over the portion around $\lambda = k_1$ the integral (IIb) is convergent. *

E_z and E_r through Π^* are found:

The expression (IIb) for Π^* will be used for the development of exact

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* See for inst. «Higher Mathematics» by Smirnov (Russian), p. 240.

formulae for a perfectly thin aerial. We shall try at first to find an expression for the electrical field. From general relations the following expressions for E_z and E_r through Π^* are found:

$$E_z = \frac{-1}{r} \frac{\partial}{\partial z} \left[r \frac{\partial \Pi^*}{\partial r} \right], \tag{III}$$

$$E_r = \frac{\partial}{\partial z} \left(\frac{\partial \Pi^*}{\partial r} \right) \tag{IIIa}$$

where Π^* is a function of r, Z given by (IIb). Let us substitute Π from (IIa) into (III), r and λ being independent values. We thus obtain:

$$E_z = \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial J_0(\lambda r)}{\partial r} \right] \frac{d\lambda}{\lambda \sqrt{\lambda^2 - k_1^2}} \left[e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} + \right.$$
$$\left. + e^{\sqrt{\lambda^2 - k_1^2}\left(z + \frac{\lambda_0}{4}\right)} \right]. \tag{IV}$$

Here the bracketed factor of the integrand is:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial J_0(\lambda r)}{\partial r} \right] = - \frac{1}{r} \frac{\partial}{\partial r} [r J_1(\lambda r)]^* = - \lambda^2 J_0(\lambda r) +$$
$$+ \frac{\lambda J_1(\lambda r)}{r} - \frac{\lambda J_1(\lambda r)}{r} = - \lambda^2 J_0(\lambda r), \tag{IV*}$$

so that

$$E_z = - \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}} \cdot e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} -$$
$$- \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}} e^{\sqrt{\lambda^2 - k_1^2}\left(z + \frac{\lambda_0}{4}\right)},$$

hence according to (I) we obtain finally:

$$E_z = - \frac{e^{jk_1 R_b}}{R_b} - \frac{e^{jk_1 R_H}}{R_H}, \tag{IVa}$$

where

$$R_b = \sqrt{r^2 + \left(z - \frac{\lambda_0}{4}\right)^2}; \quad R_H = \sqrt{r^2 + \left(z + \frac{\lambda_0}{4}\right)^2}.$$

Let us turn now to the determination of E_r. Substituting Π^* from (IIb) into (IIIa) we obtain after simple transformations:

$$E_r = \int_0^\infty d\lambda J_1(\lambda r) \left[e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} + e^{\sqrt{\lambda^2 - k_1^2}\left(z_1 + \frac{\lambda_0}{4}\right)} \right]. \tag{IVb}$$

* Here $J_1(\lambda r)$ is a first order Bessel function.

Integrating each of the right-hand integrals by parts we are able to calculate (IVb) and express it in elementary functions. And really putting in (IVb):

$$d\lambda J_1(\lambda r) = du; \quad e^{\sqrt{\lambda^2 - k_1^2}\left(z \pm \frac{\lambda_0}{4}\right)} = v;$$

$$-\frac{J_0(\lambda r)}{r} = u; \quad \left(z \pm \frac{\lambda_0}{4}\right) \frac{\lambda\, d\lambda}{\sqrt{\lambda^2 - k_1^2}} = dv,$$

we obtain:

$$E_r = -\frac{1}{r} \left| J_0(\lambda r)\, e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} + J_0(\lambda r)\, e^{\sqrt{\lambda^2 - k_1^2}\left(z + \frac{\lambda_0}{4}\right)} \right|_{\lambda=0}^{\infty} +$$

$$+ \frac{z - \frac{\lambda_0}{4}}{r} \int_0^\infty \frac{\lambda d\lambda\, J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}}\, e^{\sqrt{\lambda^2 - k_1^2}\left(z - \frac{\lambda_0}{4}\right)} +$$

$$+ \frac{z + \frac{\lambda_0}{4}}{r} \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}}\, e^{\sqrt{\lambda^2 - k_1^2}\left(z + \frac{\lambda_0}{4}\right)}.$$

The first member being equal zero we obtain finally:

$$E_r = \frac{z - \frac{\lambda_0}{4}}{r} \frac{e^{jk_1 R_b}}{R_b} + \frac{z - \frac{\lambda_0}{4}}{r} \frac{e^{jk_1 R_H}}{R_H}; \tag{IVc}$$

E_z and E_r being known, H can be calculated from the relation:

$$jk_1 H = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}.$$

Substituting E_r and E_z from (IVa) and (IVc) we obtain after reduction:

$$H = \frac{e^{jk_1 R_b}}{r} + \frac{e^{jk_1 R_H}}{r}. \tag{IVd}$$

It is quite clear that formulae (IVa), (IVc) and (IVd) expressed in the complex form are identical with formulae (14) and (16) of the above mentioned work which had been developed without use of Bessel functions.

2. Some remarks of the electromagnetic field from a half-wave aerial above a finitely conducting earth

In the first part of the present paper we obtained again a formula identical with formula (16a) of the previous work (§ 4, p. 654). According to this formula the field from the component E_z is represented by the superposition of two spherical waves issuing from both ends of the aerial.

Let us imagine now that at both ends of the aerial there are two radiating elementary dipoles — point sources. The resulting electromagnetic field from both such sources will differ in structure from the field of the half-wave aerial but as is clear from the above considerations the expression for the component E_z will be identical with that for the resulting field from the Hertzian vector for the above indicated two elementary dipoles located at both ends of the aerial. Moreover the expression for the field E_z for each of these dipoles located at both ends of the aerial: $-\frac{e^{-jk_1 R_b}}{R_b}$ and $-\frac{e^{jk_1 R_H}}{R_H}$ taken independently from one another is identical with the expression for Π of the Hertzian dipole located at the same point. This fact helped to put on a strict basis the following theorem:

The expression for the vertical component E_z^* of the electrical field from a vertical half-wave aerial above a finitely conducting plane earth is identical with the expression for the field of the Hertzian vector Π excited by two point (elementary) dipoles located at both ends of the aerial,* the electrical constants of the surrounding medium being the same in both cases.

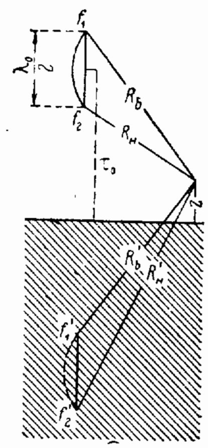

Fig. 2.

The orientation of the aerial is seen from Fig. 2.

In other words: if $E_{z_1}^*$ be the field from the vertical component of a half-wave aerial in the first medium (air), $E_{z_1}^{(b)}$ and $E_{z_1}^{(H)}$ the fields from the equivalent, point sources located at both ends of the aerial and $\Pi_{1z}^{(b)}$ and $\Pi_{1z}^{(H)}$ — the expressions for the fields from the Hertzian vectors for the elementary dipoles located at the top and the base of the aerial, then: **

$$E_{z_1}^* = E_{z_1}^{(b)} + E_{z_1}^{(H)} = \Pi_{1z}^{(b)} + \Pi_{1z}^{(H)}. \tag{V}$$

This theorem can be proved at least in two ways. This follows first of all from the fact that the full mathematical conditions for the determination of E_z^* from a half-wave aerial raised above the earth's surface are identical with those for the field Π from two elementary dipoles located at both its ends.

Let us consider these conditions. Firstly: E_z^* satisfies the wave equation:

* Which is correct up to a real constant factor depending on the radiated power.

** We have expressed our theorem with reference to the first medium but it should be borne in mind that this theorem is also valid for the second medium (earth).

$$\frac{\partial^2 E_z^*}{\partial z^2} + \frac{\partial^2 E_z^*}{\partial r^2} + \frac{1}{r}\,\frac{\partial E_z^*}{\partial r} + k^2 E_z^* = 0. \qquad (I)$$

Taking into account the boundary conditions: $E_{r_1}^* = E_{r_2}^*$ and $H_1^* = H_2^*$ for $z=0$ we obtain using Maxwellian equations the following two relations for E_z^* at the dividing surface:

$$\left.\begin{aligned} k_1^2 E_{z_1}^* &= k_2^2 E_{z_2}^* \\ \frac{\partial E_{z_1}^*}{\partial z^2} &= \frac{\partial E_{z_2}^*}{\partial z} \end{aligned}\right\} \text{ for } z=0. \qquad (II)$$

The primary excitation is expressed in the form:

$$E_{z_1}' = -\frac{e^{jk_1R_b}}{R} - \frac{e^{jk_1R_H}}{R_H}, \qquad (III)$$

and finally the conditions at infinity are:

$$E_z^* = 0 \begin{cases} \text{for } r \to \infty \text{ or for } z > 0 \;\; z \to +\infty \\ \text{and for } z < 0 \;\; z \to -\infty \end{cases} \qquad (IV)$$

The here discussed problem being linear the same above written conditions apply also to each separate equivalent «radiator» $-\frac{e^{jk_1R_b}}{R_b}$ and $-\frac{e^{jk_1R_H}}{R_H}$. If so, then all the four conditions fully coincide with those for the function Π for an elementary dipole located at the same point, * and they determine the single-valued and only solution. Thus the problem of the vertical component of the electrical field from the electrical vector of a half-wave vibrator (consisting of an infinite number of elementary dipoles) is mathematically identical with the problem of the field from the Hertzian vector for a pair of elementary dipoles acting at both ends of the vibrator.

Not touching upon the very difficult discussion of the exact solution of such problem we shall only point out that this solution was first obtained by Hörschelmann in 1911. ** As is well known this solution is expressed in a definite manner through Sommerfeld's solution for the field

* See for inst. B. A. Wwedenski, The principles underlying the theory of propagation of radio waves (Russian), 1934, pp. 88—89, and the footnote on p. 91. In the literature on the subject the expression for the primary radiation is usually taken with the sign «plus» $\left(+\frac{e^{jk_1R}}{R}\right)$. We have adopted the sign «minus». Such difference is of no importance since it only leads to the reverse sign of the corresponding solution which will be taken into account below.

** See: von Hörschelmann, JdTT, **5**, 1911. A solution of the same form for a raised dipole had been obtained later by K. Niessen though in a different way (K. Niessen, Ann. d. Phys., 1933, p. 899).

B. van der Pol investigated the expression for the field from a raised dipole from the viewpoint of its physical interpretation. B. van der Pol, Physics, **8**, 1935, p. 843—853.

A solution of this problem was also given by H. Weyl, Ann. d. Phys., **60**, 1919.

from the Hertzian vector of a dipole located directly upon the dividing surface. The expression for Π_{1z} from a raised vertical dipole is obtained if in formula (8a) in the above mentioned work (p. 26) we put $\beta = 0$, i. e. if we remove the horizontal part of the idealized aerial discussed by Hörschelmann. We shall write this solution putting $A = 1$ (A is proportional to the radiation intensity) and change the sign of the right part to the reverse, according to the sign «minus» in our expression for the primary radiation:

$$\Pi_{1z} = -\int_0^\infty \frac{J_0(\lambda r)\, e^{\sqrt{\lambda^2 - k_1^2}(z-\tau)}\lambda d\lambda}{\sqrt{\lambda^2 - k_1^2}} + \int_0^\infty \frac{J_0(\lambda r)\, e^{-\sqrt{\lambda^2 - k_1^2}(z+\tau)}\lambda d\lambda}{\sqrt{\lambda^2 - k_1^2}} - 2\int_0^\infty \frac{k_2^2 J_0(\lambda r)\, e^{-\sqrt{\lambda^2 - k_1^2}(z+\tau)}\lambda d\lambda}{N}. \tag{Va}$$

Here $N = k_1^2\sqrt{\lambda^2 - k_2^2} + k_2^2\sqrt{\lambda^2 - k_1^2}$ and τ is the height of the dipole above the earth's surface.

Applying (Va) to two dipoles located at both ends of the aerial we obtain equality (V) in the expanded form:

$$E_{1z}^* = -\frac{e^{jk_1 R_b}}{R_b} + \frac{e^{jk_1 R_b'}}{R_b'} - k_2^2\int_0^\infty \frac{J_0(\lambda r)\, e^{-\sqrt{\lambda^2 - k_1^2}\left(z+\tau_0+\frac{\lambda_0}{4}\right)}\lambda d\lambda}{N} - \frac{e^{jk_1 R_H}}{R_H} + \frac{e^{jk_1 R_H'}}{R_H'} - k_2^2\int_0^\infty \frac{J_0(\lambda r)\, e^{-\sqrt{\lambda^2 - k_1^2}\left(z+\tau_0+\frac{\lambda_0}{4}\right)}}{N}\lambda d\lambda. \tag{Vb}$$

The values of R_b, R_b', R_h, R_h', τ_0, λ in (Vb) are clear from Fig. 2. The first three terms on the right are identical with the expression for the field Π_{1z}^* from a dipole located at point f_1 and the second three terms correspond to the point f_2 (Fig. 2), the above mentioned factors being determined from the following relations:

$$R_b = \sqrt{r^2 + \left(z - \tau_0 - \frac{\lambda_0}{4}\right)^2}, \qquad R_H = \sqrt{r^2 + \left(z - \tau_0 + \frac{\lambda_0}{4}\right)^2},$$

$$R_b' = \sqrt{r^2 + \left(z + \tau_0 + \frac{\lambda_0}{4}\right)^2}, \qquad R_H' = \sqrt{r^2 + \left(z + \tau_0 - \frac{\lambda_0}{4}\right)^2}.$$

The validity of formula (V—Vb) shall now be proved by deriving it from integration along the aerial of Hörschelmann's solution for a raised dipole.

When establishing in the first part of the present paper the exact formula for an aerial in a homogenous medium we began with the integration along the aerial of the Hertzian vector for an elementary dipole, and having obtained in this way the vector Π * (IIb), we determined E_z from (III). It thus appeared that the final result of the above operations was expressed as a sum of the fields from two elementary dipoles acting at both ends of the aerial.

The here discussed problem being linear we have the right to apply the same method of determining the vertical component of the electrical field also to the case of a finitely conducting earth. *

Following this path we shall integrate Hörschelmann's vector with respect to the «elevation» τ over the whole length of the aerial multiplying it previously by $\dfrac{\cos \frac{2\pi}{\lambda_0}(\tau - \tau_0)}{k_1}$.

In air we thus obtain:

$$\Pi_{1z} = -\frac{1}{k_1}\left\{ \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}} \int_{\tau_0 - \frac{\lambda_0}{4}}^{\tau_0 + \frac{\lambda_0}{4}} e^{\sqrt{\lambda^2 - k_1^2}(z - \tau)} \cos\left[\frac{2\pi}{\lambda_0}(\tau - \tau_0)\right] d\tau + \right.$$

$$+ \int_0^\infty \frac{\lambda d\lambda J_0(\lambda r)}{\sqrt{\lambda^2 - k_1^2}} \int_{\tau_0 - \frac{\lambda_0}{4}}^{\tau_0 + \frac{\lambda_0}{4}} e^{-\sqrt{\lambda^2 - k_1^2}(z + \tau)} \cos\left[\frac{2\pi}{\lambda_0}(\tau - \tau_0)\right] d\tau -$$

$$\left. - 2k_2^2 \int_0^\infty \frac{\lambda d\lambda}{N} J_0(\lambda r) \int_{\tau_0 - \frac{\lambda_0}{4}}^{\tau_0 + \frac{\lambda_0}{4}} e^{-\sqrt{\lambda^2 - k_1^2}} \cos\left[\frac{2\pi}{\lambda_0}(\tau - \tau_0)\right] d\tau \right\}. \qquad \text{(VI)}$$

The factor $\frac{1}{k_1}\cos\left[\frac{2\pi}{\lambda_0}(\tau - \tau_0)\right]$ is introduced into formula (VI) from the same considerations as in formula (II) (see explanation to formula Ia).

Here τ is the elevation of the middle of the aerial above the earth's surface. Substituting in (VI) the values of the integrals with respect to τ by the use of (IIa) we obtain:

$$\Pi_{1z}^* = -\int_0^\infty \frac{d\lambda J_0(\lambda r)}{\lambda\sqrt{\lambda^2 - k_1^2}} \left[e^{\sqrt{\lambda^2 - k_1^2}\left(z + \tau_0 - \frac{\lambda_0}{4}\right)} + e^{\sqrt{\lambda^2 - k_1^2}\left(z - \tau_0 + \frac{\lambda_0}{4}\right)} \right] +$$

* If only the assumed above distribution of the current along the aerial remains unchanged.

$$+\int_0^\infty \frac{d\lambda J_0(\lambda r)}{\lambda\sqrt{\lambda^2-k_1^2}}\left[e^{-\sqrt{\lambda^2-k_1^2}\left(z+\tau_0-\frac{\lambda_0}{4}\right)}+e^{-\sqrt{\lambda^2-k_1^2}\left(z+\tau_0+\frac{\lambda_0}{4}\right)}\right]-$$

$$-\int_0^\infty \frac{2k_2^2 d\lambda J_0(\lambda r)}{N\lambda}\left[e^{-\sqrt{\lambda^2-k_1^2}\left(z+\tau_0-\frac{\lambda_0}{4}\right)}+e^{-\sqrt{\lambda^2-k_1^2}\left(z+\tau_0+\frac{\lambda_0}{4}\right)}\right]. \quad \text{(VIa)}$$

The unknown quantity $E^*_{z_1}$ is determined from:

$$E^*_z=-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \Pi^*_{1z}}{\partial r}\right] \quad \text{(VII)}$$

where Π^*_{1z} is given in (VIa). The operation of expanding the right side of (VII) is identical with that of (III). This operation here also consists in the determination of $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial J_0(\lambda r)}{\partial r}\right]$ which as is seen from (IV) is equal simply to $-\lambda^2 J_0(\lambda r)$. Thus the transition from (VIa) to (VII) is fulfilled by the substitution in (VIa) of $J_0(\lambda r)$ by $\lambda^2 J_0(\lambda r)$ as a result

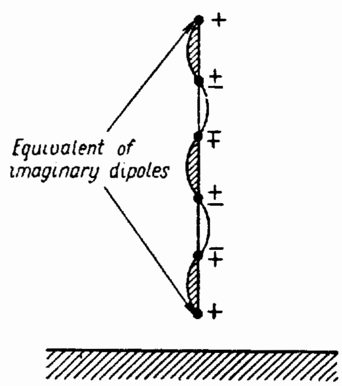

Fig. 3.

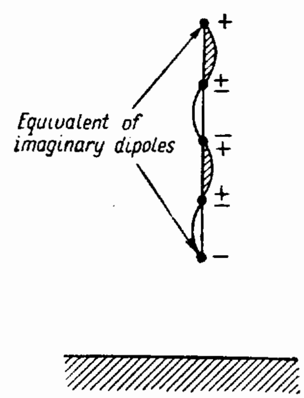

Fig. 4.

of which (as is easily seen) (Vb) is obtained again., i. e. the validity of the above formulated theorem is proved again. In conclusion the following formula for $E^*_{z_1}$ for the particular case of the field at the dividing surface will be given:

$$E^*_{z_1}=-\int_0^\infty \frac{k_2^2 J_0(\lambda r)}{N}\left[e^{-\sqrt{\lambda^2-k_1^2}\left(\tau_0+\frac{\lambda_0}{4}\right)}+e^{-\sqrt{\lambda^2-k_1^2}\left(\tau_0-\frac{\lambda_0}{4}\right)}\right]\lambda d\lambda=$$

$$=-2\int_0^\infty \frac{k_2^2}{N}J_0(\lambda r)e^{-\sqrt{\lambda^2-k_1^2}\,\tau_0}\cos\left(\frac{\pi}{2}\sqrt{1-\frac{\lambda^2}{k_1^2}}\right)\lambda d\lambda.$$

The above given theorem can easily be generalized for any even or odd harmonic to which the aerial may be excited. Such generalization could be proved mathematically in the same manner as above but it is possible to do it without carrying out any calculations. Let us discuss at first an aerial excited to an odd harmonic (Fig. 3). (Here an aerial excited to the fifth harmonic is shown.) This aerial is divided by the current nodes into simple half-wave aerials, the adjacent ones vibrating with opposite phases. On the basis of this fact and also of the above theorem the equivalent circuit of this aerial can be represented as is shown on the right of Fig. 3. From this circuit follows that the field E_z^* from the aerial is identical with the field from the Hertzian vector for a group of elementary dipoles of equal intensity located at the current nodes, the ends of the aerial bearing each a single dipole of equal phases these being equal to the phase of the medium half-wave. It is just these two dipoles which excite the field from the Hertzian vector identical with the field E_{z1}^* from the whole aerial as at each of the remaining nodes there is a pair of dipoles of opposite phases whose fields eliminate one another.

It could be shown in the same way that the field from the vertical electrical component of an aerial excited to an even harmonic is identical with the field from the Hertzian vector for two elementary dipoles located at both ends of the aerial but in distinction from the first case here the phases of both dipoles are opposite in accordance with the signs of the phases of the extreme half-waves (see Fig. 4).

It is evident that in case of an aerial excited to one of its overtones the extreme «representative» dipoles should be referred to the frequencies of the harmonics and not of the fundamental tone.

The further development of the here discussed problem on propagation of electromagnetic waves radiated by conductors (aerials) of finite length particularly by half-wave vibrators above a finitely conducting earth will be given in another work.

The author takes the opportunity of expressing his sincere gratitude to prof. N. D. Papalexi for his interest in and kind attention to the present work.

Moscow, Physical Institute
of the Academy of Sciences of the USSR
Oscillation Laboratory.
Received August 1937.