III. *An Historical Note on a Paradox in Electrodynamics.*

By IAN A. ROBERTSON, M.A., B.Sc., Assistant in Natural Philosophy in the University of Glasgow *.

[Received July 5, 1944.]

AN investigation of the forces between two electrons, or charged particles, leads to a paradox concerning these forces. As this paradox has been the subject of two recent papers, one in the 'Philosophical Magazine,' August 1941, and the other in the 'American Journal of Physics,' December 1942, and an editorial in the 'Wireless Engineer,' March 1944, it may be of interest to consider this problem in relation to its historical background. In this note it will be shown that the problem was present from the beginning of electrodynamics, that Ampère considered it and used it in his fundamental formula, and that Ampère's theory is still superior to any given in the recent papers considered in connection with this problem.

The paradox will now be given in terms of the classical theory of electricity, as is given in Joos' Theoretical Physics. According to the Biot–Savart Law, a current element or slowly moving electron gives rise at a position r to a magnetic field H. This field is given by

$$H=\frac{i(ds\times r)}{r^3}=\frac{e(v\times r)}{r^3}, \quad \ldots \ldots \ldots \quad (1)$$

where an electron of charge e moving with velocity v corresponds to a current element ids, and where $(v\times r)$ represents the vector product of v and r. This field H acts upon another electron e' having velocity v' with a force given by

$$F=e'(v'\times H)=\frac{ee'(v'\times(v\times r))}{r^3}. \quad \ldots \ldots \quad (2)$$

Now consider two electrons moving at a certain instant, in mutually perpendicular directions, as shown in fig. 1.

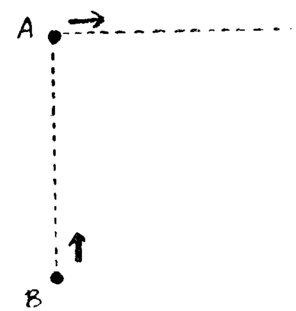
The electron at A is moving with velocity v and the electron at B is moving with velocity v', perpendicular to v. When the above formula is applied it is found that there is a force on the electron at B of magnitude $\frac{ee'\,vv'}{r^2}$ acting in a direction normal to the velocity, but that there is no corresponding force on the electron at A. Hence there is a failure of Newton's Third Law when the Biot–Savart Law is applied to electrons.

Now this problem was present from the beginning of electrodynamics.

* Communicated by Professor E. Taylor Jones.

In 1823 Ampère published his *Mémoire* on the law of force between current elements, or as they could now be called, electrons. This theory was based on postulates which Ampère attempted to establish by experiment, but as Maxwell said of these experiments : " We can scarcely believe that Ampère really discovered the law of action by means of the experiments which he describes. We are led to suspect, what, indeed, he tells us himself, that he discovered the law by some process which he has not shown to us. . . . " The postulate which concerns our problem is given on p. 202 of the 'Mémoire de L'Académie' (1823), and may be translated thus : *an element of one current has no effect on an element of another current which lies in the plane bisecting the former at right angles.* Thus for electrons or current elements in the special case under consideration, there is, according to Ampère, who assumed Newton's Laws, no force on either of the electrons and therefore no failure of the action–reaction law. The reasons that Ampère gave for using this postulate can hardly be considered satisfactory. He stated on p. 202 : " En effet, les deux

Fig. 1.



moitiés du premier élément produisent sur le second des actions égales, l'une attractive et l'autre répulsive, parce que dans l'une de ces moitiés le courant va en s'approchant et dans l'autre en s'éloignant de la perpendiculaire commune. Or, ces deux forces égales font un angle qui tend vers deux angles droits à mesure que l'élément tend vers zéro. Leur résultante est donc infiniment petite par rapport à ces forces, et doit par conséquent être négligeé dans le calcul."

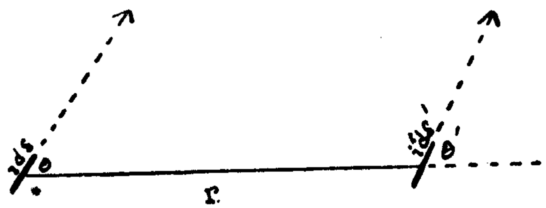
Yet it was from postulates like this, that Ampère deduced the law of force between two current elements, namely :

$$F=C\times\frac{ii'\,ds\,ds'}{r^2}\,(2\sin\theta\sin\theta'\cos w-\cos\theta\cos\theta'), \quad \ldots \quad (3)$$

where the force F acts entirely along the line joining the elements, and where θ and θ' are the angles between the directions of ds *and* ds' and the line r joining them, and w is the angle between the plane containing ds and r and the plane containing ds' and r, and C is a constant depending upon the units chosen. This formula has borne the test of experiment. Clerk Maxwell described it as " perfect in form and unassailable in accuracy and it (Ampère's theory) is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electrodynamics" (vol. ii. p. 175).

If we apply Ampère's formula to our particular problem we have $\theta=\pi/2$, $\theta'=\pi$ or 0, and therefore $F=0$, showing again that if Ampère's formula is used no paradox arises. It has already been pointed out that Ampère's formula gives results in keeping with experiment, the paradox arises because Biot's rule and not Ampère's is used in classical electrodynamics. In his *Mémoire* of 1823, on p. 280, Ampère begins his criticism of Biot's formula, which gives the magnitude and direction of the force exerted by an element of a circuit on a magnetic pole, on the grounds that a magnetic pole is more complex than a simple element of a circuit and a fundamental law should be between elements of the same nature; moreover, the force acting on the pole is at right angles to the plane containing the element and the line joining it to the pole, whereas the reaction to this force is an equal and opposite force on the element of the circuit, but acting in a direction parallel to the force on the pole, thus giving rise to a

Fig. 2.



couple. Ampère goes on to state, p. 326: "J'ai toujours regardé cette hypothèse des couples primitifs comme absolument contraire aux premières lois de la mécanique." On the other hand, Ampère was aware that Biot's rule agreed with his own, when integrated around a closed circuit. Now Biot's law leads to a violation of Newton's laws for isolated elements, yet it is Biot's law which is usually applied to electrons even to this day (*cf.* Jeans' 'Electricity,' par. 628).

There is another interesting difference between Ampère's formula and the Biot–(Savart) law which requires consideration in the light of modern theories, and that is the force between two electrons moving in the same straight line. If Ampère's formula, given above, is applied with $\theta=\theta'=0$, we find that there is an electromagnetic force of repulsion between the elements equal to

$$-C\times\frac{ii'\,ds\,ds'}{r^2}, \qquad (4)$$

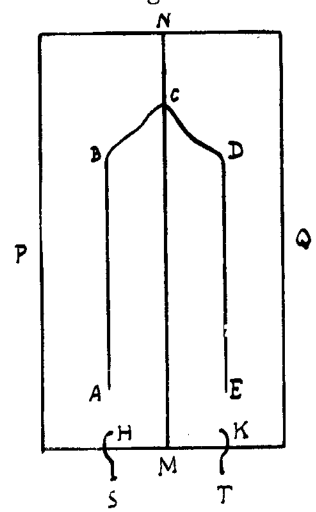
while Biot's law gives no force at all between the electrons in that position.

Ampère devised an experiment to demonstrate this repulsion. The description of his apparatus is given on p. 211 of his *Mémoire*, and may be summarized thus:

A horizontal vessel PQ is divided by an insulating partition MN into

two compartments. These compartments are filled with mercury and are connected by means of the wire ABCDE. This wire has two branches, AB and DE, floating on the surface of the mercury, and joined by the part BCD, which crosses over the partition MN. The wire is covered with insulating material except at its ends A and E. When the mercury-filled compartments are joined at H and K with the terminals S and T of a battery, a current flows, partly through the mercury, partly through the wire ABCDE, and the wire glides over the surface of the mercury away from the points H and K. Ampère concludes: "Quelle que soit la direction du courant, on voit toujours les deux fils AB, ED marcher parallèlement à la cloison MN en s'éloignant des points H et K, ce qui indique une répulsion pour chaque fil entre le courant établi dans le mercure et son prolongement dans le fil lui-même." This experiment

Fig. 3.



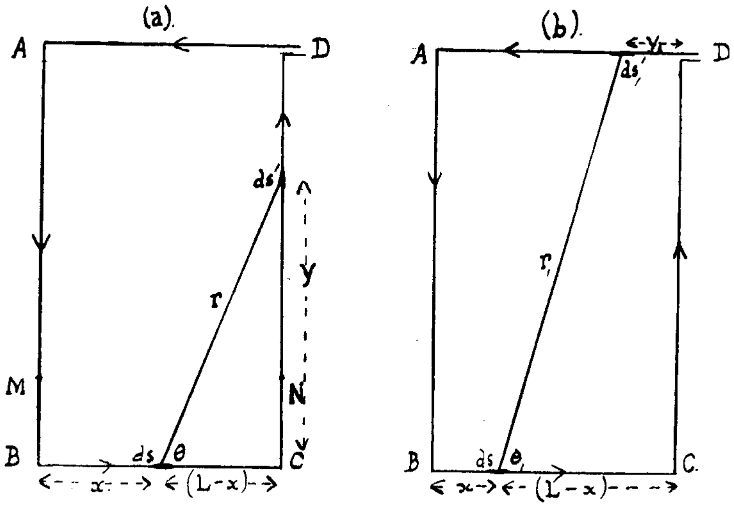
is not conclusive proof that there is a magnetic repulsion between elements moving in the same straight line. The experimental fact that the wire glides away from the points H and K is usually explained thus:—"the circuit will so arrange itself as to include the maximum flux, the current being kept constant." Of course, such a statement in no way disproves Ampère's theory that a repulsion exists.

A somewhat similar experiment was described by W. F. Dunton in 'Nature,' August 1937. In this case a current is sent round a rectangle ABCD (fig. 4 (*a*)).

M and N are mercury cups which allow MBCN to move, while the portion NDAM is firmly held. Dunton attached the middle of BC to a beam balance and measured the outward force exerted upon it. Dunton devised this experiment in the hope of deciding between Neumann's and Faraday's law of induction, but that controversy does not concern this paper. Nevertheless Dunton's experiment has been used by S. B. L.

Mathur ('Philosophical Magazine,' August 1941) as experimental evidence that Biot's law is correct and that Newton's third law is wrong. However, on careful examination of Mathur's paper, excellent reasons can be deduced for discarding Biot's law. Mathur begins by stating the paradoxical case considered in this note, pointing out that by Biot's law action is not always equal and opposite to reaction, and hence if Biot's law is used, it is wrong to use Newton's third law. Ignoring Newton's laws, Mathur calculates by Biot's law the outward force on the side BC of the rectangle ABCD. If $AB=180r$, $BC=60r$, where r is the radius of the conductor of which the circuit is made, Mathur finds this force to be $6{\cdot}9i^2$, where i is the current flowing round the rectangle. Then it is pointed out that Dunton had found experimentally that under the same conditions the force is $8{\cdot}7i^2$. One would think that the difference between $6{\cdot}9i^2$ and

Fig. 4.



$8{\cdot}7i^2$ would arouse suspicion that there was something wrong either with Biot's law or Dunton's experiment, but Mathur concludes "However, as it is evident, it is not the Biot–Savart law that is in error; but it leads to wrong results when the inapplicability of Newton's third law is not recognized."

As the problems connected with the forces in a rectangular circuit, one side of which is mechanically separable from the other three, are of practical importance, and have aroused a large amount of discussion in connection with the laws of induction, it may not be out of place to calculate the outward force on one side by Ampère's formula; after all, Ampère was the first of the "circuit-breakers." Ampère's formula, as given above, is

$$F=C\times ii'\frac{ds\,ds'}{r^2}\,(2\sin\theta\sin\theta'\cos w-\cos\theta\cos\theta').$$

Now $\cos\epsilon=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos w$, where ϵ is the angle between the directions of ds and ds'.

Hence $$F=C\times\frac{ii'\,ds\,ds'}{r^2}\,(2\cos\epsilon-3\cos\theta\cos\theta'). \quad (3)$$

Referring again to fig. 4, consider an element of current ds in BC and another element ds' in CD (the mercury cups M and N being placed at B and C), the force between these two elements lies along the line r joining them. Now BC is perpendicular to CD, therefore $\cos\epsilon=0$, and from the geometry of the figure $\theta'=\left(\theta+\frac{3}{2}\pi\right)$; hence there is a force of repulsion between the elements equal to

$$F_1=C\times\frac{i^2\,ds\,ds'}{r^2}\,[-3\cos\theta\sin\theta]$$

if we take $C=-1$, then $F_1=+\dfrac{i'\,ds\,ds'}{r^2}\,3\cos\theta\sin\theta$. Thus the outward force on ds, that is, the component force acting perpendicular to ds, due to ds', is equal to

$$\frac{i'\,ds\,ds'}{r^2}\,3\cos\theta\sin^2\theta.$$

Therefore, the outward force on BC due to CD is equal to

$$i^2\int_0^d ds'\int_a^{L-a}\frac{3\cos\theta\sin^2\theta}{r^2}\,ds, \quad (5)$$

where d is the length of the side BA, L is length of the side BC, and a is the radius of the wire, which is assumed small compared with d and L. Similarly, the outward force on BC due to AB is likewise given by the same integral, while the outward force on BC due to AD is given by

$$i^2\int_a^{L-a} ds_1'\int_a^{L-a}\frac{(2\sin\theta_1-3\cos^2\theta_1\sin\theta_1)}{r_1^2}\,ds, \quad (6)$$

where ds lies along BC and ds_1' along AD, and r_1 is the distance between ds and ds_1' (fig. 4 (*b*)). Thus the total outward force on BC due to the other three sides is given by

$$6i^2\int_a^d ds'\int_a^{L-a}\frac{\cos\theta\sin^2\theta}{r^2}ds+i^2\int_a^{L-a}ds'\int_a^{L-a}\frac{(2\sin\theta_1-3\sin^2\theta_1\cos\theta_1)\,ds}{r_1^2}, \quad (7)$$

where $\cos\theta=\dfrac{L-x}{r}$, $\cos\theta_1=\dfrac{L-x-y_1}{r_1}$,

$\sin\theta=\dfrac{y}{r}$, $\sin\theta_1=\dfrac{d}{r}$,

$r^2=(L-x)^2+y^2$, $r_1^2=(L-x-y_1)^2+d^2$,

$ds=dx,\ ds'=dy$, $ds=dx,\ ds'=dy_1$.

Substituting these values in the above integral (7) we obtain

$$6i^2\int_0^d y^2\,dy\int_a^{L-a}\frac{(L-x)\,dx}{[(L-x)^2+y^2]^{5/2}}$$
$$+i^2\int_a^{L-a}dy_1\int_a^{L-a}\frac{\left[2\left(\frac{d}{r}\right)-3\left(\frac{L-x-y}{r_1}\right)^2\frac{d}{r_1}\right]dx}{r_1^{\,2}},$$

that is

$$6i^2\int_0^d y^2\,dy\int_a^{L-a}\frac{(L-x)\,dx}{[(L-x)^2+y^2]^{5/2}}$$
$$+2i^2\int_a^{L-a}dy_1\int_a^{L-a}\frac{d.(dx)}{r_1^{\,3}}-3i^2\int_a^{L-a}dy_1\int\frac{[(L-x-y)^2.d]\,(dx)}{r_1^{\,5}},$$

and when this is integrated, we obtain

$$2i^2\left[\log\frac{L-a}{a}+\log\frac{d+\sqrt{d^2+a^2}}{d+\sqrt{d^2+(L-a)^2}}-\left(1+\frac{a^2}{d^2}\right)^{-\frac{1}{2}}\right.$$
$$\left.+\left(1-\frac{(L-a)^2}{d^2}\right)^{-\frac{1}{2}}+\left(1+\frac{(L-2a)^2}{d^2}\right)^{\frac{1}{2}}-\left(1+\frac{(L-2a)^2}{d^2}\right)^{-\frac{1}{2}}\right]. \quad . \quad . \quad . \quad (8)$$

If $d=180a$ and $L=60a$, we obtain the outward force on BC, due to the other three sides, equal to $8{\cdot}24i^2$, which is fairly close to Dunton's value $8{\cdot}7i^2$.

It may well be remarked that the calculation of such expressions by Ampère's original formula is laborious, but there is only one other method for calculating such forces on the sides of a rectangle, and that can only be accomplished by calculating the self-inductance of the rectangle, a process quite as complicated as the above calculation, and involving artifices which seem hardly satisfactory (*cf.* Jeans' 'Electricity,' par. 505).

Other experiments on the forces of rectangular circuits have been described by F. F. Cleveland ('Philosophical Magazine,' 1936). These experiments prove conclusively that action and reaction are equal and opposite for the mechanically separable parts of a rectangular circuit, thus showing again that the Biot–Savart law is wrong in this instance. Cleveland suggests that more use should be made of Ampère's formula and gives the expression for the force, on one side of a rectangular circuit, exerted by the other three sides. Unfortunately the expression given by Cleveland is invalidated by algebraic slips, and the expression as it stands would give the impression that the outward force on BC (fig. 4) tends to infinity as the sides AB and CD increase in length, whereas for a very long rectangle the force on the side BC is equal to

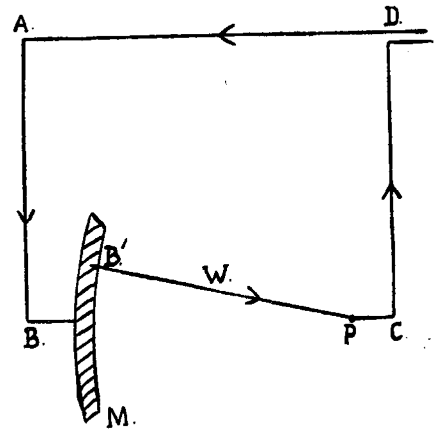
$$2i^2\log\frac{L-a}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

as can be seen at once from expression (8).

In calculating the outward force on the side BC (fig. 4), no account has been taken of the " corner effect " at B and C. This question and many

others of a similar nature were studied, fourteen years before Dunton's experiment, in a most interesting paper on Electromagnetic Forces in the 'Transactions of the American Institute of Electrical Engineers' (Feb. 1923), by Carl Hering. Part of Hering's work can be stated in his own words, thus : " Researches with high current densities in such mobile conductors as liquids and arcs, have brought out some heretofore unnoticed forces, the existence of some of which had been denied." One of the conclusions of Hering's researches, which is of special interest in connection with this paper, is the longitudinal electromagnetic force acting on current elements or electrons. Now, it has already been shown that such a force was one of the basic principles of Ampère's theory, and further experiments tending to prove the existence of such a force are of the greatest importance. Hering devised many experiments to demonstrate this longitudinal force, but there is one, so simple and yet so extraordinary, that it might well rank

Fig. 5.



alongside Ampère's original experiments. Here is a description of the experiment.

A current flows round a circuit ABCD (fig. 5). Part of the side BC is the horizontal wire W, pivoted at P, the other end of W being free to move over mercury in a curved trough M.

When Hering asked numerous physicists what would be the direction of motion of the wire W, they stated without exception that the wire would move outwards, the reason of course being " the circuit will arrange itself so as to include the maximum flux," but this, however, is contrary to experimental fact, the wire moves inwards. Indeed, the fact that the wire moves inwards is so contrary to preconceived notions, that one almost invariably suspects a trick, and soon Carl Hering had plenty of critics, but there was not one who repeated his experiment who did not also find that the wire W moves inwards, provided that the sides AB and DC are not small compared with AD.

The result can be readily explained by Ampère's theory. There is a longitudinal repulsion between the elements of the current (*i. e.* the electrons) in a metallic wire; these forces practically cancel each other, but in a liquid like mercury this longitudinal repulsion is able to push the free end of the wire W inwards. The wire W will eventually be in equilibrium under the repulsive force inwards from the forces in the mercury, and the outward force due to the rest of the circuit CDAB. When the wire W is placed on the other side of the "dead point" B, it is moved outwards under the action of two forces, one due to the mutual repulsion of the electrons in the mercury, the other due to the rest of the circuit.

There is one comment on Hering's experiment from the same journal, by John Mills, which is worth quoting: "Imagine billions of electrons rushing along from the negative terminal of the battery, along the connecting wire and plunging into the liquid of the trough. Then remember that each of these electrons is about $\frac{1}{1845}$ of the mass of the hydrogen atom. Remember that for the currents used there would be about a thousand billion billion of these electrons taking that plunge each second. Now I simply raise the question: is it inconceivable that, as they do plunge, they would kick back on the slider ? I do not believe it is inconceivable and I cannot see—and here is where I differ and am unpopular—I cannot see that flux and other Maxwellian concepts enter into the problem at all."

However, there were very few of Hering's critics who agreed with his views on longitudinal magnetic forces, although not one of them could show that no longitudinal forces existed; there were many, however, who attempted to explain Hering's experimental facts in terms of electromagnetic induction, etc. John H. Morecroft was without doubt successful in giving an alternative explanation to this experiment of Hering. Morecroft repeated the experiment using heavy currents and observed the equilibrium position of the wire W; then he measured the self-inductance of the circuit by a resonance method, using a frequency of four million cycles per second. Morecroft found that the self-inductance of the circuit decreased as the wire W was varied from its original equilibrium position; and hence the problem could be explained in terms of self-inductance, but this in no way disproves the first explanation based on the assumption that a longitudinal force exists.

As already stated, no account was taken of the "corner effect" at B and C, in fig. 4, in the calculation of the outward force on BC. If there were mercury cups at B and C (neglecting those at M and N), there would be an extra outward force on BC due to longitudinal repulsive forces arising in the mercury.

The paradox under consideration was considered in great detail in a long mathematical investigation, involving the special theory of relativity, time of propagation of the field, etc., by J. M. Keller in the 'American Journal of Physics,' December 1942. Concerning this problem Keller states "one can restore the third law to complete respectability and validity by introducing the artifice of the electromagnetic field, to which

one attributes a momentum density $\frac{1}{4\pi c}$ E×H. The field has just as much claim to reality as the original concept of momentum." Keller accepts the classical results, *i. e.* the Biot–Savart law; this, of course, leads to the paradox described at the beginning of this paper, but Keller attempts to surmount that difficulty by the use of a "field" which is solely due to the electrons themselves! One may well ask what is the difference between the charge on a particle and the "field" due to that charge; but that difference is not explained by Keller. Keller's argument may be all right for two vortex filaments in a liquid like water, which has inertia, but if it is to hold for charged particles in free space, then these particles must be assumed to be imbedded in an æther, also having inertia. Poynting believed that such an æther existed and derived an expression for its momentum density, and it is on this momentum density that Keller has based his paper. Now Poynting's theorem is a useful mathematical expedient for calculating energy radiated from wireless ærials, etc., but the large number of experiments devised during the last fifty years, for the purpose of detecting such an æther, leave no shadow of doubt that Poynting's method is devoid of any physical reality. The fact that Keller showed that the force on the electron at B is equal to $-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathrm{E}\times\mathrm{H})$, where E and H are the electrical and magnetic fields at B, in no wise solves the paradox. One rather remarkable feature of Keller's paper is the great use made of the notation of the special theory of relativity, despite the fact that an æther is implied although not stated; however, towards the end of this paper the assumption is made that the electrons are going very slowly with uniform velocity, and hence all factors such as $\left(1-\left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}}$ can be equated to unity.

While investigating the difficulties arising from these problems, I mentioned the paradoxical case to Professor G. W. O. Howe, who published an account of the problem and his conclusions upon it as an Editorial in the 'Wireless Engineer,' March 1944. Professor Howe concluded that there was no magnetic force on either electron in the particular position, by virtue of displacement currents surrounding each electron and causing the electron to act similarly to a toroidal magnet. This conclusion satisfies the particular problem, because if there is no magnetic force on either electron there is no failure of Newton's third law, and it also agrees with Ampère's original postulate. As can be gathered from the article, Professor Howe does not regard his explanation as final, but rather as an attempt to arouse interest in a paradox which is not easily solved. It is only right to remark that Professor Howe's conclusion is a direct contradiction of the Biot–Savart law, which is the one used by electrical engineers.

It has already been noted in this paper that Ampère also had difficulty in giving reasons to justify his postulate on this particular problem, but as he used this postulate to derive a formula which gives results in keeping with experiment, it can be safely said that Ampère's postulate is in keeping with experimental evidence. One cannot divorce this paradox from the rest of electrodynamics, as Professor Howe seems to imply in his Editorial (p. 107): "It must be emphasised that we are not here concerned with forces between moving electrons in general, but only with the special case illustrated in fig. 1."

Ampère was anxious that his law of force between elements of currents should be in keeping with Newton's law of motion, and in that he was almost completely successful, but in the year 1823, the Conservation of Energy was not known as a universal principle of nature, and Ampère did not consider his theory in terms of that important concept. For instance, if we try to find the energy of a system consisting of two Ampèrean elements, we are driven to conclude that the energy depends upon the angles which the elements make with the line joining them; yet, if this is so, couples will arise between the elements, when the angles are varied; and this is contrary to Ampère's original postulate that between two elements there is only a force along the line joining them. Nevertheless, the potential energy of two closed circuits is correctly given by Ampère's formula, as F. E. Neumann showed in 1845; and it is only fair to say that Biot's formula, likewise, gives the same result for two closed circuits.

Now, although Ampère's formula is not perfect, it is surely superior to Biot's, the one at present in use. It is indeed strange why Ampère's formula is so often misrepresented; for example, in Jeans' 'Electricity,' par. 497, we find: "Clearly the resultant is a force at right angles both to OP and to ds and of amount

$$\frac{i\,ds\sin\theta}{r^2},$$

where θ is the angle between OP and ds. [$OP=r$.]

"Thus the total force at P may be regarded as made up of contributions such as $\frac{i\,ds\sin\theta}{r^2}$ for each element of the circuit. This is known as Ampère's law." Now, from an historical point of view, the law given is not Ampère's but Biot's, and moreover it was severely criticised by Ampère. In the light of modern theories this criticism by Ampère is worth reading. The reason why Biot's formula has been preferred to Ampère's may be due to the fact that Biot's formula lends itself to simpler calculations, and, over closed circuits, where both formulæ give identical results, that is an advantage.

The object of this paper has been to draw attention to a formula in existence since the beginning of electrodynamics, superior to, although a little more complicated than, the one at present in use, and particularly suited to dealing with this paradox which has aroused a great deal of discussion. Moreover, this formula suggests that the magnetic force between electrons has a component parallel to the direction of the motion of the electrons, and although there are many who do not like this idea, it does not appear to have ever been proved wrong, and there are many experiments which can be much more easily explained in terms of this force, but further research is urgently required into that important aspect of the problem.

I wish to express my thanks to Professor Taylor Jones for much encouragement and advice, and also to Professor G. W. O. Howe for the loan of many papers and for valuable criticism. I wish it to be understood that I am solely responsible for the opinions and criticism expressed in this paper.

References.

A. Ampère, *Mémoire de L'Académie royale des sciences de L'Institut de France*, vol. vi. p. 175 (1823).
F. F. Cleveland, Phil. Mag. vol. xxi. p. 416 (Feb. 1936).
W. F. Dunton, 'Nature,' vol. cxl. p. 245 (Aug. 1937).
C. Hering, Trans. A.I.E.E. vol. xlii. p. 139 (Feb. 1923).
G. W. O. Howe, 'Wireless Engineer,' vol. xxi. Editorial (March 1944).
J. H. Jeans, 'Electricity and Magnetism,' § 497 (1925).
G. Joos, 'Theoretical Physics,' p. 447.
J. M. Keller, 'American Journal of Physics,' vol. x. p. 302 (Dec. 1942).
S. B. L. Mathur, Phil. Mag. vol. xxxii. p. 171 (1941).
Clerk-Maxwell, 'Electricity and Magnetism' (1873).
A. O'Rahilly, 'Electromagnetism,' chap. iv.
I. A. Robertson, "Magnetic Moments," Phil Mag. vol. xxxiv. p. 182 (March 1943).
J. J. Thomson, British Association Report, p. 98 (1885).

IV. *Construction of Groups of Commutative Functions.*

By Ludwik Silberstein, Ph.D.*

[Received January 11, 1944.]

Two functions, f and g, of a single variable x are said to be *commutative* or to commute with each other if, operationally,

$$fg=gf,$$

that is, $f[g(x)]=g[f(x)]$, for all x.

Let $g(x)$ be given and let it be required to find $f(x)$ as an analytical function commuting with $g(x)$. If we write $y=g(x)$, then the equation to be solved for f is

$$f(y)=g[f(x)],$$

whence, assuming the differentiability of both functions,

$$f'(y)\,g'(x)=g'[f(x)]f'(x),$$

$$f''(y)\,g'^2(x)+f'(y)\,g''(x)=g''[f(x)]f'^2(x)+g'[f(x)]f''(x),$$

* Communicated by the Author.