Self-Energy and Stability of the Classical Electron*

F. Rohrlich
Department of Physics and Astronomy, State University of Iowa, Iowa City, Iowa
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The classical theory of the electron, as proposed by Abraham and Lorentz, is usually presented as beset by the difficulty that the momentum and velocity of its Coulomb field are incorrectly related kinematically: $p=\frac{4}{3}m_s v$, where m_s is the electromagnetic mass defined by the electromagnetic self-energy. This problem also persists in the relativistic theory. It is shown here that the difficulty is eliminated from the relativistic theory by treating the integrals over the electromagnetic field in a relativistic fashion, i.e., taking note of their dependence on the motion of the electron. The surface dependence of the integrals representing the electromagnetic momentum and energy of the particle is essential and occurs whenever the matter tensor is not introduced. The nonrelativistic limit of this formulation then also leads to the correct relationship $p=m_s v$. The corrected Abraham-Lorentz theory still contains the stability problem, but this problem is no longer related to the transformation properties. It can be removed by renormalization.

1. HISTORY AND IMPORTANCE OF THE PROBLEM

THE first theory of an elementary particle was the theory of the electron, since it was the only elementary particle known at that time. It was developed in a number of papers by M. Abraham and by H. A. Lorentz, based on Maxwell's theory of electricity and magnetism. In its "final" form it was presented by Lorentz in an encyclopedia article[1] and in his Columbia Lectures[2] of 1906, and by Abraham in book form.[3] It was reviewed by Pauli[4] and v. Laue[5] in their treatises on relativity. All these are classics today. From them it found its way into many standard texts on electricity and magnetism.

This theory is still unsurpassed by any elementary particle theory today, as far as structure and mathematical description is concerned, as well as to agreement with experiments—within its limits of applicability. Not that the theory is perfect, closed, or mathematically satisfactory, but the other theories we have today are beset by basically the same difficulties and in many ways start with the same physical picture of a particle on one hand, a field on the other, and an interaction between the two. Of course, this situation is not very surprising, because in last analysis these other theories are modeled after the Abraham-Lorentz theory.

The comparison just made seems not quite valid in one respect of major importance: Quantum electrodynamics, at present our best description of the electron as far as experimental agreement is concerned, does seem to have overcome a basic difficulty present in the Abraham-Lorentz theory, viz., it succeeds in reducing the self-energy problem and the stability problem of the electron to the problem of mass and charge. A brief explanation of this point is perhaps in order here.

Just like the classical electron theory, quantum electrodynamics is not a theory which attempts to explain why the electron has a certain mass and charge as found by experiments. In fact, it breaks down as soon as the question of the magnitude of these quantities is asked. The theory is thus phenomenological in this respect: The observed values of mass and charge must be fed into the theory. If this is granted, the self-energy problems can be eliminated by reducing them to the mass and charge problem. The same is true for the stability problem which will be discussed later on.

* The subject matter of this paper was first presented as part of a lecture series at the Theoretical Physics Institute at the University of Colorado, Boulder, during the summer of 1959.

[1] H. A. Lorentz, *Enzyklopädie der Mathematishen Wissenschaften* (B. G. Teubner, Leipzig, 1904), Vol. V, No. 14.

[2] H. A. Lorentz, *The Theory of Electrons* (Dover Publications, New York, 1952), 2nd ed.

[3] M. Abraham, *Theorie der Electrizität* (B. G. Teubner, Leipzig, 1905), Vol. 2, 1st ed.

[4] W. Pauli, "Relativitätstheorie," *Enzyklopädie der Mathematishen Wissenschaften* (B. G. Teubner, Leipzig, 1921), Vol. V, No. 19, translated as *Theory of Relativity* (Pergamon Press, New York, 1958).

[5] M. v. Laue, *Das Relativitätsbrinzip* (B. G. Teubner, Leipzig, 1911).

The Abraham-Lorentz theory also requires the knowledge of the observed electron mass and charge. But its self-energy problem is apparently not reducible to the mass problem, as in quantum electrodynamics. This is in contradistinction to a theory developed by Dirac[6] in 1938 in which this reduction is carried out. These theories have therefore been always regarded as basically different.

One purpose of the following two sections is to bridge this apparent gap, at least as far as the self-energy problem is concerned. One can show that also in all other respects the Dirac theory is simply a further development of the Abraham-Lorentz theory, but this would exceed the scope of the present remarks. However, in comparing the two theories one must keep in mind that the Dirac theory describes a point electron, whereas the Abraham-Lorentz theory can refer to an electron of finite *or* zero radius.

2. ELECTRON SELF-ENERGY

The point electron is defined as the limit (as the radius approaches zero) of an electron of finite radius. For a relativistic theory it is essential that such an electron be described in a relativistic way. For example, a spherical electron must contract, when in motion, according to the Lorentz transformation.

Although the particular model is irrelevant for the point in question, let us assume, for definiteness, that our electron when at rest, is a sphere of radius a whose charge e is entirely on the surface. The electromagnetic field inside will then be zero and outside it will be the Coulomb field

$$\mathbf{E}=(e/r^2)\hat{\mathbf{r}}, \quad \mathbf{H}=0, \tag{1}$$

with the unit vector $\hat{\mathbf{r}}=\mathbf{r}/r$. The momentum and energy of any electromagnetic field is best defined in terms of the symmetric electromagnetic energy-momentum tensor $\mathsf{T}^{\mu\nu}$ whose space part[7] (T^{ik}) forms a dyadic, called the stress tensor density

$$\mathsf{T}=(1/4\pi)(\mathbf{EE}+\mathbf{HH})-\mathbf{1}U, \quad U=(1/8\pi)(E^2+H^2), \tag{2}$$

whose time part

$$\mathsf{T}^{00}=-U, \tag{3}$$

and whose mixed parts are the components of the Poynting vector $\mathbf{S}$,

$$\mathsf{T}^{0k}=-(1/c)S^k, \quad \mathbf{S}=(c/4\pi)\mathbf{E}\times\mathbf{H}. \tag{4}$$

In the Abraham-Lorentz theory the energy and momentum of the Coulomb field surrounding a uniformly moving electron are defined by

$$W=-\int \mathsf{T}^{00}d^3x=\int U d^3x$$

$$\text{(Abraham-Lorentz)} \tag{5}$$

$$p^k=-\frac{1}{c}\int \mathsf{T}^{0k}d^3x=\frac{1}{c^2}\int S^k d^3x.$$

Let us use this definition for the particle at rest. It gives, after integration from a to ∞.

$$W=\frac{1}{8\pi}\int_a^\infty \frac{e^2}{r^4}d^3x=\frac{e^2}{2a}, \quad \text{(at rest)} \tag{6}$$

$$\mathbf{p}=0.$$

The self-energy W due to the Coulomb field is written as

$$W=m_s c^2, \tag{7}$$

which defines the electromagnetic mass

$$m_s=e^2/2ac^2. \tag{8}$$

The latter is a relativistic invariant.

Let us now assume that the electron is moving with constant velocity $\mathbf{v}$. The corresponding energy and momentum are obtained from those at rest by a Lorentz transformation

$$W'=\gamma(W+\mathbf{p}\cdot\mathbf{v}), \quad \gamma=(1-v^2/c^2)^{-\frac{1}{2}}$$
$$\mathbf{p}'=\gamma(\mathbf{p}+W\mathbf{v}/c^2)+(\gamma-1)\hat{\mathbf{v}}\times(\hat{\mathbf{v}}\times\mathbf{p}),$$

which yields, by means of (6),

$$W'=\gamma m_s c^2$$
$$\mathbf{p}'=\gamma m_s\mathbf{v}. \tag{9}$$

This is exactly the result that one wants to obtain. It gives the correct relativistic relationship between the momentum and the velocity which kinematically means that the Coulomb field is carried along by the electron in a relativistically invariant way.

Why, then, did Abraham and Lorentz obtain a different result? The answer is simple: The defini-

[6] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

[7] Our relativistic notation is $x^\mu=(ct,\mathbf{r})=(x^0,x^1,x^2,x^3)$; $A_\mu B^\mu=\mathbf{A}\cdot\mathbf{B}-A^0B^0$; $i, k=1, 2, 3$; $\mu, \nu=0, 1, 2, 3$.

tions (5) are not Lorentz invariant, i.e., their form changes when a Lorentz transformation is applied. Though this is no criticism for a nonrelativistic theory, the situation leads to an inconsistency in the relativistic case where Lorentz and Abraham took great pains to describe the contraction of the spherical shape.

What is the physical reason behind this lack of invariance? The integrals in (5) are to be carried out from the surface of the electron to infinity. But this surface does not remain spherical when seen by an observer in motion relative to the electron. The world-line of the electron then does not remain orthogonal to a space-like hypersurface σ (e.g., $t=$constant). Lorentz invariance requires this orthogonality and therefore the integrals (5) are not covariant.

Let us try to see this situation quantitatively. For this purpose it is convenient to define the vector

$$d\sigma^\mu = n^\mu d\sigma(x). \tag{10}$$

n^μ is the (timelike) unit vector normal to the surface at the point x and directed into the future light cone. This means $n_\mu n^\mu = -1$, $n^0 > 0$. $d\sigma$ is an invariant infinitesimal element of this surface. In the rest system of the electron $n^k = 0$, $n^0 = 1$, and $d\sigma = d^3x$ for a plane hypersurface. The energy and momentum in Eq. (9) form a four-vector which can be written in manifestly covariant form

$$p^\mu = \left(\frac{1}{c}W, p\right) = -\frac{1}{c}\int_\sigma \mathbf{T}^{\mu\nu} d\sigma_\nu. \tag{11}$$

The definition (5), however, is

$$\left(\frac{1}{c}W, p\right) = -\frac{1}{c}\int \mathbf{T}^{\mu 0} d^3x, \tag{5'}$$

and this definition would follow from (11) only if the integral were independent of σ.

One can prove that an integral of the form (11) is independent of the hypersurface σ if and only if

$$\partial_\mu \mathbf{T}^{\mu\nu} = 0. \tag{12}$$

This equation is exactly the differential form of the conservation laws of energy and momentum for the electromagnetic field. But it is valid *only in the region* $r > a$, i.e., outside the electron where no matter is present. Thus, the surface over which the integral (11) is extended can be chosen arbitrarily for $r > a$ only. For $r \leqslant a$ it is determined by the relativistic change of the shape of the electron, which change in turn depends on the motion of the electron. *Even in the limit* $a \to 0$ will the integral depend on the motion of the point electron electron.

An alternative definition of P^μ which would be of the form (11) but surface *independent* could clearly be given by means of the *total* energy momentum tensor (including the matter tensor) whose divergence vanishes *everywhere*. However, this approach is not followed usually, despite its popularity in quantum field theory. The reason for this reluctance of introducing a matter tensor lies mainly in the attempt to separate these problems from the question of the equations of motion. The latter leads to difficulties of its own that should not be confused with the issues presently under discussion.

Returning to the definition (11), let us choose σ to be a plane hypersurface, i.e., n^μ independent of x; this choice will be used exclusively from here on. Then the direction of n^μ will be determined by the motion of the electron via

$$n^\mu = v^\mu / c, \tag{13}$$

where v^μ is the velocity four-vector

$$v^\mu = (c\gamma, \mathbf{v}\gamma), \quad \mathbf{v} = d\mathbf{r}/dt. \tag{14}$$

This is intuitively clear, since v^μ is the only dynamical quantity which is a timelike vector of fixed magnitude with $v^0 > 0$ (as n^μ was defined). But it can also be proven formally by asking for the conditions on n^μ such that P^μ as defined in (11) satisfy

$$P^\mu = m_s v^\mu. \tag{15}$$

These conditions lead uniquely to (13).

We can summarize this discussion by saying that the definition (5) is incorrect and should be replaced by (11) which becomes (for a plane hypersurface)

$$P^\mu = -\frac{1}{c}\int \mathbf{T}^{\mu\nu} v_\nu d\sigma, \tag{16}$$

i.e.,

$$\begin{aligned} W &= \gamma \int U d\sigma - \gamma \int \mathbf{S}\cdot\mathbf{v} d\sigma \\ \mathbf{p} &= \gamma \int \mathbf{S} d\sigma + \frac{\gamma}{c^2}\int \mathbf{T}\cdot\mathbf{v} d\sigma. \end{aligned} \tag{17}$$

Since the volume element $d^3x=\gamma d\sigma$, we see that these expressions differ from the Abraham-Lorentz definition (5) by additional terms.

In the nonrelativistic limit (17) becomes ($v\ll c$)

$$W=\int Ud^3x, \qquad \mathbf{p}=\int \mathbf{S}d^3x+\frac{1}{c^2}\int \mathbf{T}\cdot\mathbf{v}d^3x \qquad (NR) \quad (18)$$

because the second term in W is negligible, but the second term in $\mathbf{p}$ is not negligible, as we shall now verify by explicit calculation.

For slow (nonrelativistic) motion

$$\mathbf{E}=(e/r^2)\hat{\mathbf{r}}, \quad \mathbf{H}=(1/c)\mathbf{v}\times\mathbf{E}. \qquad (19)$$

Using the explicit expression (2) for $\mathbf{T}$ we find

$$\frac{1}{c^2}\int \mathbf{T}\cdot\mathbf{v}d^3x=\frac{1}{4\pi c^2}\int(\mathbf{EE}\cdot\mathbf{v}-\tfrac{1}{2}\mathbf{v}E^2)d^3x =\frac{e^2}{4\pi c^2}\int_r^\infty\frac{dr}{r^2}\int(\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}-\tfrac{1}{2}\mathbf{v})d\Omega.$$

The integration over the solid angle $d\Omega=\sin\vartheta d\vartheta d\varphi$ leads to $-4\pi\mathbf{v}/6$ as can easily be verified.[8] Thus,

$$\frac{1}{c^2}\int \mathbf{T}\cdot\mathbf{v}d^3x=-\frac{e^2}{6c^2}\mathbf{v}\int_a^\infty\frac{dr}{r^2} =-\frac{e^2\mathbf{v}}{6ac^2}=-\tfrac{1}{3}m_s\mathbf{v}. \qquad (20)$$

Now, the Abraham-Lorentz definition (5) for the momentum [which is the first term of our definition (18)] yields in the nonrelativistic case

$$\mathbf{p}_{A-L}=\tfrac{4}{3}m_s\mathbf{v}, \qquad (21)$$

as is proven in many texts, following the original work by Abraham.[9] The relativistically correct definition has a nonrelativistic limit (18) which differs from $\mathbf{p}_{A-L}$ exactly by the addition of the integral (20). Thus,

$$\mathbf{p}=\mathbf{p}_{A-L}-\tfrac{1}{3}m_s\mathbf{v}=m_s\mathbf{v}. \qquad (22)$$

This, in turn is a special case of (15). We have thus shown that for a uniformly moving slow electron the definition (18) for the momentum yields the correct result, and that the second term in this definition cannot be neglected. The definition of *energy* in (18) is the same as that proposed by Abraham and Lorentz. In the relativistic case the correct definitions are provided by (16).

The generalization of the covariant definition of momentum (16) to accelerated motion is implied in Dirac's classical electrodynamics.[6] It adds little to the points made in the present considerations.

3. STABILITY OF THE ELECTRON

As is pointed out in standard texts, and as was first observed by Abraham,[10] the definitions (5) which lead to the unacceptable result (21) also lead to the necessity of the existence of nonelectromagnetic forces which are required *both* for the stability of the electron and for the Lorentz invariance of the theory. These forces were introduced by Poincaré and are known by his name.[11]

Intuitively, the situation seems trivially simple: A surface charge on a sphere would "fly apart" unless held together by some attractive forces. No such forces, however, appear in a purely electromagnetic theory. Quantitatively, the situation is described by the electromagnetic stress tensor

$$\Theta^{ik}=\mathbf{T}^{ik}d^3x \quad (i,k=1,2,3). \qquad (23)$$

If the system were in equilibrium, Θ^{ik} would have to vanish in the rest system. This is not the case. A simple calculation shows that in the rest system

$$\Theta^{ik}=0 \quad i\neq k; \quad \Theta^{ii}=-\tfrac{1}{3}m_sc^2, \qquad (24)$$

indicating that the electromagnetic stresses are not compensated and the electron is not stable. Poincaré simply postulated attractive forces corresponding to stresses which would exactly balance these and establish equilibrium. Such forces must evidently be of nonelectromagnetic nature. Mathematically, the Poincaré tensor plays the same role as the matter tensor mentioned earlier.

[8] For example, multiplication of the angular integral by a constant vector perpendicular to $\mathbf{v}$ yields zero for this integral. Thus, the integration must give a vector parallel to $\mathbf{v}$. It is now sufficient to compute this integral times $\mathbf{v}$, which is $-4\pi v^2/6$ and gives the stated result.

[9] M. Abraham, Ann. Physik **10**, 105 (1903). See, for example, W. Panofsky and M. Phillips, *Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

[10] M. Abraham, Physik Z. **5**, 576 (1904).

[11] H. Poincare, R. C. Circ. Mat. Palermo **21**, 129 (1906).

It assures that the integral be independent of the surface, since it is chosen to complement the electromagnetic tensor to yield a divergence free result in all space.

The Abraham-Lorentz definition (5) of energy and momentum of the Coulomb field leads to a very confusing situation: It provides a link between the transformation properties of these quantities and the instability of the charge. The factor $\frac{4}{3}$ in (21) seems linked with the nonvanishing result (24). This can be seen by carrying out a Lorentz transformation on $\mathbf{T}^{\mu 0}$ in (5) or (5′). One finds for motion in the x direction:

$$W'=\gamma W-\gamma\left(\frac{v}{c}\right)^2\int \mathbf{T}^{11}d^3x$$
$$p'=\gamma\frac{W}{c^2}v-\gamma\frac{v}{c^2}\int \mathbf{T}^{11}d^3x, \tag{25}$$

where W and $\mathbf{T}^{11}$ refer to the rest system, so that $W=m_s c^2$. This demonstrates that the correct relationship (15) (without the $\frac{4}{3}$ factor) can be obtained if and only if $\Theta^{11}=0$ in the rest system. If this conclusion were true it would mean that a relativistic theory cannot be constructed on purely electromagnetic interactions: The separation of electromagnetic and nonelectromagnetic Poincaré forces is not Lorentz invariant, and conversely, the theory would no longer diverge in the point limit if it were Lorentz invariant. This conclusion is obviously false, as is evident from the existence of a divergent but Lorentz invariant quantum electrodynamics.

Indeed, our definition (16) no longer permits such a conclusion. The same Lorentz transformation applied to (16) gives

$$W'=\gamma W$$
$$p'=\gamma(W/c^2)v, \tag{26}$$

and all stress tensor integrals exactly cancel in the process. Equation (26) holds *despite* the fact that $\Theta^{11}\neq 0$. We now have, as in quantum electrodynamics, a *relativistically invariant* classical electron theory, but which diverges in the point limit, and which yields an unstable charge. *The stability problem is not related to the transformation properties of the theory.*

Again, just as in quantum electrodynamics, the relativistic invariance of the theory permits one in a unique way to eliminate all divergences in terms of the mass and charge problem. This procedure is today well known in quantum field theory as renormalization; it can be carried through in classical just as in quantum electrodynamics. In fact, historically, the former was renormalized first (Dirac, 1938).

The details of classical renormalization theory exceeds the present discussion,[12] but when it is carried out consistently it yields a renormalized stress tensor which vanishes identically. Thus, the renormalized classical electron is *stable*, just as in renormalized quantum electrodynamics.

It remains to explain why, after renormalization, the electron no longer "flies apart," since no attractive forces have been introduced. How can renormalization play the same role as the Poincaré "glue" played previously?

This seems indeed to be a baffling situation. But what makes the electron unstable in the first place? It is the (repulsive) Coulomb force of one part acting on another part of the charge. The covariant formulation permits a separation of the field of the electron into a part which acts on other charges and a part which acts on itself. The latter part is removed from the theory by renormalization. No part of the renormalized electron can act on another part of it, very much within the spirit of regarding the electron as "elementary." This makes the self-stress vanish not because of cancellation with another stress (as in the Poincaré model), but because there are no self-interactions in a classical renormalized theory. We emphasize that renormalization is only possible when the self-interaction can be separated in a relativistically invariant way. This was not possible with the Abraham-Lorentz definition of self-energy momentum.

We conclude that, after correction of the definitions of energy and momentum of the Coulomb field, the classical electron theory exhibits exactly the same structure as quantum electrodynamics, both as to the mass problem and as to the stability of the charge.

[12] See, e.g., S. N. Gupta, Proc. Phys. Soc. (London) A64, 50 (1951).