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Electromagnetic Momentum, Energy, and Mass*

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Electromagnetic systems of finite mass ($m>0$) and of zero mass ($m=0$) are considered. For both cases the electromagnetic energy and momentum are computed and are shown to lead to essentially different formulas. An invariant expression for the electromagnetic mass in $m>0$ systems is derived. All $m>0$ results are then specialized to the electrostatic case. Historical and present day confusion in the literature is discussed.

1. INTRODUCTION

The momentum four-vector of classical electromagnetic systems as well as their electromagnetic mass are topics of repeated discussion in this Journal and elsewhere. Most recently, two papers by Butler[1] published in this Journal were added to this literature. Unfortunately, both existing textbooks and original papers are not always free of errors or misleading statements on this matter.

A good deal of this confusion is due to the fact that errors made by great physicists of the past are taken over into textbooks in an uncritical way. We shall return to this point in Sec. 5 of this paper.

The results obtained here are not new, but the approach is rather unconventional and hopefully, pedagogical and transparent. We shall first give various classifications of systems so that it will be quite clear which physical systems we are talking about (Sec. 2). Section 3 is devoted to systems of finite mass $m>0$; this is the heart of the paper. Then come systems with $m=0$ (Sec. 4). The rest of the paper involves historical remarks and discussion (Secs. 5 and 6).

Since the electromagnetic field is an intrinsically relativistic object, we use the language of the special theory of relativity. We choose the diagonal metric tensor $g_{00}=-1$, $g_{kk}=+1$ ($k=1, 2, 3$), so that the vector x has the components $x^\mu=(t, \mathbf{x})$ and $x_\mu=(-t, \mathbf{x})$, $\mu=0, 1, 2, 3$.

2. CLASSIFICATION OF SYSTEMS

There are many ways in which physical systems can be classified. The following classifications will be relevant to our present interest:

(a) Macroscopic and microscopic systems. Microscopic systems of charged particles (for example point charges) lead to difficulties if described by the usual classical electromagnetic theory (infinite Coulomb self-energy, etc.). While a new formulation of the theory is free of such difficulties,[2] we shall *not* be concerned with microscopic systems here. Therefore, there will then also be no question concerning the applicability of classical theory (as compared to quantum theory).

(b) One distinguishes open and closed systems. If a physical system is removed from all other matter and fields so that there is no interaction whatsoever between this system and the rest of the universe, we call it closed. Otherwise, it is open. Obviously, some or all of the conservation laws valid for a closed system may be violated if this system is opened by bringing it in interaction with other systems.

The most important invariance property for the present consideration is translation invariance. In *integral* form this invariance insures the existence of a momentum four-vector P which is a constant of the motion (i.e., each component P^μ is a constant of the motion). In *differential* form it states that the energy tensor T is divergence free,[3]

$$\partial_\mu T^{\mu\nu}=0. \tag{2.1}$$

If the system consists of two components, e.g., a source and a field, then these two components are in general not separately invariant. Their individual energy tensors, $\Pi^{\mu\nu}$ and $\Theta^{\mu\nu}$, say, are not separately divergence free. Instead, we have

$$\partial_\mu\Pi^{\mu\nu}+\partial_\mu\Theta^{\mu\nu}=0. \tag{2.2}$$

(c) According to special relativity, all forms of energy have a mass equivalent. But not all systems have a *rest* mass: A unidirectional electromagnetic wave carries energy but has no rest mass. One can therefore classify systems by their rest mass. We shall distinguish $m=0$ and $m>0$ below.[4] This is equivalent to a classification by the total momentum four-vector: $m=0$ corresponds to a null vector P (i.e., $P_\mu P^\mu=0$) while $m>0$ corresponds to a timelike $P(P_\mu P^\mu<0)$. The rest mass m can in fact be defined by $m^2=-P_\mu P^\mu$.

3. SYSTEMS WITH $m>0$

A. The General Case

One of the most basic properties of a closed physical system is translation invariance. This is deduced from the fact that the physical description does not depend on the choice of origin of the coordinate frame (origin of the three space coordinates or of the time coordinate). Closely associated with this invariance property is the existence of a four-vector P which is the (conserved) linear four-momentum of the system. Mathematically, P is the infinitesimal generator of the translation transformation.

Translation invariance is common to Poincaré invariance (invariance under the inhomogeneous Lorentz transformations) and to Galilean invariance (invariance under the inhomogeneous Galilean transformations). The existence of P is therefore very deeply rooted in physics and is not an outgrowth of special relativity. The transformation properties of P under motion with constant velocity of course do depend on whether the relativistic (Poincaré) or the nonrelativistic (Galilean) description is used.

The rest mass of a physical system as mentioned earlier, is given by its momentum P through

$$P_\mu P^\mu=-m^2. \qquad (3.1)$$

Since $m^2>0$ is assumed here, this equation makes P a timelike vector in Minkowski space. Such vectors come in two types, depending on whether they point into the future or into the past light cone. The positivity of the total energy of a system leads to the physical requirement

$$P^0>0. \qquad (3.2)$$

In a given coordinate frame S, the components of P, i.e., P^μ, shall be denoted by $E=P^0$ and $\mathbf{P}=(P^1, P^2, P^3)$. Every timelike vector can be transformed to a rest system S_0 which is characterized by $\mathbf{P}_{(0)}=0$. From (3.1) and (3.2) follows $E_{(0)}=m$.

The three-dimensional spatial volume $V_{(0)}$ of the system in S_0 (which may be infinitely large) can be thought of as composed of small cells each of which carries a certain amount of energy. If $U_{(0)}$ is the energy density in the frame S_0, then

$$E_{(0)}=\int U_{(1)}(x)\,dV_{(0)}, \qquad \mathbf{P}_{(0)}=0. \qquad (3.3)$$

We shall now show that the four-vector transformation property of P together with (3.3) imply that P in any frame S must have the form

$$P^\mu=\int T^{\mu\nu}(x)\,d^3\sigma_\nu(x), \qquad (3.4)$$

where $d^3\sigma^\mu$ are the components of the infinitesimal volume element of a three-dimensional spacelike hyperplane. The tensor T is called the energy (or energy-momentum) tensor. It is actually (dimensionally) an energy *density* tensor.

In Minkowski space the volume $V_{(0)}$ is the three-dimensional hyperplane $t=\text{const}$. It has a unit normal which is the timelike four-vector $n_{(0)}$ with components $n_{(0)}{}^\mu=(1, \mathbf{0})$ so that

$$n_{(0)}{}^\mu n_{(0)\mu}=-1.$$

In general, a spacelike hyperplane with normal vector n and $n_\mu n^\mu=-1$ has the infinitesimal element

$$d^3\sigma^\mu=n^\mu d^3\sigma, \qquad (3.5)$$

where $d^3\sigma=-\eta_\mu d^3\sigma^\mu$ is an invariant. This element transforms like a four-vector. In S_0,

$$d^3\sigma_{(0)}{}^\mu=n_{(0)}{}^\mu d^3\sigma_{(0)}=(dV_{(0)}, \mathbf{0}).$$

Thus, $d^3\sigma=dV_{(0)}$. Finally, since $d^3\sigma^\mu$ is a vector, $T^{\mu\nu}$ must be a tensor.

It follows that (3.4) can be written

$$P^\mu=\int P^\mu(x)\,d^3\sigma, \qquad (3.6)$$

where

$$P^\mu(x)=T(x)^{\mu\nu}n_\nu \qquad (3.7)$$

is the *momentum density four-vector* in the reference frame S. In S_0 it reduces to $\mathbf{P}_{(0)}(x)$ ($\neq 0$ in general) and $P_{(0)}{}^0(x)=E_{(0)}(x)=U_{(0)}(x)$. We make the important observation that in any frame $S\neq S_0$ the energy component $P(x)^0$ of the momentum density four-vector is *not* $T(x)^{00}$ but is $T^{0\nu}n_\nu$ and in general n_ν has nonvanishing components for $\nu=1, 2, 3$.

A Lorentz transformation on the unit normal vector $n_{(0)}{}^\mu$ of the $t=\text{const.}$ hyperplane in S_0 shows that in a frame S that moves with velocity $\mathbf{v}$ relative to S_0 we have the components [using $\gamma \equiv (1-\mathbf{v}^2)^{-1/2}$]

$$n^\mu = (\gamma, \gamma\mathbf{v}) = v^\mu. \qquad (3.8)$$

This can also be seen from the observation that both n and v are timelike unit vectors, $n_\mu n^\mu = -1$, $v_\mu v^\mu = -1$ and that in $S_{(0)}$ $n_{(0)}{}^\mu = (1, \mathbf{0})$ and $v_{(0)}{}^\mu = (1, \mathbf{0})$. Thus, (3.4) can also be written

$$P^\mu = M^{\mu\nu} v_\nu, \qquad (3.9)$$

with

$$M^{\mu\nu} = \int T^{\mu\nu}(x) d^3\sigma. \qquad (3.10)$$

We shall call this tensor the *mass tensor*. From the identification of the total mass m with the total energy (divided by $c^2=1$) in the rest frame S_0 we have

$$m = P_{(0)}{}^0 = -v_{(0)\mu} P_{(0)}{}^\mu = -v_{(0)\mu} M_{(0)}{}^{\mu\nu} v_{(0)\nu} = -M_{(0)}{}^{00}, \qquad (3.11)$$

and therefore in any system S

$$m = -v_\mu P^\mu = -v_\mu M^{\mu\nu} v_\nu. \qquad (3.12)$$

As a physical requirement, the mass tensor must be *symmetric*. Otherwise, $M^{\mu\nu}$ would contain a covariant part (i.e., the antisymmetric part) which does not contribute to the mass.

Given the symmetric mass tensor $M^{\mu\nu}$ we can define the rest frame as that frame in which

$$M_{(0)}{}^{k0} = 0 \qquad (k=1, 2, 3). \qquad (3.13)$$

That S_0 exists is ensured by the timelike nature of the four momentum. The mass m is then defined by the M^{00} component in S_0 according to (3.11).

Alternatively, we can say that the eigenvalue equation

$$M^\mu{}_\nu \psi^\nu = \lambda \psi^\mu \qquad (3.14)$$

has a solution for a real positive eigenvalue $\lambda = m$ and an associated eigenvector $\psi^\mu = v^\mu$, according to (3.12). In the rest frame S_0 this is easily verified to be correct; a Lorentz transformation to S then confirms it in general. We conclude therefore that for any frame S

$$P^\mu = m v^\mu \qquad (3.15)$$

with m given by (3.12).

B. The Electromagnetic Case

To proceed further, additional information about the system is necessary. Only now do we specify our energy tensor as that of an electromagnetic field. When the system is macroscopic, as we assume, the total energy tensor will consist of a purely electromagnetic part, $\Theta^{\mu\nu}$ and a remaining tensor which describes nonelectromagnetic forces and matter.

If we restrict our attention to the electromagnetic field which is in general only part of a physical system, this *partial physical system* will be open in the sense of Sec. 2. The presence of a source of the field (e.g., an electric charge) requires a separation of the electromagnetic field produced by this source (e.g., the Coulomb field) from the matter of this source (the "bare" source). This separation is very artificial and unphysical for an individual electron which is considered to be "all electromagnetic," but it is quite reasonable in macroscopic physics where bodies can be charged and discharged at will.

It follows that the situation is of a type characterized by Eq. (2.2). If we wish to restrict our attention to the electromagnetic field only, we can still construct a covariant momentum four-vector if we follow the procedure of the previous section and integrate over the hyperplane characterized by the rest system. We shall return to this point at the beginning of Sec. 6.

The electromagnetic momentum four-vector is then given by

$$P^\mu = \int \Theta^{\mu\nu}(x) v_\nu d^3\sigma, \qquad (3.16)$$

which is valid relative to any Lorentz frame S and where $\Theta(x)$ is the electromagnetic energy tensor. If the system contains no fields inside matter ($\epsilon=1$, $\mu=1$ in Gaussian units) there are only two three-vector fields, $\mathbf{E}$ and $\mathbf{B}$. The antisymmetric field tensor has components $F^{\mu\nu}$ with $F^{ij}=B_k$ (i, j, k, being a cyclic permutation of 1, 2, 3) and $F^{0k}=E_k$. In terms of $F^{\mu\nu}$ we have the well-known expression[5]

$$\Theta^{\mu\nu} = (4\pi)^{-1}(F^{\mu\alpha}F_\alpha{}^\nu + \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}). \qquad (3.17)$$

This symmetric tensor can be written in terms of $\mathbf{E}$ and $\mathbf{B}$ as

$$\Theta^{00} = -(8\pi)^{-1}(E^2+B^2) \equiv -U;$$
$$\Theta^{0k} = -(4\pi)^{-1}(\mathbf{E}\times\mathbf{B})_k \equiv -(\mathbf{S})_k;$$
$$\Theta^{kl} = (\mathsf{T})^{kl} = (4\pi)^{-1}[\mathbf{EE}+\mathbf{BB}-\tfrac{1}{2}\mathsf{1}(E^2+B^2)]_{kl}. \qquad (3.18)$$

The characterization of the rest frame S_0, (3.13), therefore becomes

$$\int \mathbf{S}_{(0)} dV_{(0)} = 0 \tag{3.19}$$

and expresses the fact that in this frame the *total* integrated Poynting vector vanishes.

The mixed electromagnetic mass tensor

$$M^\mu{}_\nu = \int \Theta^\mu{}_\nu(x) d^3\sigma \tag{3.20}$$

has the total electromagnetic mass m as an eigenvalue and v^μ as eigenvector, according to (3.12) to (3.15).

Since the trace of Θ vanishes,

$$\Theta_\mu{}^\mu = 0, \tag{3.21}$$

as is evident from (3.17), the remaining three eigenvalues of this 4×4 tensor $M^\mu{}_\nu$ must add up to $-m$. In S_0 these are the eigenvalues of the integrated Maxwell stress tensor $\Theta^k{}_l = \Theta^{kl}$ which is given by (3.18) in dyadic form. They are not of interest here.

We are thus ensured the existence of an electromagnetic momentum four-vector of the form (3.15) with the electromagnetic mass

$$m = (8\pi)^{-1} \int (\mathbf{E}_{(0)}{}^2 + \mathbf{B}_{(0)}{}^2) dV_{(0)} \tag{3.22}$$

as follows from (3.11) and (3.18). This is the electromagnetic rest energy ($c=1$!) of the system [compare (3.3)].

When (3.15) is written as an integral over the momentum density (3.16), one obtains the components of the electromagnetic momentum four-vector in terms of U, $\mathbf{S}$, and T from (3.18) and (3.8)

$$P^0 = \gamma \int (U - \mathbf{v}\cdot\mathbf{S}) d^3\sigma = (\gamma/4\pi) \times \int [\tfrac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) - \mathbf{v}\cdot\mathbf{E}\times\mathbf{B}] d^3\sigma, \tag{3.23}$$

$$\mathbf{P} = \gamma \int (\mathbf{S} + \mathbf{v}\cdot\mathsf{T}) d^3\sigma = (\gamma/4\pi) \times \int [\mathbf{E}\times\mathbf{B} + \mathbf{v}\cdot\mathbf{E}\mathbf{E} + \mathbf{v}\cdot\mathbf{B}\mathbf{B} - \tfrac{1}{2}\mathbf{v}(\mathbf{E}^2+\mathbf{B}^2)] d^3\sigma. \tag{3.24}$$

From these equations we learn that, when referred to the invariant three-dimensional integration volume $d^3\sigma = dV_{(0)}$, the energy density is *not* $U(x)$ or $\gamma U(x)$, but in the frame S

$$P^0(x) = \gamma[U(x) - \mathbf{v}\cdot\mathbf{S}(x)], \tag{3.25}$$

$\mathbf{v}$ being the velocity of S relative to the rest frame S_0. Similarly, the momentum density in frame S is not the Poynting vector $\mathbf{S}$ or $\gamma\mathbf{S}$, but

$$\mathbf{P}(x) = \gamma[\mathbf{S}(x) + \mathbf{v}\cdot\mathsf{T}(x)]. \tag{3.26}$$

There are many erroneous statements in the literature concerning these densities. As we shall see below, $P^0(x) = U$, and $\mathbf{P}(x) = \mathbf{S}(x)$ holds only under very special circumstances when one is dealing with pure radiation (no matter) and unidirectional energy flow, i.e., when the total electromagnetic rest mass m vanishes.

The Eqs. (3.23) and (3.24) are not new,[6] but they are quite general and therefore deserve more attention than is given to them in the standard literature.

C. The Mass Invariant and the Electrostatic Case

The electromagnetic mass m is an invariant and, because of the assumed translation invariance, is in fact a constant (i.e., time independent). This invariance is not evident from (3.22) but (3.12) tells us that

$$m = -v_\mu M^{\mu\nu} v_\nu = -\int v_\mu \Theta^{\mu\nu}(x) v_\nu d^3\sigma = (4\pi)^{-1} \int (f_\alpha f^\alpha + \tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta}) d^3\sigma, \tag{3.27}$$

where

$$f^\alpha \equiv F^{\alpha\nu} v_\nu \tag{3.28}$$

is the Lorentz force four-vector per unit charge,

$$f^\alpha = (\gamma\mathbf{v}\cdot\mathbf{f};\ \gamma\mathbf{f}), \tag{3.29}$$

$$\mathbf{f} = \mathbf{E} + \mathbf{v}\times\mathbf{B}. \tag{3.30}$$

Eq. (3.27) makes the invariance of m manifest. The invariants can be expressed in terms of the three-vector fields $\mathbf{E}$ and $\mathbf{B}$,

$$f_\alpha f^\alpha = \gamma^2[\mathbf{f}^2 - (\mathbf{v}\cdot\mathbf{f})^2] = \gamma^2[\mathbf{E}^2 + (\mathbf{v}\times\mathbf{B})^2 - 2\mathbf{v}\cdot\mathbf{E}\times\mathbf{B} - (\mathbf{v}\cdot\mathbf{E})^2], \tag{3.31}$$

$$\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = -\tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2). \tag{3.32}$$

Let us now consider *electrostatic systems* which are characterized by the fact that in their rest frame $\mathbf{B}_{(0)} = 0$ everywhere. When such a system is in motion, a magnetic induction is present which is determined by the velocity $\mathbf{v}$ relative to the rest system and by the electric field at that point,[5]

$$\mathbf{B} = \mathbf{v}\times\mathbf{E}. \tag{3.33}$$

In such a system the invariants (3.31) and (3.32) simplify considerably; one finds easily

$$f_\alpha f^\alpha = \mathbf{E}^2 - \mathbf{v}^2\mathbf{E}^2 - (\mathbf{v}\cdot\mathbf{E})^2 = \mathbf{E}^2 - (\mathbf{v}\times\mathbf{E})^2, \quad (3.34)$$

and therefore,

$$f_\alpha f^\alpha = -\tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}. \quad (3.35)$$

The last equality is most easily demonstrated directly by noting that in the rest system S_0 both sides equal $\mathbf{E}_{(0)}^2$. If the two invariants are equal in one coordinate frame they are equal in all Lorentz frames.

It follows that for the electrostatic case the electromagnetic mass of the system is simply [compare (3.27), (3.35), and (3.32)]

$$m = (4\pi)^{-1}\int (-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta})d^3\sigma$$
$$= (8\pi)^{-1}\int (\mathbf{E}^2 - \mathbf{B}^2)d^3\sigma. \quad (3.36)$$

The electromagnetic momentum four-vector is, therefore, from (3.15),

$$P^\mu = -(16\pi)^{-1}\int F_{\alpha\beta}F^{\alpha\beta}v^\mu d^3\sigma \quad (3.37)$$

or

$$P^\mu = \int [(\mathbf{E}^2 - \mathbf{B}^2)/8\pi]v^\mu d^3\sigma. \quad (3.38)$$

If we use the components of v^μ we can write (3.38) as

$$P^0 = \gamma \int [(\mathbf{E}^2 - \mathbf{B}^2)/8\pi]d^3\sigma = \gamma m, \quad (3.39)$$

$$\mathbf{P} = \gamma\mathbf{v}\int [(\mathbf{E}^2 - \mathbf{B}^2)/8\pi]d^3\sigma = \gamma\mathbf{v}m. \quad (3.40)$$

It is important to emphasize that all equations from (3.33) on are valid only for *electrostatic systems*, i.e., are special cases of the general results which we discussed previously. In particular, the electromagnetic energy and momentum (3.39) and (3.40) are special cases of the generally valid equations (3.23) and (3.24). The restricted validity is due to the use of (3.33) which does not hold in general.

4. Systems with $m=0$

There are electromagnetic systems for which the total electromagnetic four-momentum is a null vector, i.e.,

$$P_\mu P^\mu = 0. \quad (4.1)$$

These systems still have positive energy

$$P^0 > 0, \quad (4.2)$$

but they have no rest mass. The mass can be defined by $P_\mu P^\mu$ as in (3.1) in which case (4.1) is the special case $m=0$. Or, it can be defined by the energy tensor integral, i.e., the mass tensor M, (3.10) and (3.12), which read for the electromagnetic case

$$m = -v_\mu v_\nu \int \Theta^{\mu\nu}(x)d^3\sigma. \quad (4.3)$$

Both definitions yield the same result if $m>0$. But when $m=0$ there is no rest frame and therefore, no timelike velocity four-vector v^μ and consequently (4.3) has no meaning.

In order to explain this point let us recall that the rest frame was defined by (3.13) provided there exists a frame in which that equation is satisfied. But if the integrated Poynting vector never vanishes, i.e., if

$$M^{k0} \equiv \int \Theta^{k0}d^3\sigma = \int (\mathbf{S}(x))_k d^3\sigma \neq 0 \quad (4.4)$$

in *every* Lorentz frame, no rest frame exists. In that case v^μ is clearly undefined.

The momentum four-vector can of course still be expressed in terms of an integral of the energy tensor Θ over a spacelike plane,

$$P^\mu = \int \Theta^{\mu\nu}(x)d^3\sigma_\nu. \quad (4.5)$$

But the normal vector n^μ of $d^3\sigma^\mu = n^\mu d^3\sigma$ is no longer related to a rest frame by (3.8). The situation here is rather different. We are now dealing with a *closed* system since there are no sources in the system. It consists entirely of electromagnetic fields. Thus,

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad (4.6)$$

as in (2.1). In that case the integral (4.5) is independent of the choice of the spacelike plane (see Sec. 6), and we can take $n^\mu = (1, \mathbf{0})$,

$$P^\mu = \int \Theta^{\mu 0}n_0 d^3\sigma = -\int \Theta^{\mu 0}dV_{(0)}.$$

In three-vector notation this means [see (3.18)]

$$P^0 = \int U dV_{(0)} = (8\pi)^{-1}\int (\mathbf{E}^2 + \mathbf{B}^2)dV_{(0)}, \quad (4.7)$$

$$\mathbf{P} = \int \mathbf{S}dV_{(0)} = (4\pi)^{-1}\int (\mathbf{E}\times\mathbf{B})dV_{(0)}. \quad (4.8)$$

These equations are generally valid for systems with $m=0$ and for all frames S. They look like (3.23) and (3.24) for $m>0$ in S_0, but these should not be confused.

Systems with $m=0$ consist of electromagnetic radiation with no sources present; the total momentum is given by the net flux of radiation. A wave train of given direction and finite cross section is a realistic physical example.

5. HISTORICAL REMARKS

In the physics literature, one often finds the expressions (4.7) and (4.8) for the electromagnetic energy and momentum of zero mass systems applied to systems with positive mass. The correct expressions for the latter case, Eqs. (3.23) and (3.24), are usually not used. This situation has a historical origin which is not generally known.

By far the most influential physicists concerned with the classical theory of electrons around the turn of the century were Lorentz and Abraham. This influence was enhanced by the fact that the latter's book on electricity and magnetism, later revised by Becker and edited under the names of both authors, became the classic text on electromagnetic theory in German-speaking countries and around the world. And it remained so for a third of this century.

For nonrelativistic motion, (3.23) and (4.7) are identical, so that no error is introduced in the energy when the $m=0$ expression is used for the $m>0$ case. But the expressions for $\mathbf{P}$, (3.24) and (4.8) differ even in the nonrelativistic limit.

The incorrect use of (4.8) for $\mathbf{P}$ in $m>0$ systems can be traced directly to Lorentz. In 1906, Lorentz delivered a series of lectures at Columbia University which were later published and which became very well known. On p. 32 of the cited edition one reads (we use our symbols in the cited equations)[7]:

> ... our discussion shows the importance of the vector $\mathbf{S}dV_{(0)}$ which has a definite direction and magnitude for every element of volume, and of the vector $\mathbf{P}=\int \mathbf{S}dV_{(0)}$ that may be derived from it by integration. Abraham [Ref. 8] of Göttingen has applied to these quantities the name of electromagnetic momentum. We may term them so, even if we do not wish to convey the idea that they represent a real momentum ....

In his lectures, Lorentz then proceeds to compute the energy and momentum of the electromagnetic field surrounding a moving charge using the above expression and is led to a result inconsistent with translation invariance. For example, in the nonrelativistic limit $\mathbf{P}=\frac{4}{3}m\mathbf{v}$, where $m=E/c^2$.

It appears, therefore, that Abraham first suggested the use of the Poynting vector as electromagnetic momentum density *in general*, after it was found to be the correct expression for radiation. This uncritical application of (4.8) to other than $m=0$ systems such as to the Coulomb field of a charged particle was taken over by Lorentz. It thus received undeserved endorsement from the highest authority in the field. As far as one can tell, the resultant inconsistencies were never blamed on Eq. (4.8) as inapplicable to Coulomb fields.

6. DISCUSSION

This section is conveniently divided into four separate parts referring to rather unconnected aspects of the preceding exposition.

(a) By a generalization of Gauss' integral theorem to four-dimensional Minkowski space one can prove the following theorem: The integral in (3.4) will be independent of the choice of the surface normal n^μ of $d^3\sigma^\mu$ if and only if the integrand $T^{\mu\nu}$ is divergence free [i.e., (2.1) holds].[9] In Sec. 4, where the system with $m=0$ is closed, this theorem applies so that n^μ on (4.5) can be chosen arbitrarily (as long as it is a timelike unit vector). On the other hand, in Sec. 3, where the electromagnetic field is *not* a closed system by itself so that (4.6) does not hold, the integral representing P^μ, Eq. (3.16) is not independent of the surface normal. Rather, the normal is fixed to the motion of the system by (3.8). This means that one integrates always over the hyperplane which is the transformed three space of the rest frame. Only in this way can this open system yield a meaningful momentum four-vector.

It is clear that the above theorem establishes a relation between the differential conservation laws [such as (4.6)] and the integral conservation laws $P^\mu=\text{const}$.

One way of stating the error made in the older literature, as given in Sec. 5, is to say that they treated the Coulomb field as if it were a closed system; they computed the momentum integral as if it were independent of the choice of the surface. Since the Coulomb field by itself is open, this is not so.

(b) In Eq. (3.14) an eigenvalue problem for $M^{\mu\nu}$ is indicated. This is a somewhat different problem from the eigenvalue problem of the energy (density) tensor $\Theta^{\mu\nu}$. The latter is more difficult.[10] It leads to the result

$$m=(16\pi)^{-1}\int\left[(F_{\alpha\beta}F^{\alpha\beta})^2+(\tfrac{1}{2}F^{\alpha\beta}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta})^2\right]^{1/2}d^3\sigma, \quad (6.1)$$

where $\epsilon_{\alpha\beta\gamma\delta}=+1,\ -1,\ 0$ depending on whether $\alpha\beta\gamma\delta$ is an even or odd permutation of 0123, or neither. The first invariant under the square root is known from (3.32). The second one is

$$\tfrac{1}{8}F^{\alpha\beta}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}=\mathbf{E}\cdot\mathbf{B}. \quad (6.2)$$

For electrostatic systems, this invariant vanishes because it vanishes in S_0. Then (6.1) reduces to (3.36). In the general case, the identity of (6.1) and (3.27) is not obvious since it holds only for the integral but not for the integrand.

(c) The considerations of Sec. 3 can be extended to dielectric media. If $\epsilon\neq 1$, $\mu\neq 1$, we shall have **D** and **H** in addition to **E** and **B**. These two three vectors form an antisymmetric tensor $H^{\mu\nu}$ analogous to $F^{\mu\nu}$. The electromagnetic energy tensor is then

$$\Theta^{\mu\nu}=(4\pi)^{-1}(F^{\mu\alpha}H_\alpha{}^\nu+\tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}H^{\alpha\beta}), \quad (6.3)$$

and the results of Sec. 3, especially (3.27) and (3.37), generalize in an obvious way.

(d) Two recent papers by Butler refer to the subject matter of the present work.[1] In B1 he claims to derive "... a new equation for the energy density dU/dV ... of a classical macroscopic charged body ...". But his equation is our (3.39) and is not new since it is a special case of (3.23) which is contained in Ref. 2 and in other places. In fact, Ref. 2 is never mentioned by this author.

In B1 we read "we now show that Rohrlich's equation ... can be simplified ...". But the author then proceeds to derive the electrostatic case from the general case, thus making a specialization rather than a simplification. But the Trouton–Noble experiment is certainly a good example of the usefulness of (3.39), and the author must be given credit for treating correctly an often confused subject matter.

The results for the momentum four-vector obtained in B1 and B2 are not new[2] and are just (3.39) and (3.40). However, the author prefers to express the invariant $d^3\sigma=dV_{(0)}$ by the Lorentz contracted volume $dV=dV_{(0)}/\gamma$, so that $d^3\sigma$ in these equations is replaced by γdV in B1 and B2. The results are, however, not general and pertain only to electrostatic systems; the *general* expressions for the electromagnetic energy and momentum were given earlier[2] and are rederived here, Eqs. (3.23) and (3.24). Furthermore, B1 and B2 do not explain why the $m=0$ expressions are not applicable to $m>0$ systems.

The history of the confusion in the literature on this topic was sketched some time ago[11] and has now been elaborated in more detail in Sec. 5.

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[1] J. W. Butler, Amer. J. Phys. **36**, 936 (1968) and **37**, 1258 (1969). These will be referred to as B1 and B2.

[2] F. Rohrlich, *Classical Charged Particles* (Addison–Wesley, Reading, Mass., 1965).

[3] The reader who is not quite sure about the covariant notation used here is referred to Ref. 2 above (Appendix 1) or any text on special relativity. In order to avoid cluttering the equations with irrelevant constants we shall measure time t by the distance a light beam would travel during t *in vacuo*. This means that effectively we put the velocity of light $c=1$.

[4] Recent speculations (at least so far unsupported by experimental evidence) concerning the existence of particles which move faster than light (tachyons) assume the possibility of purely imaginary mass, $m=i\alpha$, where α is a real number. We shall *not* consider this possibility.

[5] See, for example, Ref. 2 or J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962) who uses the same units but a different metric, or W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison–Wesley, Reading, Mass., 1962) who use different units and a different metric from ours.

[6] See for example Ref. 2, Eqs. (4-125) and (4-126).

[7] H. A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat* (Dover, New York, 1952), 2nd ed.

[8] M. Abraham, Ann. Physik **10**, 105 (1903).

[9] See Ref. 2, Appendix, Sec. A1-5.

[10] J. L. Synge, *The Special Theory of Relativity* (North-Holland, Amsterdam, 1956 and Interscience, New York, 1965).

[11] Reference 2, Chap. 2, especially pp. 13 and 17.