The flow of electromagnetic energy in the decay of an electric dipole

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In a typical dipole decay, the induction magnetic field and the radiation magnetic field are in opposing directions. The former is dominant close to the decay, and the latter is dominant far away. Thus, there exists a surface on which these two components cancel each other, so that the magnetic field reverses direction. Because the magnetic field changes direction, so also will the Poynting vector field, provided the electric field is reasonably well behaved. This defines a "causal surface," within which electromagnetic energy will collapse, and outside which the electromagnetic energy will radiate away. I present a simple example, the decay of a point electric dipole, to argue that the radiated energy comes, not necessarily from the accelerating charges themselves, but from the energy stored in the far field. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

More than a century ago, Maxwell argued that the energy density of an electromagnetic field is proportional to the squares of the magnitudes of the fields.[1–3] In modern parlance, the energy density he advocated is

$$u=u_E+u_B=\frac{1}{2}\,\epsilon_0|\mathbf{E}|^2+\frac{1}{2\mu_0}\,|\mathbf{B}|^2. \tag{1}$$

Some years later, Poynting put forward his theory of how the electromagnetic field conveys energy.[4] He argued that the power density or time rate of energy flow at a point is (again in more modern notation[5])

$$\mathbf{S}=\frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}. \tag{2}$$

Poynting showed that his formulation leads to the correct expression for Joule heating in a resistor,[6] and he successfully applied his idea to electromagnetic radiation.[7] Poynting's result was simultaneously discovered by Heaviside.[8]

Much has been written on the subject of alternate electromagnetic energy flow vectors, both in this Journal[9–14] and elsewhere.[15,16] Although of great interest, such concerns go beyond the scope of this paper.

In discussing his theory of energy flow, Poynting observed, "We ought to look for proofs at points where the energy is transformed into other modifications... ."[17] The aim of this present paper is to focus on the consequences of Poynting's original formulation when static electromagnetic energy transforms into radiant energy. This is exactly what happens in an electric decay process, e.g., when an electric dipole decays.

The radiation from an electric dipole has long been a standard topic,[18,19] and most intermediate electricity and magnetism texts discuss it. These discussions are often complemented by an analysis of the field energy.[20] The model of a decaying dipole has been used to study the radiation produced by lightning,[21] and the radiation from a current pulse has been reported on in this Journal.[22,23] Further, the power radiated by lightning is a subject of current interest.[24] Surprisingly however, the present approach of comparing the energy radiated in a dipole decay to the energy distribution in the original static field apparently is not addressed in the literature.

Some very interesting ideas may be easily understood by considering a formulation of the Biot–Savart and Coulomb laws due to Jefimenko.[25] I will begin with a discussion of Jefimenko's form of the Biot–Savart and Coulomb laws. Then, I will explain how his version of the Biot–Savart law can be used to define a surface around a decaying dipole through which there is no net energy flow. I interpret this as a causal surface separating the in-flowing energy that drives the decay from the out-flowing energy that is radiated away. Finally, I will apply the concept of a causal surface to the decay of an electric dipole.

II. THE JEFIMENKO FORM OF THE BIOT–SAVART AND COULOMB LAWS

Jefimenko has noted that the Biot–Savart law and the Coulomb law may be generalized to depend not only on the charge and current densities but also on their time derivatives.[26] His approach has been the subject of recent discussion in this Journal.[27–29] Here, I closely follow the particularly clear and succinct presentation of Griffiths and Heald.[30]

Consider the usual formula for the retarded magnetic vector potential:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{x}\,dV, \tag{3}$$

where $x\equiv|\mathbf{r}-\mathbf{r}'|$, the integral is over the volume V within which the current distribution is located, and the square brackets indicate that the current density is to be evaluated at the retarded time:

$$t_r=t-x/c. \tag{4}$$

The curl of the magnetic vector potential yields the magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\left[\frac{1}{x}\,\nabla\times[\mathbf{J}]-[\mathbf{J}]\times\nabla\left(\frac{1}{x}\right)\right]dV. \tag{5}$$

The curl of the current density depends on the field point through the retarded time, so

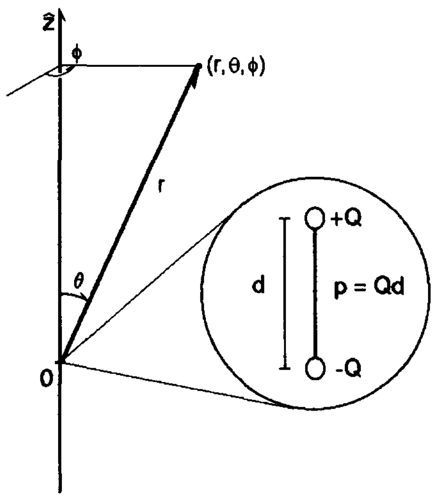

Fig. 1. The geometry of a point dipole.

$$\nabla\times[\mathbf{J}]=-\left[\frac{\partial\mathbf{J}}{\partial t}\right]\times\nabla(t_r)=\frac{1}{c}\left[\frac{\partial\mathbf{J}}{\partial t}\right]\times\hat{\mathbf{x}}. \tag{6}$$

Thus,

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\left(\frac{[\mathbf{J}]\times\hat{\mathbf{x}}}{x^2}+\frac{1}{xc}\left[\frac{\partial\mathbf{J}}{\partial t}\right]\times\hat{\mathbf{x}}\right)dV. \tag{7}$$

Similarly, one may derive an expression for the electric field using the retarded scalar potential:

$$V=\frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{x}\,dV. \tag{8}$$

The electric field is

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}=-\frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{x}\nabla[\rho]+[\rho]\nabla\left(\frac{1}{x}\right)\right]dV-\frac{\mu_0}{4\pi}\int\frac{1}{x}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dV. \tag{9}$$

The gradient of the charge density may be reexpressed:

$$\nabla[\rho]=\left[\frac{\partial\rho}{\partial t}\right]\nabla t_r=-\frac{1}{c}\left[\frac{\partial\rho}{\partial t}\right]\hat{\mathbf{x}}, \tag{10}$$

so

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\left(\frac{[\rho]}{x^2}\hat{\mathbf{x}}+\frac{1}{xc}\left[\frac{\partial\rho}{\partial t}\right]\hat{\mathbf{x}}-\frac{1}{xc^2}\left[\frac{\partial\mathbf{J}}{\partial t}\right]\right)dV. \tag{11}$$

III. THE DIPOLE FIELDS

Consider an electric dipole as in Fig. 1. One can imagine two equal and opposite charges with magnitude Q aligned with the $\hat{\mathbf{z}}$ axis and separated by a distance d. In the limit as the charges are increased and the distance simultaneously decreased so as to hold their product constant, one obtains a point electric dipole with moment

$$\mathbf{p}=(Qd)\hat{\mathbf{z}}=p\hat{\mathbf{z}}. \tag{12}$$

I will assume that the dipole decays so that the magnitude of the dipole moment is

$$p=fp_0, \tag{13}$$

where f is some function of time. Further, I will assume that the orientation of the dipole is fixed along the $\hat{\mathbf{z}}$ axis for all time.

One must exercise caution in using the dipole model, because in the limit as the two charges become arbitrarily large and arbitrarily close, the fields become singular at the origin. The dipole field is a valid approximation provided the observation distance is much further away than the characteristic size of the dipole, and provided the fields are changing with a time scale τ such that $c\,\tau$ is much greater than the characteristic size of the dipole.

There are a variety of techniques by which the fields about an electric dipole may be determined.[31,32] One way is to take advantage of Jefimenko's formulation, developed in the previous section.[33] To apply his equations, one must replace $\mathbf{J}\,dV$ by $I\,d\mathbf{l}=I\hat{\mathbf{z}}\,dz$, where

$$I=\frac{\dot{p}}{d} \tag{14}$$

and

$$\dot{I}=\frac{\ddot{p}}{d} \tag{15}$$

for $-\frac{1}{2}d\leqslant z\leqslant\frac{1}{2}d$. Solving for the magnetic field:

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_{-d/2}^{+d/2}\left(\frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{x}}}{x^2 d}+\frac{1}{xcd}[\ddot{\mathbf{p}}]\times\hat{\mathbf{x}}\right)dz \tag{16}$$

and so

$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{x}}}{x^2}+\frac{1}{xc}[\ddot{\mathbf{p}}]\times\hat{\mathbf{x}}\right). \tag{17}$$

(Note that $\mathbf{x}=\mathbf{r}-\mathbf{r}'$, and thus the unit vector $\hat{\mathbf{x}}$ is in the direction of $\mathbf{r}-\mathbf{r}'$, not along the Cartesian $\hat{\mathbf{x}}$ axis.) In the spherical coordinates of Fig. 1, and using the time dependent dipole moment of Eq. (13) this becomes

$$\mathbf{B}=\frac{\mu_0 p_0}{4\pi r}\left(\frac{[\dot{f}]}{r}+\frac{[\ddot{f}]}{c}\right)\sin\theta\hat{\boldsymbol{\phi}}. \tag{18}$$

Perhaps the easiest way to obtain the electric field is to take the curl of the magnetic field and integrate over time. For a current free region in a medium of negligible dielectric constant, Maxwell's equation

$$\nabla\times\mathbf{B}=\epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} \tag{19}$$

holds. Solving

Table I. The components of the Poynting vector field for an electric dipole.

$\mathbf{S}_{m,n}=\frac{1}{\mu_0}(\mathbf{E}_m\times\mathbf{B}_n)$	$\mathbf{B}_{\rm ind}=\frac{\mu_0 p_0[\dot f]}{4\pi r^2}\sin\theta\,\hat{\boldsymbol\phi}$	$\mathbf{B}_{\rm rad}=\frac{\mu_0 p_0[\ddot f]}{4\pi rc}\sin\theta\,\hat{\boldsymbol\phi}$
$\mathbf{E}_{\rm stat}=\frac{kp_0}{r^2}\frac{[f]}{r}(2\cos\theta\hat{\mathbf r}+\sin\theta\hat{\boldsymbol\theta})$	$\frac{kp_0^2[f][\dot f]}{4\pi r^5}(\sin^2\theta\hat{\mathbf r}-\sin 2\theta\hat{\boldsymbol\theta})$	$\frac{kp_0^2[f][\ddot f]}{4\pi r^4 c}(\sin^2\theta\hat{\mathbf r}-\sin 2\theta\hat{\boldsymbol\theta})$
$\mathbf{E}_{\rm ind}=\frac{kp_0[\dot f]}{r^2 c}(2\cos\theta\hat{\mathbf r}+\sin\theta\hat{\boldsymbol\theta})$	$\frac{kp_0^2[\dot f]^2}{4\pi r^4 c}(\sin^2\theta\hat{\mathbf r}-\sin 2\theta\hat{\boldsymbol\theta})$	$\frac{kp_0^2[\dot f][\ddot f]}{4\pi r^3 c^2}(\sin^2\theta\hat{\mathbf r}-\sin 2\theta\hat{\boldsymbol\theta})$
$\mathbf{E}_{\rm rad}=\frac{kp_0[\ddot f]\sin\theta}{c^2 r}\hat{\boldsymbol\theta}$	$\frac{kp_0^2[\dot f][\ddot f]}{4\pi r^3 c^2}\sin^2\theta\hat{\mathbf r}$	$\frac{kp_0^2[\ddot f]^2}{4\pi r^2 c^3}\sin^2\theta\hat{\mathbf r}$

$$
\begin{aligned}
\mathbf{E}&=\frac{1}{\epsilon_0\mu_0}\int\nabla\times\mathbf{B}\,dt=\frac{1}{\epsilon_0\mu_0}\int\left\{\frac{\hat{\mathbf r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta B_\phi)\right]+\frac{\hat{\boldsymbol\theta}}{r}\left[-\frac{\partial}{\partial r}(rB_\phi)\right]\right\}dt\\
&=\frac{p_0}{4\pi\epsilon_0}\int\left(\begin{array}{c}\frac{\hat{\mathbf r}}{r\sin\theta}\left(\frac{[\dot f]}{r^2}+\frac{[\ddot f]}{cr}\right)\left[\frac{\partial}{\partial\theta}(\sin^2\theta)\right]\\+\frac{\sin\theta\hat{\boldsymbol\theta}}{r}\left[-\frac{\partial}{\partial r}\left(\frac{[\dot f]}{r}+\frac{[\ddot f]}{c}\right)\right]\end{array}\right)dt\\
&=\frac{p_0}{4\pi\epsilon_0}\int\left(\begin{array}{c}\frac{\hat{\mathbf r}}{r\sin\theta}\left(\frac{[\dot f]}{r^2}+\frac{[\ddot f]}{cr}\right)(2\sin\theta\cos\theta)\\+\frac{\sin\theta\hat{\boldsymbol\theta}}{r}\left(\frac{[\dot f]}{r^2}+\frac{[\ddot f]}{cr}+\frac{[\dddot f]}{c^2}\right)\end{array}\right)dt\\
&=\frac{p_0}{4\pi\epsilon_0}\int\left[\left(\frac{[\dot f]}{r^3}+\frac{[\ddot f]}{cr^2}\right)2\cos\theta\hat{\mathbf r}+\left(\frac{[\dot f]}{r^3}+\frac{[\ddot f]}{cr^2}+\frac{[\dddot f]}{c^2 r}\right)\sin\theta\hat{\boldsymbol\theta}\right]dt
\end{aligned}
\tag{20}
$$

and so

$$
\mathbf{E}=\frac{kp_0}{r^2}\left(\frac{[f]}{r}+\frac{[\dot f]}{c}\right)(2\cos\theta\hat{\mathbf r}+\sin\theta\hat{\boldsymbol\theta})+\frac{kp_0[\ddot f]\sin\theta}{c^2 r}\hat{\boldsymbol\theta},
\tag{21}
$$

where

$$
k=\frac{1}{4\pi\epsilon_0}.
\tag{22}
$$

The term proportional to $[f]$ is the electrostatic component of the field ($\mathbf{E}_{\rm stat}$), the $[\dot f]$ term is the induction component ($\mathbf{E}_{\rm ind}$), and the $[\ddot f]$ term is the radiation term ($\mathbf{E}_{\rm rad}$). Similarly, the $[\dot f]$ term of the magnetic field is the magnetic induction field ($\mathbf{B}_{\rm ind}$), and the $[\ddot f]$ term is the magnetic radiation field ($\mathbf{B}_{\rm rad}$). A summary of these field components is also included in Table I.

IV. THE CAUSAL SURFACE

In a standard decay, the rate at which the charge redistributes itself will be proportional to the potential difference between the opposite charges, and thus, to the quantity of charge remaining:

$$
\dot{\mathbf p}\propto\mathbf{p}.
\tag{23}
$$

The expected solution is an exponential decay:

$$
\mathbf{p}=\mathbf{p}_0 e^{-t/\tau},
\tag{24}
$$

where the time constant (τ) is related to the capacitance and to the resistance of the path along which the system decays:

$$
\tau=RC.
\tag{25}
$$

The decaying dipole yields a point current proportional to the time rate of change of the electric dipole moment:

$$
\mathbf{I}\propto\dot{\mathbf p}=-\frac{\mathbf{p}_0}{\tau}e^{-t/\tau}.
\tag{26}
$$

The time rate of change of the current is in turn proportional to the second derivative of the dipole moment:

$$
\dot{\mathbf I}\propto\ddot{\mathbf p}=\frac{\mathbf{p}_0}{\tau^2}e^{-t/\tau}.
\tag{27}
$$

The crucial observation is that the displacement current is directed *opposite* to the direction of the current flow. Since both the current and the time rate of change of the current are sources of the magnetic field, it is possible to define a surface on which the magnetic field is zero. Examining Eq. (17), this surface will be the solution to

$$
\frac{[\dot{\mathbf p}]\times\hat{\mathbf x}}{x^2}+\frac{1}{xc}[\ddot{\mathbf p}]\times\hat{\mathbf x}=0.
\tag{28}
$$

If the dipole moment is of the form given by Eq. (24), then

$$
[\ddot{\mathbf p}]=-\frac{1}{\tau}[\dot{\mathbf p}],
\tag{29}
$$

and so Eq. (28) is equivalent to

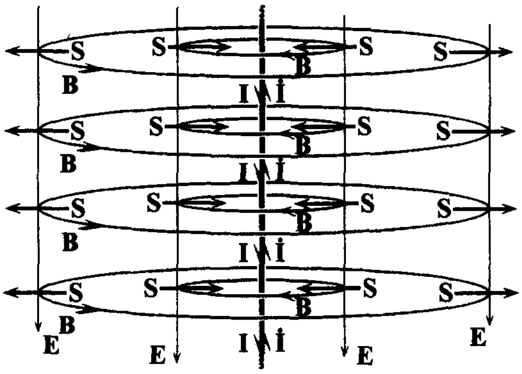

Fig. 2. A qualitative sketch of the electric, magnetic, and Poynting vector fields about an exponentially decaying current.

$$\left(\frac{1}{x^2}-\frac{1}{xc\tau}\right)([\hat{\mathbf{p}}]\times\hat{\mathbf{x}})=0. \tag{30}$$

The magnetic field is thus zero if

$$x=c\tau. \tag{31}$$

For a point decay current, the surface is a sphere[34] with radius

$$r_{\text{causal}}=c\tau. \tag{32}$$

The surface on which $|\mathbf{B}|\to 0$ defines what I call a causal surface around the electric dipole. To fix ideas, consider the behavior of the magnetic field about an exponentially decaying current. Suppose the current is directed down as shown in Fig. 2. Close to the wire, the current (or induction) term dominates and the magnetic field is clockwise (if viewed from above). At some further distance, the term due to the changing current (the radiation term) dominates and the magnetic field has the opposite orientation. Incidentally, the radiation term may also be thought of as that portion of the magnetic field attributable to the displacement current.

As the decay current flows, the electric field will be decreasing. To aid in visualizing the qualitative behavior of the fields and energy flow, assume that this decreasing electric field is directed uniformly down (an oversimplification about which I will have more to say presently). Then, close to the decay current, the Poynting vectors are directed radially inward. The electromagnetic field energy in this near region is collapsing into the decay current, expended in accelerating the charges and then to Joule heat in the wire.[35] At some further distance, the Poynting vectors are directed radially outward. The electromagnetic field energy originally in this far region is radiating away.

The causal surface separates the two regions. It partitions the electromagnetic field energy of the original static field into a portion that is absorbed by the decay current, and a portion that is radiated. Only that portion of the field energy within the causal surface can exert a causal influence on the decay current.

As noted, the assumption that the electric field is uniformly down is something of an oversimplification. When calculating the magnetic fields due to displacement currents one must avoid approximations that are incompatible with the effects being studied. For instance, French and Tessman observed how neglecting the fringing fields about an internally shorted capacitor leads to the erroneous conclusion that the displacement current "cancels out" the conduction current yielding no net magnetic field.[36] As these authors demonstrated, in many situations the magnetic field produced by the displacement current is negligible.

V. DECAY OF AN ELECTRIC DIPOLE

To further understand the relative dominance of the magnetic field components that gives rise to the causal surface, one must apply the idea to a more realistic example, such as the decay of an electric dipole.

First, I will calculate the energy radiated in the exponential decay of an electric dipole. I will show that, for the case of a simplified exponential decay (one that turns on instantaneously at time zero), the radiated energy is identically equal to the energy of that portion of the original static electric field outside the causal surface. I will also consider the implications of a finite rise time.

A. The energy radiated by an electric dipole decay

The power density will be given by the Poynting vector field [Eq. (2)]. The net Poynting vector field follows from summing the six components presented in Table I:

$$\mathbf{S}=S_r\hat{\mathbf{r}}+S_\theta\hat{\theta}, \tag{33}$$

where

$$S_r=\frac{kp_0^2\sin^2\theta}{4\pi r^2}\left(\frac{[f][\dot{f}]}{r^3}+\frac{1}{r^2c}([f][\ddot{f}]+[\dot{f}]^2)+2\,\frac{[\dot{f}][\ddot{f}]}{rc^2}+\frac{[\ddot{f}]^2}{c^3}\right) \tag{34}$$

and

$$S_\theta=-\frac{kp_0^2\sin 2\theta}{4\pi r^2}\left(\frac{[f][\dot{f}]}{r^3}+\frac{1}{r^2c}([f][\ddot{f}]+[\dot{f}]^2)+\frac{[\dot{f}][\ddot{f}]}{rc^2}\right). \tag{35}$$

In the far field, the dominant field terms are

$$\mathbf{E}_{\text{rad}}=\frac{kp_0[\ddot{f}]\sin\theta}{c^2r}\,\hat{\theta} \tag{36}$$

and

$$\mathbf{B}_{\text{rad}}=\frac{\mu_0p_0\sin\theta[\ddot{f}]}{4\pi rc}\,\hat{\phi}. \tag{37}$$

The radiated power density is

$$\mathbf{S}_{\text{rad}}=\frac{1}{\mu_0}(\mathbf{E}_{\text{rad}}\times\mathbf{B}_{\text{rad}})=\frac{kp_0^2\sin^2\theta[\ddot{f}]^2}{4\pi r^2c^3}\,\hat{\mathbf{r}}. \tag{38}$$

To obtain the total radiated power, one must integrate the Poynting vector field over the surface of a sphere centered at the dipole and with a radius R sufficiently large that the far field approximations may be used. Then by integrating with respect to the retarded time for all times, one obtains the radiated energy:[37]

$$U_{\rm rad}=\int_{-\infty}^{\infty}\left(\oint \mathbf{S}\cdot\hat{\mathbf{n}}d\sigma\right)dt_r$$

$$=\int_{-\infty}^{\infty}\left(\int_0^{2\pi}\int_0^{\pi}\frac{kp_0^2[\ddot{f}]^2\sin^2\theta}{4\pi R^2c^3}\sin\theta R^2\,d\theta\,d\phi\right)dt_r$$

$$=\frac{kp_0^2}{2c^3}\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]_0^{\pi}\int_{-\infty}^{\infty}[\ddot{f}]^2dt_r$$

$$=\frac{2kp_0^2}{3c^3}\int_{-\infty}^{\infty}[\ddot{f}]^2dt_r=\frac{2kp_0^2}{3c^3}\int_{-\infty}^{\infty}\ddot{f}^2\,dt. \qquad (39)$$

The physical time and retarded time differ by the same constant factor at every point on the surface over which the integral extends, so due to the infinite limits of integration, one may freely translate between these integration variables.

If the dipole decays exponentially starting at time $t=0$, then

$$f=f_1=\begin{cases}1, & t\leqslant 0\\ \exp\left(-\frac{t}{\tau}\right), & t\geqslant 0\end{cases}, \qquad (40)$$

and the Poynting vector field for the dipole is

$$\mathbf{S}=\frac{kp_0^2e^{-t/\tau}}{4\pi c^3\tau^4r^5}\begin{pmatrix}(r-c\tau)(r^2-c\tau r+c^2\tau^2)\sin^2\theta\hat{\mathbf{r}}\\ +c\tau(r-c\tau)^2\sin 2\theta\hat{\boldsymbol{\theta}}\end{pmatrix} \qquad (41)$$

for $t>0$. Of course, a real dipole cannot *instantaneously* begin an exponential decay as is implied by Eq. (40), but an analysis of the implications of a finite rise time will be postponed to a later section.

B. The energy in the static field of an electric dipole

The original static electric field is

$$\mathbf{E}=\frac{kp_0}{r^3}(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}). \qquad (42)$$

The electric field energy density is

$$u_E=\frac{1}{2}\epsilon_0|\mathbf{E}|^2=\frac{\epsilon_0}{2}\left(\frac{kp_0}{r^3}\right)^2(3\cos^2\theta+1). \qquad (43)$$

The causal surface is obtained by solving for where the magnetic field goes to zero:

$$r_{\rm causal}=-c\frac{\dot{f}}{\ddot{f}}. \qquad (44)$$

If the dipole decays exponentially with time dependence f_1 [Eq. (40)], then

$$r_{\rm causal}=c\tau \quad (t>0). \qquad (45)$$

Note that the causal surface is constant at positive times for this exponential decay. The energy stored in the electric field outside of the causal surface is

$$U_{\rm outside}=\int_{c\tau}^{\infty}\int_0^{\pi}u_E2\pi\sin\theta r^2\,d\theta\,dr$$

$$=\pi\epsilon_0\int_{c\tau}^{\infty}\left(\frac{kp_0}{r^2}\right)^2\int_0^{\pi}(3\cos^2\theta+1)\sin\theta\,d\theta\,dr$$

$$=\pi\epsilon_0\int_{c\tau}^{\infty}\left(\frac{kp_0}{r^2}\right)^2[-\cos^3\theta-\cos\theta]_0^{\pi}dr$$

$$=4\pi\epsilon_0\int_{c\tau}^{\infty}\left(\frac{kp_0}{r^2}\right)^2dr=\frac{1}{3}\frac{kp_0^2}{c^3\tau^3}. \qquad (46)$$

Compare this to the energy radiated by the exponentially decaying dipole. Evaluating Eq. (39) for the exponential decay described by f_1 [Eq. (40)] yields

$$U_{\rm rad}=\frac{2kp_0^2}{3c^3}\int_0^{\infty}\ddot{f}^2\,dt=\frac{2kp_0^2}{3c^3}\int_0^{\infty}\left(\frac{1}{\tau^4}e^{-2t/\tau}\right)dt$$

$$=\frac{2kp_0^2}{3c^3}\left[-\frac{1}{2\tau^3}e^{-2t/\tau}\right]_0^{\infty}=\frac{1}{3}\frac{kp_0^2}{c^3\tau^3}. \qquad (47)$$

The energy radiated by an exponentially decaying dipole is thus identically equal to the static field energy stored outside the causal surface.

To make sure that the radial component of the Poynting vector changes sign at $x=c\tau$, one must make certain that $\mathbf{E}_\theta$ does not change sign, but this is obvious from Eq. (21). Thus, the electric field is sufficiently well behaved so as not to affect the conclusions regarding the causal surface that were derived from the reversal of the magnetic field.

C. The effect of a finite rise time

The exponential decay time dependence f_1 [Eq. (40)] makes an interesting pedagogical example, but it is not truly physical. The preceding analysis has neglected the fact that $\dot{f}$ is discontinuous at time $t=0$, and so $\ddot{f}$ becomes large without limit. In other words, an "infinite" amount of energy will be radiated. This may be qualitatively understood by realizing that the causal surface is undefined at this time, so we may suppose that this infinite radiated energy comes from the infinite energy of the singular fields at the origin. Nevertheless, these infinities make a simple quantitative comparison untenable.

In this section, I will discuss what conclusions may be drawn in the case of a finite rise time, and explore two alternate dipole decay functions.

One way to introduce a finite rise time is to choose the following time dependence for the current:

$$\dot{f}=\dot{f}_2=\frac{1}{\exp(t/\tau)+\exp(-t/\sigma)}. \qquad (48)$$

This function describes a rise over a time scale of order σ, followed by an exponential decay with time constant τ. This might be applicable to the decay of a dipole whose size is of order $c\sigma$. The rise time will be chosen so that $\sigma\ll\tau$. Note that if $\sigma=\tau$,

$$\dot{f}_2=\operatorname{sech}\left(\frac{t}{\tau}\right). \qquad (49)$$

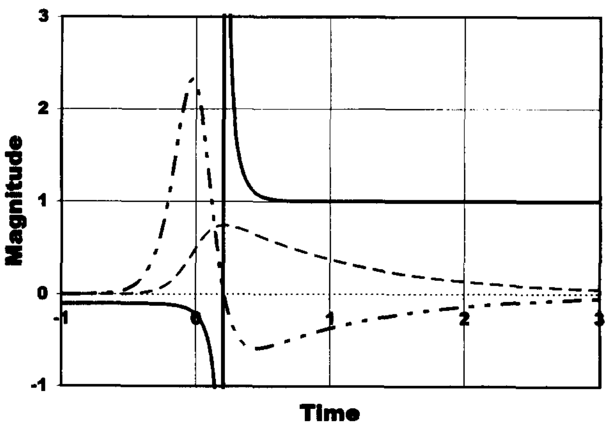

Fig. 3. The dashed line denotes $\dot{f}_2$ and the double dot dashed line is $\ddot{f}_2$ where τ=1.0 and σ=0.1. The solid line is the radius of the causal surface (r_{causal}). The causal radius is asymptotic to $c\tau$ for positive time, and to $-c\sigma$ for negative time. The radiated power is proportional to $(\ddot{f}_2)^2$, so there is a pulse of radiation associated with the rise and another associated with the decay. The energy in the decay pulse appears to come from the region outside the asymptotic causal surface.

The second derivative is

$$\ddot{f}=\ddot{f}_2=\frac{(1/\sigma)\exp(-t/\sigma)-(1/\tau)\exp(t/\tau)}{[\exp(t/\tau)+\exp(-t/\sigma)]^2}. \tag{50}$$

The radius of the causal surface is

$$r_{\text{causal}}=-c\,\frac{\dot{f}_2}{\ddot{f}_2}=c\,\frac{\exp(t/\tau)+\exp(-t/\sigma)}{(1/\tau)\exp(t/\tau)+(1/\sigma)\exp(-t/\sigma)}. \tag{51}$$

The causal radius is displayed in Fig. 3 along with $\dot{f}_2$ and $\ddot{f}_2$. I am not certain what physical significance can be ascribed to a negative causal surface, except for the observation that the radial energy non-negative everywhere (outward or asymptotically zero) during the first or rise pulse of radiation, even arbitrarily close to the dipole.

The power radiated is proportional to $\ddot{f}_2^2$, so there will be two pulses of radiation, one associated with the rise, and the other with the decay. Observe that the causal radius is strictly greater than zero for all times during the second or decay pulse. Since there is no net energy flux through this surface, the energy radiated away presumably comes from the energy stored in the far field, outside the asymptotic causal radius.

The causal radius depicted in Fig. 3 exhibits some curious behavior. Just as the decay pulse begins, the causal radius is infinitely large. Thus initially, there is a net inward flow of energy no matter how far away the observation point. If the observation point is outside the asymptotic causal radius (i.e., if $r>c\tau$), this net inward flow will quickly reverse to become a net outward flow.

A quantitative comparison is possible with the dipole time dependence:[38]

$$f=f_3=\frac{1}{2}\left(1-\tanh\frac{t}{\tau}\right). \tag{52}$$

Evaluating the derivatives,

$$\dot{f}_3=-\frac{1}{2\tau}\,\text{sech}^2\,\frac{t}{\tau} \tag{53}$$

and

$$\ddot{f}_3=\frac{1}{\tau^2}\,\text{sech}^2\,\frac{t}{\tau}\,\tanh\frac{t}{\tau}. \tag{54}$$

The causal surface is

$$r_{\text{causal}}=-c\,\frac{\dot{f}_3}{\ddot{f}_3}=\frac{c\tau}{2}\coth\frac{t}{\tau}. \tag{55}$$

For times $t \gg \tau$, the asymptotic causal radius is

$$\lim_{t\to\infty} r_{\text{causal}}=\lim_{t\to\infty}\frac{c\tau}{2}\coth\frac{t}{\tau}=\frac{c\tau}{2}. \tag{56}$$

Following Eq. (46), the energy stored outside this volume is

$$U_{\text{outside}}=4\pi\epsilon_0\int_{c\tau/2}^{\infty}\left(\frac{kp_0}{r^2}\right)^2 dr=\frac{8}{3}\,\frac{kp_0^2}{c^3\tau^3}. \tag{57}$$

Using Eq. (38), the energy radiated is

$$\begin{aligned}U_{\text{rad}}&=\frac{2kp_0^2}{3c^3}\int_{-\infty}^{\infty}\ddot{f}^2\,dt\\&=\frac{2kp_0^2}{3c^3}\int_{-\infty}^{\infty}\left(\frac{1}{\tau^4}\,\text{sech}^4\,\frac{t}{\tau}\,\tanh^2\frac{t}{\tau}\right)dt\\&=\frac{2kp_0^2}{3c^3\tau^3}\int_{-\infty}^{\infty}\left(\frac{\sinh^2(t/\tau)}{\cosh^6(t/\tau)}\right)\frac{1}{\tau}\,dt\\&=\frac{2kp_0^2}{3c^3\tau^3}\left[\frac{1}{120}\,\text{sech}^5\,\frac{t}{\tau}\left(\sinh\frac{5t}{\tau}+5\sinh\frac{3t}{\tau}\right.\right.\\&\qquad\left.\left.-20\sinh\frac{t}{\tau}\right)\right]_{-\infty}^{\infty}=\frac{8kp_0^2}{45c^3\tau^3}.\end{aligned} \tag{58}$$

This is a factor of 15 less than the energy outside the asymptotic causal surface. Since the causal radius is decreasing to its asymptotic value, however, one expects the energy stored outside the asymptotic causal surface to be an upper bound on the energy radiated. Much of the energy originally outside the asymptotic causal surface was apparently pulled into the decay current.

VI. DISCUSSION AND CONCLUSIONS

This paper introduces a remarkable consequence of Poynting's theorem: the concept of a causal surface that partitions regions of electromagnetic energy flows. By applying this idea to the simple pedagogical example of an exponentially decaying dipole, I demonstrated that the original static field energy may be partitioned into a radiated portion and an absorbed portion. Further, I showed exact agreement between the amount of energy stored outside the causal surface, and the amount of energy radiated by the system. I also showed for the more physically meaningful case of a finite rise time that there is a rise pulse of radiation and a decay pulse. The decay pulse originates outside the causal surface, and so presumably consists of energy that was originally stored in the static field outside this surface.

This paper is certainly not intended to be a thorough examination of the concept of electromagnetic energy and the difficulties involved in its localization. Nevertheless, the present results are highly suggestive, and given the premise that Poynting's interpretation is correct, a couple of conclusions follow.

First, the energy that is emitted in the form of radiation *does not* necessarily originate from the immediate vicinity of the accelerating charges themselves. In at least some situations, it comes from energy stored in free space in the initial static electric field. This is, of course, reminiscent of the 19th century notion of an "æther." (By using this term, I mean only to suggest the active role played by "empty space;" nothing like the prerelativistic notion of a primary inertial frame is intended.)

To the modern eye, the mechanical models proposed in the 19th century appear quaint if not ridiculous. However, one need not invoke any specific mechanical model to appreciate and accept Maxwell's conviction that electromagnetic energy is stored, *somehow*, by whatever physical process underlies electromagnetic fields.[39]

This first result should be neither surprising nor unexpected. Poynting's interpretation followed from Maxwell's conviction that the æther is to be viewed as "a receptacle of energy,"[40] storing the electromagnetic energy according to Eq. (1). The proper application of Poynting's theorem should yield results consistent with the premises upon which it rests.

Even accepting the premises of Maxwell and Poynting, there is an alternate interpretation of my so-called causal surface. One might argue that although the causal surface has no *net* energy flow, there exist inward and outward flows of energy that exactly cancel each other out. I tend to discount this possibility however, because it would require an arbitrary appeal to multiple "brands" of electromagnetic energy flows.[41] Such a proliferation of hypothetical processes should be entertained only as a last resort, not as "physical first aid."

Second, the present line of inquiry offers an interesting physical system against which to test our notions of what an electromagnetic field is and what physical processes underlie it. For instance, how are we to account for the influx of electromagnetic energy that precedes the outflowing decay pulse of f_2 even arbitrarily far into the radiation zone?

In any event, the present analysis offers an interesting starting point for understanding electromagnetic energy transfer, and for further such inquiries on the nature of electromagnetic field energy.

Note added in proof. After this paper was accepted for publication, Peter Miloni brought to my attention some similar work done by L. Mandel, ["Energy Flow from an Atomic Dipole in Classical Electrodynamics," J. Opt. Soc. Am., August 1972, pp. 1011–1012]. In this letter, Mandel derives an expression for the radial energy flow equivalent (to within a constant) to my Eq. (34). He applies his results to an estimation of the lifetime of an excited atom, and notes that "No more energy flows from the dipole into the near field during the decay phase, although energy emerges in the far field during this time." Mandel interprets this as the energy of excitation residing in the near field and gradually flowing into the far field as the dipole decays. Mandel does not, however, identify the specific surface through which there is no net energy flow.

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[1]This idea was originally due to William Thompson (Lord Kelvin). See Sir Edmund Whittaker, *A History of Theories of Aether and Electricity* (Harper and Brothers, New York, 1960), Vol. 1, pp. 218–222.

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[3]James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Academic Reprints, Stanford, CA, 1953), Chap. XI. Originally published 1892.

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[31]See Ref. 19. Larmor's expressions for the electric (p. 511) and magnetic (p. 509) fields are equivalent to Eqs. (21) and (18) here.
[32]See Ref. 20.
[33]See Ref. 25, Sec. 16.8. Note that Jefimenko assumes a sinusoidally varying current or $f(t)=\cos \omega t$ in the present notation.
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[41]Referring to "brands" of electromagnetic energy is particularly appropriate, since "brand" has the additional meaning of a torch; that is, a source of electromagnetic radiation.

Magnetism and mirror symmetry

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Symmetries play an important role in physics. In this article, we examine the relationship between magnetism and mirror symmetry. The reflection properties of the vector potential $\mathbf{A}(t)$ and the magnetic induction $\mathbf{B}(t)$ are obtained, and these properties are used to analyze some simple circuits. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

While learning magnetism, students encounter a basic example—the axial **B** field of a circular loop. Usually, the professor teaches this example by taking the axis of the loop as the z axis, applying the Biot–Savart law to the loop, calculating some integrals, and finally getting $B_x=B_y=0$.[1] The clever professor then remarks that, because currents on opposite sides of the loop produce axial fields with equal and opposite xy components, rotational symmetry about the z axis causes all such components to cancel. Therefore, the axial **B** field of a circular loop is along the loop's axis.

The professor's remark exemplifies the symmetry arguments so widely used by physicists. Indeed, symmetries are important in physics,[2] and many physicists consider symmetry arguments more aesthetically appealing than doing laborious calculations. Actually, the circular loop possesses a symmetry more clever than the professor might realize. That symmetry is mirror symmetry.[3,4]

II. THE B FIELD AS SEEN IN A MIRROR

In electromagnetism, mirror symmetry is more fundamental than rotational symmetry. Three observations motivate this hypothesis. First, rotation about a fixed axis can be decomposed into two reflections.[5] Second, when transformed by the reflection group, solutions of Maxwell's equations exhibit a definite parity.[6] Third, reflection is conceptually related to the method of electromagnetic images,[7] so we can exploit our geometric intuition.

Before discussing mirror symmetry, let us consider some general aspects of reflection. Define R as the reflection operator with respect to the yz plane. Then the mirror image of a point $\mathbf{r}=(x,y,z)$ is

$$\mathsf{R}\mathbf{r}=(-x,y,z). \tag{1}$$

Note that R is a linear isometry, i.e.,

$$\mathsf{R}(c_1\mathbf{r}_1+c_2\mathbf{r}_2)=c_1\mathsf{R}\mathbf{r}_1+c_2\mathsf{R}\mathbf{r}_2; \tag{2}$$

$$|\mathsf{R}\mathbf{r}|=|\mathbf{r}|. \tag{3}$$

We omit other trivial properties of R.

Next, consider the relation between R and the cross product. By writing the cross product as a determinant, then noting that the determinant is linear in its first column,[8] we can easily show

$$\mathsf{R}\mathbf{r}_1\times\mathsf{R}\mathbf{r}_2=-\mathsf{R}(\mathbf{r}_1\times\mathbf{r}_2). \tag{4}$$

Therefore, to obtain the cross product of two mirror images, we first reflect the original cross product in the mirror, then we invert that reflection. This "reflect-and-invert" algorithm has an unexpected effect. Since our right hand becomes our left hand in a mirror, R transforms the right-hand rule into

