SUMMARY.

The aim of the paper has been the establishment of a system of internal ballistics, based on the assumption of the particular forms we have recently put forward for the equation of state of propellant gases and for the rate of burning law of stabilized colloidal propellants. The main analysis has centred round the conditions obtaining during the propellant combustion phase, and these are chiefly discussed in Sections III. and IV. An abridged procedure has been outlined in Section V. together with a brief reference to particular features of the motion both during and immediately subsequent to propellant combustion. The analysis has been specifically restricted to supposedly known conditions, no attempt being made to deal with wider applications, an adequate discussion of which would compel as a preliminary the introduction and detailed examination of several essential criteria not required for the immediate purposes in view.

In conclusion, we desire to express our thanks to Captain A. C. Goolden, R.N., for his encouragement and interest in the work, and to acknowledge our indebtedness to the Director of Artillery, the Director of Naval Ordnance, and the Ordnance Committee for their courtesy in sanctioning publication.

LIX. *The Electromagnetic Field of a moving uniformly and rigidly Electrified Sphere and its Radiationless Orbits.* *By* G. A. SCHOTT, *F.R.S., University College of Wales, Aberystwyth* *.

1. IT is very generally assumed that an electrified body in accelerated motion necessarily loses energy by radiation at a rate proportional to the square of its acceleration. In the course of an investigation carried out recently, I have discovered a case in which this assumption is certainly false on the basis of the Electromagnetic Theory of Maxwell and its further developments by Larmor and H. A. Lorentz. In order to make quite clear what is involved let us consider a concrete example: imagine a metal sphere suspended by a fine metal wire in such a manner that it can be earthed or insulated at will. Surround it by a closely fitting insulating coating, *e. g.*, two thin hollow hemispheres of ebonite fitted together, and then place around the whole and concentric with it a larger insulated metal sphere made of two hemispheres with a very small hole through which the suspending wire passes without touching the outer sphere. Now connect the outer metal sphere to one pole of a battery and the inner one momentarily to the other pole, and again insulate the latter sphere. This receives a charge which resides on the ebonite in contact with it according to the theory of Maxwell and a well known experiment of Faraday. Remove the outer metal hemispheres and also the ebonite hemispheres. Joining the latter together we obtain a very nearly uniformly charged insulating sphere, and, if the ebonite insulated perfectly, the charge would remain uniform however the sphere moved about as a whole. This charged ebonite sphere is a concrete example, realized approximately, of what is meant in the present paper. It will be shown that, if the centre of such a uniformly and rigidly electrified sphere describes a closed orbit of any form with a suitably chosen period, and the sphere rotates with such an angular velocity that every point of it describes an equal and parallel orbit, then the electromagnetic field due to the sphere at a sufficient distance is a static field, and therefore no energy will be radiated to infinity. The motion of the sphere must be one of pure translation without any spin.

* Communicated by the Author.

2. In order to prove this result we start from the following expressions for the scalar potential ϕ and vector potential $\mathbf{a}$ *

$$\phi=\frac{1}{2\pi}\int de\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\mu(t-\tau-\mathrm{R})}d\tau\, d\mu/\mathrm{R} \quad . \quad . \quad . \quad (1)$$

$$\mathbf{a}=\frac{1}{2\pi}\int de\int_{-\infty}^{\infty}\int_{-}^{\infty} e^{i\mu(t-\tau-\mathrm{R})}\mathbf{v}\,d\tau\, d\mu/\mathrm{R}, \quad . \quad . \quad (2)$$

* Schott, *Ann. der Physik*, xxvi. p. 637 (1907); 'Electromagnetic Radiation,' ch. II. §9, eqs. (12) and (13).

where **v** is the common vector velocity of every element de of the charge on the sphere, and R is the distance of the field point from de, both at the time τ. To simplify printing and writing the unit of time has been taken to be the time taken by an electromagnetic disturbance to travel unit distance. To return to the usual unit we replace t and τ by ct and $c\tau$, and **v** by **v**$/c$.

We assume that the integration with respect to de can be interchanged with those with respect to t and μ; the validity of this change of order of integration, as well as the character of the integrals (1) and (2), is discussed elsewhere.

Let the total charge on the sphere be e, the radius a, the distance of the field-point from the centre at time τ, r, and the angle between r and the radius to the element de, γ. Since there is symmetry for the sphere and its charge about r, we may write

$$R^2=a^2+r^2-2ar\cos\gamma,\quad de=\tfrac{1}{2}e\sin\gamma d\gamma=eRdR/2ar\ . \quad . \quad (3)$$

Changing the order of integration in (1) and using (3) we get

$$\phi=\frac{e}{4\pi a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mu(t-\tau)}d\tau\, d\mu}{r}\int_{R_1}^{R_2}e^{-i\mu R}dR$$

$$=\frac{e}{4\pi a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mu(t-\tau-R_1)}-e^{i\mu(t-\tau-R_2)}}{i\mu r}d\tau\, d\mu,$$

$$=\frac{e}{2\pi a}\int_{-\infty}^{\infty}\frac{d\tau}{r}\int_{0}^{\infty}\frac{\sin\mu(t-\tau-R_1)-\sin\mu(t-\tau-R_2)\, d\mu}{\mu}, \quad . \quad . \quad . \quad (4)$$

where R_1 and R_2 are the least and greatest distances of the field-point from the sphere, both taken positively. The last expression is derived by dividing the range of integration for μ into two, from $-\infty$ to 0 and from 0 to ∞, and changing the sign of μ in the first half. In the same way we obtain from (2) an expression for **a** differing from (4) only by an additional factor **v** in the integral with respect to τ.

3. In order to make further progress we must consider the kinematics of the problem briefly. As the centre

of the sphere describes its orbit, whatever it be, the sphere sweeps out a region of space bounded by the envelope of the moving sphere; this envelope can also be generated by a great circle of the sphere, whose plane always remains normal to the path of the common centre. Two cases arise: in one the envelope does not intersect itself, and consists of a single sheet of tubular form, either extending to infinity or forming a closed ring; in the other it intersects itself and consists of two sheets, an outer sheet generated by the outer part of the moving great circle, and an inner sheet generated by the inner part. Any field-point, outside the single sheet of the envelope in the first case, or the outer sheet in the second, is such that $r>a$ for every position of the moving sphere; any field-point inside the inner sheet of the envelope in the second case is such that $r<a$ for every position of the moving sphere. Every other field-point is such that $r>a$ for some positions of the moving sphere, and $r<a$ for others. The first two types of field-point are comparatively easy to deal with, especially the first ($r>a$ always); we shall call this type an outer field-point, and restrict our investigation to it, because it leads to general and interesting results.

4. *Outer field-point*: $r>a$, $R_1=r-a$, $R_2=r+a$.
The arguments of the two sines in (4) become

$$\mu(t+a-\tau-r) \quad \text{and} \quad \mu(t-a-\tau-r)$$

respectively, and the corresponding integrals with respect to μ are discontinuous Dirichlet integrals, whose values are

$$\pm\pi/2 \quad \text{for} \quad \tau+r\lessgtr t+a, \quad \text{or} \quad t-a,$$

as the case may be. They are easily reduced by means of the substitution

$$\sigma=\tau+r,\ \frac{d\sigma}{d\tau}=1+\frac{dr}{d\tau}=K,\ d\tau=\frac{d\sigma}{K}, \quad . \quad . \quad (5)$$

where K is a Döppler factor and is positive so long as the velocity v of the sphere is less than c, the velocity of light. We shall suppose that this is so, and then σ increases continually from $-\infty$ to ∞ as τ increases between the same limits. We obtain at once from (4)

$$\phi=\int_{t-a}^{t+a}\frac{e\, d\sigma}{2aKr}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and in precisely the same way

$$\mathbf{a}=\int_{t-a}^{t+a}\frac{e\mathbf{v}d\sigma}{2a\mathrm{K}r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Returning to the usual units by replacing t and σ by ct and $c\sigma$ and $\mathbf{v}$ by $\mathbf{v}/c$ we obtain

$$\phi=\int_{t-a/c}^{t+a/c}\frac{ecd\sigma}{2a\mathrm{K}r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6')$$

$$\mathbf{a}=\int_{t-a/c}^{t+a/c}\frac{e\mathbf{v}d\sigma}{2a\mathrm{K}r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7')$$

where now we have

$$\sigma=\tau+\frac{r}{c},\ \mathrm{K}\frac{d\tau}{d\tau}=1+\frac{dr}{cd\tau}. \quad . \quad . \quad . \quad . \quad . \quad (5')$$

5. These expressions can be written in another more suggestive form. Let τ_1 be the time at which a disturbance would have to be emitted from the centre in its position Q_1 in order to reach the field-point P at the time $t-a/c$, and let τ_2, Q_2 correspond similarly to the time $t+a/c$, just as τ corresponds to σ. Also let

$$\mathrm{Q}_1\mathrm{P}=r_1,\ \mathrm{Q}_2\mathrm{P}=r_2\,;$$

then by (5′)

$$\tau_1+\frac{r_1}{c}=t-\frac{a}{c}, \quad \tau_2+\frac{r_2}{c}=t+\frac{a}{c}. \quad . \quad . \quad . \quad (8)$$

Further, let ds be the element of arc described by the centre in the interval from τ to $\tau+d\tau$, and $\mathbf{ds}$ the same element regarded as a vector. Then we have

$$\frac{cd\sigma}{\mathrm{K}}=cd\tau=\frac{cds}{v}=\frac{ds}{\beta},\quad \frac{\mathbf{v}d\sigma}{\mathrm{K}}=\mathbf{v}d\tau=\mathbf{ds},\ \beta=v/c$$

as usual. Then (6′), (7′) and (8), give

$$\phi=\int_{\mathrm{Q}_1}^{\mathrm{Q}_2}\frac{eds}{2a\beta r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\mathbf{a}=\int_{\mathrm{Q}_1}^{\mathrm{Q}_2}\frac{e\mathbf{ds}}{2ar}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In words, ϕ is the electrostatic potential at P due to a distribution of charge along the arc $\mathrm{Q}_1\mathrm{Q}_2$ of the orbit of the centre of the sphere with line density $e/2a\beta$, whilst $\mathbf{a}$ is the vector potential of the same arc when it carries a uniform current of strength $e/2a$.

Since the positions Q_1, Q_2 of the centre of the sphere depend on the time t as well as on the coordinates of the field-point P in accordance with (8), ϕ and $\mathbf{a}$ will both depend on the time in general. But, when the orbit of the centre is periodic, ϕ and $\mathbf{a}$ can become independent of the time, although they will still depend on the coordinates of the field-point and the nature of the orbit.

6. *Periodic orbit.*—The orbit is necessarily a closed curve fixed in position; let the time of one revolution be T. Obviously the values of ϕ and $\mathbf{a}$ will generally repeat themselves after any interval which is an integral multiple of T, for the end points Q_1, Q_2 of the arc concerned in (9) and (10) will be the same as they were originally. Moreover, we see from (8) that Q_2 will coincide with Q_1, whenever $2a$ is equal to $c\mathrm{T}$, or an integral multiple j thereof.

By considering this periodic orbit as the limit of j ultimately coincident turns of a spiral, we deduce that, when $2a = jc\mathrm{T}$, where j is an integer,

$$\phi=j\oint\frac{eds}{2a\beta r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\mathbf{a}=j\oint\frac{e\mathbf{ds}}{2ar}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where each contour integral is taken once round the orbit.

We can confirm these results by deriving them directly from (6′) and (7′) by a very convenient, though not rigorous, symbolic method. For the sake of brevity write

$$\mathrm{F}(t)=\int^{t}\frac{cd\sigma}{\mathrm{K}r}=\mathrm{D}^{-1}\left[\frac{1}{\mathrm{K}r}\right],$$

where D denotes the operator d/cdt with the usual unit of time. Then we obtain from (6′)

$$\phi=\frac{e}{2a}\left\{\mathrm{F}\left(t+\frac{a}{c}\right)-\mathrm{F}\left(t-\frac{a}{c}\right\}=\frac{e}{2a}(e^{a\mathrm{D}}-e^{-a\mathrm{D}})\mathrm{F}(t)\right.$$

$$=\frac{\sinh a\mathrm{D}}{a\mathrm{D}}\left[\frac{e}{\mathrm{K}r}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Similarly (7′) gives

$$\mathbf{a}=\frac{\sinh a\mathrm{D}}{a\mathrm{D}}\left[\frac{e\mathbf{v}}{c\mathrm{K}r}\right] \quad . \quad . \quad . \quad . \quad (14)$$

In words, we can obtain the retarded point potentials due to the uniformly and rigidly charged sphere, with charge e and radius a, in purely translatory motion, by operating with $\sinh a\mathrm{D}/a\mathrm{D}$ on the corresponding retarded point potentials due to a charge e concentrated at the centre.

Since the operator $\sinh a\mathrm{D}/a\mathrm{D}$ is an even function of $a\mathrm{D}$, we can replace $a\mathrm{D}$ by $i\omega a/c$, when the function operated on is a simple harmonic function of ωt.

Now, when the motion of the centre is periodic with a period T, the point potentials can be expanded in Fourier series of the types

$$\left[\frac{e}{\mathrm{K}r}\right]=\sum_{n=0}^{\infty}\left(\Phi_n \cos\frac{2\pi nt}{\mathrm{T}}+\Phi_n' \sin\frac{2\pi nt}{\mathrm{T}}\right) \quad . \quad (15)$$

$$\left[\frac{e\mathbf{v}}{c\mathrm{K}r}\right]=\sum_{n=0}^{\infty}\left(\mathbf{A}_n \cos\frac{2\pi nt}{\mathrm{T}}+\mathbf{A}_n' \sin\frac{2\pi nt}{\mathrm{T}}\right), \quad . \quad (16)$$

where the coefficients are functions of the coordinates of the field-point and of the parameters defining the orbit.

Then we obtain from (13) . . . (16)

$$\phi=\sum_{n=0}^{\infty}\frac{\sin(2\pi na/c\mathrm{T})}{2\pi na/c\mathrm{T}}\left(\Phi_n \cos\frac{2\pi nt}{\mathrm{T}}+\Phi_n' \sin\frac{2\pi nt}{\mathrm{T}}\right. \quad (11)$$

$$\mathbf{a}=\sum_{n=0}^{\infty}\frac{\sin(2\pi na/c\mathrm{T})}{2\pi na/c\mathrm{T}}\left(\mathbf{A}_n \cos\frac{2\pi nt}{\mathrm{T}}+\mathbf{A}_n' \sin\frac{2\pi nt}{\mathrm{T}}\right. . \quad (18)$$

When $\mathrm{T}=2a/cj$, the factor outside the brackets in (17) and (18) reduces to $\sin\pi nj/\pi nj$; it vanishes when j is an integer for all values of n except zero, and then reduces to unity. Hence in this case we obtain

$$\phi=\Phi_0=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\frac{edt}{\mathrm{K}r}=\frac{jc}{2a}\int_0^{\mathrm{T}}\frac{ed\tau}{r}=j\oint\frac{eds}{2a\beta r},$$

$$\mathbf{a}=\mathbf{A}_0=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\frac{e\mathbf{v}dt}{c\mathrm{K}r}=\frac{j}{2a}\int_0\frac{e\mathbf{v}d\tau}{r}=j\oint\frac{e\mathbf{ds}}{2ar},$$

which are identical with (11) and (12).

7. Since the line density of the distribution along the orbit is $e/2a\beta$, the total charge for j turns is

$$j\oint\frac{eds}{2a\beta}=j\int_0^{\mathrm{T}}\frac{ecd\tau}{2a}=\frac{jec\mathrm{T}}{2a}=e,$$

as might have been expected. Moreover, the total current in the j coincident turns is $je/2a$, when the orbit is only reckoned once. Hence we can express our result in the following words :—

When the centre of a uniformly and rigidly charged sphere, with charge e and radius a, in purely translatory motion, describes a closed orbit periodically in a time $2a/cj$, where j is any integer, the electromagnetic field at every outer point is a static field.

The electrostatic potential of this field is the same as that due to a charge e distributed along the orbit of the centre with a linear density varying inversely as the velocity of the centre, which is the same as that of every point of the sphere, and the magnetic field is the same as that due to a uniform steady current of strength $je/2a$ flowing round the orbit of the centre in the direction of its motion.

At great distances from the orbit the electric force varies inversely as the square of the distance and the magnetic force inversely as its cube, so that the total flow of energy across a large sphere enclosing the orbit is zero in the limit as the radius approaches infinity, and the radiation from the moving sphere is zero, as usually defined, although there is acceleration.

This is true only when the motion of the sphere is one of pure translation ; spin always produces radiation.

It is noteworthy that the dimensions of these radiationless orbits are generally small compared with the diameter of the sphere. In fact, the perimeter l of the orbit with period T is given by

$$l=\int_0^{\mathrm{T}}vd\tau=\bar{v}\mathrm{T}=2a\bar{\beta}/j, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $\bar{\beta}$ is the time average of v/c along the orbit. When this is less than unity, as we have supposed, the perimeter of the orbit is less than the diameter of the sphere and may be much less, if β is small, or j large. Thus the orbit of the centre lies entirely inside the sphere, in fact, inside the inner sheet of the envelope, which has two sheets.

8. *Application to problems of atomic structure.*—Having now established the principal result of our investigation, we may perhaps be permitted to indulge in a little

speculation, and, though models of the atom and its constituents, especially classical ones, are out of fashion, enquire whether such models, constructed out of charged spheres, like the one considered above, may not, after all, be of use in the elucidation of atomic problems. The chief difficulty has always been that stationary electric charges cannot form a stable system, whilst charges moving in closed orbits cannot be permanent in the presence of the radiation hitherto always supposed to accompany them; but a charged sphere, such as ours, is not necessarily subject to the latter objection. Obviously it does not help to account for Bohr's radiationless electron orbits, for, if one of our spheres were used as a model of the electron, the radiationless orbits of its centre would be far too small, since they would lie entirely inside the electron, as we saw in the last section.

But this very fact suggests that, if two of our spheres were taken as models of the electron and proton, it might prove possible to use them to construct a permanent model of the neutron, possibly also permanent models of atomic nuclei; for we require radiationless orbits of nuclear dimensions, which can be provided by our spheres, if they are of such dimensions, and have no spin. In this connexion the experimental fact that the electron loses its spin, when it enters and exists inside the nucleus, is very suggestive.

Thus take a uniformly and rigidly charged sphere, with charge e and radius a, as model of the proton, and a similar sphere, with charge $-e$ and radius a', as model of the electron, both without any spin. Suppose that under their mutual attraction their centres describe periodic orbits about their common centroid in a periodic time T. From what we proved above it follows that they will not lose any energy by radiation if

$$T=2a/cj=2a'/cj',$$

where j and j' are integers. For this to be possible a and a' must be commensurable and we must have

$$j':j=a':a=m:m',$$

apart from relativistic corrections, where m and m' are the masses of the proton and electron, which are approximately as 1850 : 1. The average linear dimensions of the two orbits must be less than $a/\pi j$ and $a'/\pi j'$, the greatest values being less than the radius of the proton, and therefore extremely small. Thus the accelerations are exceedingly great, but nevertheless there is no loss of energy due to radiation. The orbits and the proton itself are completely inside the electron—in fact, inside the inner sheet of the two-sheeted envelope of the electron. If under these conditions the mutual attraction proves to be such that a periodic motion of the type contemplated is possible, we shall have a stable and permanent model of the neutron, which could, however, be disrupted by disturbing forces powerful enough to pull the proton out of the electron. The study of such a model must be left for a future investigation; if it should prove successful, we might be able to attack the problem of the structure of more complicated nuclei with some hope of success.

LX. *On the Definition of Distance in General Relativity.*

By I. M. H. ETHERINGTON *.

§ 1. *Introduction.*

IN recent papers Professor E. T. Whittaker † and H. S. Ruse ‡ have discussed the problem of defining, in a general Riemannian space-time, the concept of distance between two particles, as distinct from that of interval (or integrated line-element) between two events. The problem has also been considered by Dr. R. C. Tolman § with reference to particular metrics. Ruse's procedure is purely mathematical, being a natural extension of the concept of spatial distance in Special Relativity. Whittaker and Tolman, on the other hand, related their definitions to the astronomical methods of calculating great distances, such as those of the extragalactic nebulæ. These methods depend ultimately on a comparison of absolute and apparent brightness, it being assumed that brightness decreases with the square of the distance; or, alternatively, on a similar

* Communicated by Prof. E. T. Whittaker, Sc.D., F.R.S.

† Proc. Roy. Soc. A, cxxxiii. p. 93 (1931).

‡ Proc. Roy. Soc. Edinburgh, lii. p. 183 (1932).

§ (i.) Astrophys. Journ. lxix. p. 245 (1929). (ii.) Proc. Nat. Acad. Sci. xvi. p. 511 (1930).