given out in quanta, which represent the differences between the energies in two steady states of motion.

When, however, the numerical value of the appropriate constants in the formula of Ritz is considered, it is found that the magnetic forces set up by the atom are not in themselves sufficient to account for more than a small fraction of the effect that would be necessary to give the observed distribution of lines in spectral series.

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VI. *On the Motion of the Lorentz Electron.* *By* G. A. Schott, *B.A., D.Sc., Professor of Applied Mathematics, University College of Wales, Aberystwyth**.

DURING a theoretical investigation of the origin of X-rays I found it necessary to take into account the effect on the motion of the electron of the reaction due to its own radiation, and from this point of view examined some simple cases of motion in order to gain a clear idea of the result to be expected. The following communication includes these preliminary studies, but is also intended to serve as an introduction to a more complete investigation to be published later.

The Equations of Motion and Energy of the Electron.

1. The vector-equation of motion of the electron may be written in the following form †

$$\dot{\mathbf{G}}-\mathbf{K}=\mathbf{F} \quad \text{. . . . . (1)}$$

where

$$\mathbf{G}=\frac{m\mathbf{v}}{\sqrt{(1-\beta^2)}}, \quad \text{. . . . . (2)}$$

$$\mathbf{K}=\frac{2e^2\ddot{\mathbf{v}}}{3c(c^2-v^2)}+\frac{2e^2(\mathbf{v}\ddot{\mathbf{v}})\mathbf{v}}{3c(c^2-v^2)^2}+\frac{2e^2(\mathbf{v}\dot{\mathbf{v}})\dot{\mathbf{v}}}{c(c^2-v^2)^2}+\frac{2e^2(\mathbf{v}\dot{\mathbf{v}})^2\mathbf{v}}{c(c^2-v^2)^3}. \quad (3)$$

G denotes the electromagnetic momentum of the electron in the form due to Lorentz, **K** the reaction due to radiation, *i. e.* the radiation pressure in the form due to Abraham ‡, and **F** the external mechanical force. If we accept the Principle of Relativity for accelerated as well as for uniform

* Communicated by the Author.

† Schott, 'Electromagnetic Radiation,' pp. 175, 176, 246 (quoted below as E. R.).

‡ Abraham, *Theorie der Elektrizität*, ii. p. 123.

motion, the expression (2) for the electromagnetic momentum follows as a matter of course, but the expression (3) for the radiation pressure requires a special hypothesis to justify its introduction. It must, however, be borne in mind that the deduction of the Lorentz momentum, as for instance by Planck *, also implies the existence of a kinetic potential, and that this has only been defined for reversible changes, whilst accelerated motions of an electron involve radiation and therefore are irreversible. If, on the other hand, we adopt the usual equations of the Electron Theory of Larmor and Lorentz together with the hypothesis that the electron occupies a finite though small region of space, whether surface or volume, then the terms on the left of (1) represent merely the first two terms of an infinite series. If a be a length of the same order of magnitude as the linear dimensions of the electron, and l a second length of the order of the radii of curvature and of torsion of its path and of the distance within which its speed is doubled, this series proceeds according to ascending powers of a/l, and converges with sufficient rapidity only when a/l is small compared with $1-\beta^2$. When the acceleration of the electron becomes very large, or its velocity nearly equal to that of light, the series fails entirely; indeed it is probable that under these conditions the usual definition of the electromagnetic mass, implied in (2), can no longer be upheld. For the rigid spherical electron of Abraham this has been proved definitely by Sommerfeld †; he shows that when the velocity of a uniformly accelerated electron is equal to that of light, the largest term in the mechanical force on it due to its own charge is proportional to the square root of the acceleration when the latter is small. Unfortunately Sommerfeld's method cannot easily be extended to the case of the Lorentz electron, so that it is impossible to be quite sure of what happens here, but it does not seem likely that the result would be very different. However that may be, it is clear that the expressions (1), (2), and (3) must be used with caution in cases where the velocity may be expected to approach that of light, or in very strong electric or magnetic fields, where the acceleration and curvature of the path of the electron may reach large values. Thus we must be careful in using them for an electron which approaches very closely to the nucleus of Rutherford's model atom, and in all problems of a similar kind. May not the failure of the

* Planck, *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1907, p. 8.

† "Zur Elektronentheorie," *Göttinger Nachrichten*, 1904, p. 411.

ordinary mechanics and older electrodynamics so often alluded to by present-day investigators of theories of the atom, be after all due to neglect of proper precautions and to unjustifiable usage of confessedly imperfect analytical expressions as much as to defects in the fundamental principles of the electron theory?

2. The equation of energy may be derived from the equation of motion by multiplying it scalarly by the velocity $\mathbf{v}$; after a few simple algebraic transformations it is obtained in the following form *:

$$\dot{\mathrm{T}}-\dot{\mathrm{Q}}+\mathrm{R}=(\mathbf{v}\mathbf{F}), \quad \ldots \quad (4)$$

where

$$\mathrm{T}=c^2m\left\{\frac{1}{\sqrt{(1-\beta^2)}}-1\right\}, \quad \ldots \quad (5)$$

$$\mathrm{Q}=\frac{2ce^2(\mathbf{v}\dot{\mathbf{v}})}{3(c^2-v^2)^2}, \quad \ldots \quad (6)$$

$$\mathrm{R}=\frac{2ce^2}{3}\left\{\frac{\dot{\mathbf{v}}^2}{(c^2-v^2)^2}+\frac{(\mathbf{v}\dot{\mathbf{v}})^2}{(c^2-v^2)^3}\right\}. \quad \ldots \quad (7)$$

Here T denotes the kinetic energy of the electron and is given by (5) in the usual form; $(\mathbf{vF})$ gives the rate of working of the mechanical force; the remaining terms in (4) are derived from the radiation pressure. Of these R is essentially positive and denotes the irreversible rate of loss of energy due to radiation; the expression (7) is the well-known one due to Liénard. On the other hand, $\dot{\mathrm{Q}}$ represents a reversible rate of loss of energy; hence $-\mathrm{Q}$ must be regarded as work stored in the electron in virtue of its acceleration, so that we may speak of it as acceleration energy. Its existence is a direct consequence of a mechanical theory of the æther †.

3. In order to simplify the equations as much as possible it is convenient to introduce a new system of units; we shall choose the

new unit of length $=2e^2/3c^2m=1{\cdot}83 \,.\, 10^{-13}$ cm.,
" " time $=2e^2/3c^3m=6{\cdot}1 \,.\, 10^{-24}$ sec.,
" " velocity $=c=3 \,.\, 10^{10}$ cm./sec.
" " force $=3c^4m^2/2e^2=4{\cdot}3 \,.\, 10^{6}$ dyne,
" " energy $=c^2m=7{\cdot}88 \,.\, 10^{-7}$ erg.

The numerical values given in the last column have been

* E. R. pp. 176, 177.

† E. R. p. 9.

calculated for the electron with $e=4{\cdot}65\,.\,10^{-10}$ E.S.U. and $e/cm=1{\cdot}77\,.\,10^7$. When the new units are used we must replace the factor m in (2), $2e^2/3c$ in (3), c^2m in (5), and $2ce^2/3$ in (6) and (7) by unity, the quantity β in (2) and (5) by v, and the velocity c in the expression c^2-v^2 in (3), (6), and (7) by unity.

4. *Introduction of a new time-variable.*—Using the new units we put *

$$\tau=\int_0^t \sqrt{(1-v^2)}\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We shall use an accent to denote differentiation with respect to the new time-variable τ, but for the sake of brevity shall use the symbol $\mathbf{w}$ to denote the velocity relative to τ. Then we find in succession

$$\mathbf{v}=\mathbf{w}\sqrt{(1-v^2)}=\frac{\mathbf{w}}{\sqrt{(1+w^2)}},\text{ whence }\sqrt{(1-v^2)}=\frac{1}{\sqrt{(1+w^2)}},$$

$$\dot{\mathbf{v}}=\frac{\mathbf{w}'}{1+w^2}-\frac{(\mathbf{w}\mathbf{w}')\mathbf{w}}{(1+w^2)^2},$$

$$\ddot{\mathbf{v}}=\frac{\mathbf{w}''}{(1+w^2)^{3/2}}-\frac{(\mathbf{w}\mathbf{w}'')\mathbf{w}+3(\mathbf{w}\mathbf{w}')\mathbf{w}'+\mathbf{w}'^2\mathbf{w}}{(1+w^2)^{5/2}}-\frac{4(\mathbf{w}\mathbf{w}')^2\mathbf{w}}{(1+w^2)^{7/2}}.$$

Substituting these values in the expressions (2), (3), (5), (6), and (7), we find

$$\mathbf{G}=\mathbf{w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\mathbf{K}=\frac{\mathbf{w}''}{\sqrt{(1+w^2)}}-\left\{\mathbf{w}'^2-\frac{(\mathbf{w}\mathbf{w}')^2}{1+w^2}\right\}\frac{\mathbf{w}}{\sqrt{(1+\mathbf{w}^2)}}, \quad (10)$$

$$\mathrm{T}=\sqrt{(1+w^2)}-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\mathrm{Q}=\frac{(\mathbf{w}\mathbf{w}')}{\sqrt{(1+w^2)}}=\mathrm{T}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\mathrm{R}=\mathbf{w}'^2-\frac{(\mathbf{w}\mathbf{w}')^2}{1+\mathbf{w}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

With these values the equation of motion (1) becomes

$$\mathbf{w}'-\mathbf{w}''+\mathrm{R}\mathbf{w}=\mathbf{F}\sqrt{(1+w^2)}. \quad . \quad . \quad . \quad . \quad (14)$$

Similarly the equation of energy (4) becomes

$$\mathrm{T}'-\mathrm{T}''+\mathrm{R}(\mathrm{T}+1)=(\mathbf{w}\mathbf{F}). \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The curious similarity of form of the last two equations is worthy of remark.

* E. R. p. 292.

Rectilinear Motion.

5. As an example of the use of the equations we have obtained, we shall now consider the case where the electron moves in a straight line under the action of an electrostatic field in the same direction. We shall take the straight line as the axis of x, so that $\mathbf{w}'=w'=x'$. Then we find from (13) and (14) respectively

$$\mathrm{R}=\frac{w'^2}{1+w^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$w'-w''+\frac{ww'^2}{1+w^2}=\mathrm{F}\sqrt{(1+w^2)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In order to reduce these equations to a simpler form we write

$w=\sinh\chi$, whence $v=\beta=\tanh\chi$, and $\mathrm{T}=\cosh\chi-1$, (18)

(16) and (17) now give

$$\mathrm{R}=\chi'^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\chi'-\chi''=\mathrm{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When F is known as a function of τ, (20) may be solved at once in the form

$$\chi=\int_0^\tau \mathrm{F}\,d\tau-\epsilon^\tau\int_0^\tau \mathrm{F}\epsilon^{-\tau}d\tau+\mathrm{A}+\mathrm{B}\epsilon^\tau, \quad . \quad . \quad (21)$$

where A and B are arbitrary constants to be determined from the initial conditions. A third arbitrary constant will be introduced when we determine x from the differential equation $\dot{x}=w=\sinh\chi$, but we may make this constant zero by choosing the origin of coordinates so that x vanishes when t and τ vanish.

6. *Determination of the arbitrary constants A and B.*—One relation can be obtained at once between A and B, for we are at liberty to choose the origin of time so that v, and therefore also χ, vanishes when $\tau=0$. This condition with (21) gives

$$\mathrm{A}+\mathrm{B}= \;, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Substituting for A in (21) we obtain

$$\chi=\int_0^\tau \mathrm{F}\,d\tau-\epsilon^\tau\int_0^\tau \mathrm{F}\epsilon^{-\tau}\,d\tau+\mathrm{B}(\epsilon^\tau-1). \quad . \quad . \quad (23)$$

Bearing in mind our choice of the origins of space and time and using (8) and (18), we find

$$x=\int_0^\tau \sinh\chi\,d\tau, \quad \ldots\ldots \quad (24)$$

$$t=\int_0^\tau \cosh\chi\,d\tau. \quad \ldots\ldots \quad (25)$$

We have now fully utilized the initial conditions so far as they relate to the initial values of the coordinate and the velocity of the electron, but there still remains an arbitrary element—the arbitrary constant B in (23) to be determined. Here we are brought face to face with one point of difference between the ordinary mechanics of Newton and the electron mechanics founded on the electron theory. Very slight consideration shows that the presence of the third arbitrary constant is due to the fact that the equation of motion of the electron, (1), or (14), or (17), when regarded as a differential equation for the coordinate, is of the third order, and that the differential coefficient of the third order arises from the radiation terms. It is important to bear in mind that these terms must be present whether we adopt the Theory of Relativity for accelerated motions, or base our mechanics on the hypothesis of the extended electron; only in the latter case every additional term of higher order which we introduce into our equation of motion brings with it another arbitrary constant. These additional arbitrary elements, in so far as they must be determined by the initial conditions, represent the effect on the motion of the electron of its past history, a point which I have emphasized on previous occasions *. Unfortunately, the past history is unknown in many problems, and therefore we are compelled to make some additional hypothesis to overcome the difficulty. We must choose it so as to preserve the continuity of the electron mechanics with the ordinary mechanics, which we know suffices in all cases where the velocity of the electron is infinitely small compared with that of light: thus the proper hypothesis suggests itself, namely, that in these cases Newton's Laws of Motion hold without alteration. Hence we assume provisionally:

When the velocity of the electron is zero, its acceleration is equal to the external mechanical force per unit mass.

This hypothesis has the advantage, as we shall see later, that it leads to simple results which can be controlled by experiment.

* Schott, *Annalen der Physik*, 1908, p. 63; E. R. p. 155.

7. In order to apply the new hypothesis to our problem we must find the acceleration. From (18) together with the expressions given in § 4 we obtain

$$\dot{v}=\frac{w'}{(1+w^2)^2}=\operatorname{sech}^3\chi\cdot\chi'. \quad \ldots \quad (26)$$

Again we find from (23)

$$\chi'=\left\{B-\int_0^\tau F\epsilon^{-\tau}d\tau\right\}\epsilon^\tau. \quad \ldots \quad (27)$$

Hence applying our new hypothesis we obtain

$$B=\chi_0'=F_0, \quad \ldots\ldots \quad (28)$$

where the suffix is used to denote initial values. Substituting this value in (23) ... (25) we get finally

$$\chi=\int_0^\tau F d\tau-\epsilon^\tau\int_0^\tau F\epsilon^{-\tau}d\tau+F_0(\epsilon^\tau-1), \quad \ldots \quad (29)$$

$$x=\int_0^\tau \sinh\left\{\int_0^\tau F d\tau-\epsilon^\tau\int_0^\tau F\epsilon^{-\tau}d\tau+F_0(\epsilon^\tau-1)\right\}d\tau, \quad (30)$$

$$t=\int_0^\tau \cosh\left\{\int_0^\tau F d\tau-\epsilon^\tau\int_0^\tau F\epsilon^{-\tau}d\tau+F_0(\epsilon^\tau-1)\right\}d\tau. \quad (31)$$

We also find from (19) and (29)

$$R=\left\{F_0-\int_0^\tau F\epsilon^{-\tau}d\tau\right\}^2\epsilon^{2\tau}. \quad \ldots\ldots \quad (32)$$

In order better to appreciate the import of our hypothesis we shall now apply the solutions (29) ... (32) to the particular case of a uniform force.

8. *Example—Motion of a Lorentz electron in a uniform electrostatic field parallel to the line of motion.* I have already treated this example elsewhere *, but without taking the radiation pressure into account.

In the present problem F is a constant, so that we may omit the zero suffix as no longer necessary. Then we find

$$\chi=F\tau,\ Fx=\cosh\chi-1,\ Ft=\sinh\chi,\ R=F^2. \quad (33)$$

Eliminating χ between the second and third of these equations, we obtain precisely the same relation between x and t as we do when we neglect radiation. This surprising result is a direct consequence of the hypothesis of § 6; in order to

* E. R. p. 181.

understand this better we must examine the energy relations of the electron.

From (11) we obtain by means of (18) and (33)

$$T = \cosh\chi - 1 = Fx. \quad \ldots \quad (34)$$

This equation shows that the whole of the work done by the external field is converted into kinetic energy of the electron, just as if there had been no radiation at all. None of it is radiated.

Again, from (12) we find by means of (33) and (34)

$$Q = T' = \sinh\chi \cdot \chi' = F^2 t = Rt. \quad \ldots \quad (35)$$

Thus we see that the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were an internal compensation amongst the different parts of the radiation pressure, which causes its resultant effect to vanish.

The total energy radiated is on the present hypothesis only a very small fraction of the kinetic energy, unless the external force be exceptionally large. From (33) ... (35) we find by means of (18)

$$\frac{Rt}{T} = \frac{F\sinh\chi}{\cosh\chi - 1} = F\sqrt{\frac{1+\sqrt{(1-v^2)}}{1-\sqrt{(1-v^2)}}}. \quad \ldots \quad (36)$$

In applying this equation we must bear in mind that we are using the new units of § 3; hence when we return to C.G.S. units we must replace F by $2e^2F/3c^4m^2 = 2e^3X/3c^4m^2$, where X is the electric force in E.S.U. From the value of the new unit of force given in § 3, viz. $4{\cdot}3 \,.\, 10^6$ dyne, and that of e, viz. $4{\cdot}65 \,.\, 10^{-10}$ E.S.U., we find that F in (36) is equal to $1{\cdot}08 \,.\, 10^{-14}$, and that Rt/T is about $4{\cdot}03 \,.\, 10^{-14}$ when X is 30,000 volt/cm. and v or β is 0·5.

9. In order to test the truth of the hypothesis of § 3 we must examine what happens when it fails. Still confining our attention to the case of an electron moving in a uniform electric field along the line of motion, let us return to equations (18) and (23) ... (25), which are true quite independently of the hypothesis in question. Bearing in mind that F as well as B is a constant, we see that we may write instead of (28)

$$B = F(1+\delta), \quad \ldots \quad (37)$$

where δ is another constant, *i. e.* a quantity independent of χ or v, but generally a function of F. We may regard δ as a measure of the deviation of our hypothesis from the truth.

We now find instead of (33)

$$\chi = F\{\tau + \delta(\epsilon^\tau - 1)\}, \quad R = F^2\{1+\delta\epsilon^\tau\}^2, \quad \ldots \quad (38)$$

while we have as before

$$v = \beta = \tanh\chi, \quad x = \int_0^\tau \sinh\chi\, d\tau, \quad t = \int_0^\tau \cosh\chi\, d\tau.$$

Changing the independent variable from τ to χ we obtain

$$\left.\begin{aligned}
Fx &= \int_0^\chi \frac{\sinh\chi d\chi}{1+\delta\epsilon^\tau} = \int_0^\chi \frac{\sinh\chi d\chi}{1+\delta-\tau+\chi/F}, \\
Ft &= \int_0^\chi \frac{\cosh\chi d\chi}{1+\delta\epsilon^\tau} = \int_0^\chi \frac{\cosh\chi d\chi}{1+\delta-\tau+\chi/F}, \\
R &= F^2\{1+\delta\epsilon^\tau\}^2 = F^2\{1+\delta-\tau+\chi/F\}^2, \\
\int_0^t R\,dt &= F\int_0^\chi \{1+\delta\epsilon^\tau\}\cosh\chi d\chi = F\int_0^\chi \{1+\delta-\tau+\chi/F\}\cosh\chi d\chi.
\end{aligned}\right\} \quad (39)$$

The equations (38) and (39) show that the analytical character of the solution is completely altered by the failure of the hypothesis under consideration; what change will be produced in the numerical results depends on the magnitudes of β, F, and δ. In estimating this change we must bear in mind that what we measure by experiment is the increase of velocity produced in a measured distance by a field of known strength, and perhaps in certain cases the total energy radiated in the process. Knowing β and therefore χ we can calculate x and the energy radiated by means of (39); but in order to measure β independently of the hypothesis to be tested we must not use a deflexion method, either with an electric or a magnetic field, because that would again involve the hypothesis and require very troublesome calculations. We must measure the kinetic energy, *e. g.* by a thermopile, and thence calculate χ and β by means of (18).

When the exponential term in (38) for χ is negligible in comparison with the first, we have the case already considered in § 8; for the sake of brevity we shall speak of it as the Newtonian motion. On the other hand, when the exponential term preponderates we have another extreme case, which we shall call the exponential motion and shall now examine.

10. *The exponential motion.*—We retain only the exponential term in (38), and accordingly only the term χ/F in the expression $1+\delta-\tau+\chi/F$, which occurs in (39). Then the denominators in the integrals for x and t vanish at the

lower limit, so that t becomes infinite although x remains finite. For this reason it is convenient to extend the integrals from a finite lower limit χ_0 to the upper limit χ_1, the suffixes $_0$ and $_1$ being used to indicate initial and final values respectively. Using the notation of the exponential integral we find from (39)

$$\left.\begin{aligned} x_1-x_0&=\int_{\chi_0}^{\chi_1}\frac{\sinh\chi}{\chi}d\chi=\tfrac{1}{2}\{\mathrm{Ei}(\chi_1)-\mathrm{Ei}(\chi_0)-\mathrm{Ei}(-\chi_1)+\mathrm{Ei}(-\chi_0)\}\\ t_1-t_0&=\int_{\chi_0}^{\chi_1}\frac{\cosh\chi}{\chi}d\chi=\tfrac{1}{2}\{\mathrm{Ei}(\chi_1)-\mathrm{Ei}(\chi_0)+\mathrm{Ei}(-\chi_1)-\mathrm{Ei}(-\chi_0)\}\\ \mathrm{R}=\chi^2,\ \int_{t_0}^{t_1}\mathrm{R}dt&=\chi_1\sinh\chi_1-\cosh\chi_1-\chi_0\sinh\chi_0+\cosh\chi_0\ .\ \ . \end{aligned}\right\}\quad(40)$$

These expressions involve neither F nor δ, but only χ_0 and χ_1, so that in this extreme case of the exponential motion the result depends only on the initial and final velocities of the electron, and not at all on the strength of the field or on the precise value of δ. This fact of itself is sufficient to prove that the exponential motion is not realisable experimentally, at any rate not with the electric fields at our command; a numerical example may make this clearer.

Let us take the case of an electron which has its speed increased by an electric field of 27,700 volt/cm. (giving F equal to 10^{-14}) from $\beta_0=0{\cdot}01$ to $\beta_1=0{\cdot}30$, *i. e.* from $\chi_0=0{\cdot}01$ to $\chi_1=0{\cdot}31$.

With the help of tables of the exponential integral * and of the hyperbolic functions we obtain the following results for the two limiting motions:—

	Newtonian motion.		Exponential motion.	
	Units of § 3.	C.G.S. units.	Units of § 3.	C.G.S. units.
x_1-x_0 ...	$4{\cdot}8\,.\,10^{12}$	$0{\cdot}88$	$0{\cdot}302$	$5{\cdot}5\,.\,10^{-14}$
t_1-t_0 ...	$3{\cdot}05\,.\,10^{13}$	$1{\cdot}86\,.\,10^{-10}$	$3{\cdot}458$	$2{\cdot}1\,.\,10^{-23}$
$\int_{t_0}^{t_1}\mathrm{R}dt$ ...	$3{\cdot}05\,.\,10^{-15}$	$2{\cdot}4\,.\,10^{-21}$	$0{\cdot}049$	$3{\cdot}9\,.\,10^{-8}$

A comparison of the numbers in the last four columns of this table shows conclusively the enormous difference between the two limiting motions, and there can be no question that the Newtonian motion is in far better agreement than the exponential motion with what we know from experience. Even if the hypothesis of § 6 be not exactly true, its deviation

* Dale, 'Tables of Mathematical Functions,' p. 85 and p. 64.

from the truth, as measured by the number δ, must be exceedingly small. In order to obtain some idea of its amount we must study the general motion of § 9 a little more fully.

11. *The limits of accuracy of the hypothesis.*—As we have already remarked in § 9, the theoretically best method of testing the hypothesis in question depends upon a comparison of the kinetic energy, T, acquired by the electron with the work, Fx, done by the external field. We see from (18) and (39) that T differs from Fx by a finite amount, the difference being derived from the acceleration energy of the electron. Suppose then that as a result of experiment we find

$$\mathrm{T}=\cosh\chi-1=(1+f)\mathrm{F}x=(1+f)\mathrm{F}\int_0^\tau\sinh\chi dt,\qquad(41)$$

where f is a number, which is probably a small fraction with the same sign as δ. We must express δ in terms of f by means of (38), (39), and (41). Let us substitute for χ in (41) its expression in terms of τ and δ given by (38), expand both sides of the equation in ascending powers of $\mathrm{F}\delta\epsilon^\tau$ by means of Taylor's theorem and integrate with respect to τ. Rearranging the terms according to powers of $\mathrm{F}\delta\epsilon^\tau$ we find

$$\mathrm{F}\delta\,\frac{(1+f\mathrm{F}^2)\epsilon^\tau\sinh\mathrm{F}(\tau-\delta)-(1+f)\mathrm{F}\epsilon^\tau\cosh\mathrm{F}(\tau-\delta)+(1+f)\mathrm{F}\,(\cosh\mathrm{F}\delta+\mathrm{F}\sinh\mathrm{F}\delta)}{1-\mathrm{F}^2}$$

$$-\mathrm{F}^2\delta^2\,\frac{(4+f\mathrm{F}^2)\epsilon^{2\tau}\cosh\mathrm{F}(\tau-\delta)-2(1+f)\mathrm{F}\epsilon^{2\tau}\sinh\mathrm{F}(\tau-\delta)-(1+f)\,\mathrm{F}(2\sinh\mathrm{F}\delta+\mathrm{F}\cosh\delta)}{2(4-\mathrm{F}^2)}$$

$$\ldots=f\{\cosh\mathrm{F}(\tau-\delta)-1\}-(1+f)\{\cosh\mathrm{F}\delta-1\}.\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\qquad(42)$$

We must combine this equation with (38) so as to eliminate τ and determine δ, but the calculation is so difficult that the result will hardly repay the labour expended; hence we shall content ourselves with finding limits for δ.

We first observe that the series on the left side of (42), being derived from exponential series by integration, is absolutely convergent for all values of $\mathrm{F}\delta\epsilon^\tau$, and that the coefficients of all powers of Fδ increase with τ provided that $\tanh\mathrm{F}(\tau-\delta)$ is greater than fF, a condition which is satisfied in actual experiments on account of the smallness of F. Hence the first term on the left, which for such values of τ has the sign of δ, is less than the right-hand member when δ is positive, and of course f also positive, but is greater (numerically) when δ, and of course f, is negative. Thus when δ is positive, we can obtain an upper limit for its value by omitting all positive terms in the factor of $F\delta$ and all negative ones in the right-hand member of the equation. In this way we find

$$F\delta\epsilon^{\tau}\{\tanh F(\tau-\delta)-(1+f)F\}<f\{1-\operatorname{sech} F(\tau-\delta)\}.$$

This expression can be simplified very considerably without raising the limit appreciably in any actual experiment. In fact we see from (38) that $F(\tau-\delta)$ is less than χ or $\tanh^{-1}\beta$, whence we easily prove that $\operatorname{sech} F(\tau-\delta)$ is greater than $\sqrt{(1-\beta^2)}$, and $\tanh F(\tau-\delta)$ greater than $\beta-F\delta\epsilon^{\tau}$, so that

$$F\delta\epsilon^{\tau}\{\beta-(1+f)F-F\delta\epsilon^{\tau}\}<f\{1-\sqrt{(1-\beta^2)}\}.$$

From this equation we find, again making use of (38), that

$$\left.\begin{aligned} \delta\epsilon^{\delta} &< (\beta/2F)\,\epsilon^{\beta/2F}\left(\frac{1-\beta}{1+\beta}\right)^{1/2F}, \\ \text{provided that}\quad f &< \frac{\{\beta-(1+f)F\}^2}{4\{1-\sqrt{(1-\beta^2)}\}}. \end{aligned}\right\} \quad (43)$$

Of course, as we have stated above, (43) presupposes that δ is positive.

12. Hitherto no experiments appear to have been made in which both the kinetic energy and the work done by the external field have been measured directly as our investigation supposes, but in the course of some determinations of e/m the fall of potential has been measured directly, while the speed of the electron has been determined, usually by means of the deflexion produced by a known magnetic field. The calculation of the speed, and hence of the kinetic energy, from the magnetic deflexion involves an error due to the radiation, presumably of the same order as f but unknown, so that experiments of this kind cannot be expected to supply us with an accurate value of δ. Nevertheless they may be expected to give us some information as to its order of magnitude.

One of the latest determinations of this kind has been made by Hupka * for velocities ranging from one quarter to one half of the velocity of light and falls of potential from 4000 to 20,000 volt/cm. measured to within about 1 in 400. Assuming e/m to be $1{\cdot}77\,.\,10^7$, Hupka calculated the velocity β from the measured fall of potential by means of the Lorentz formula (18) for the kinetic energy, of course neglecting the effect of radiation which we wish to estimate. In his experiments he measured the magnetic force required to produce a prescribed radius of curvature in the path of the electron, and compared their product with the ratio $\beta/\sqrt{(1-\beta^2)}$ to which it should be proportional for the Lorentz electron. This proportionality was found to hold throughout the whole range of the measurements to within about 1 in 4000. It is obvious that this constancy of the ratio of the two quantities to be compared could only be possible either if the hypothesis were nearly true, or if in the event of its failure the errors compensated each other exactly. Of course it is extremely improbable that the effect of radiation on the kinetic energy should balance its effect on the magnetic deflexion so as to produce exact compensation, but in the absence of a complete theory of the magnetic deflexion absolute certainty is impossible. We may, however, draw the conclusion that the number f, which measures the difference between the kinetic energy and the work done by the external field, is of the same order of magnitude as the errors in Hupka's experiments. By far the greatest error is that in the determination of the fall of potential, given above as 1 in 400; hence we conclude that f is about $1/400$.

* Hupka, *Ann. der Phys.* 1910 (1), p. 169.

From six experiments with about equal falls of potential we find that the fall of potential used by Hupka for a velocity $\beta=0{\cdot}5$ is nearly 20,000 volt/cm., which corresponds to $F=7{\cdot}2\,.\,10^{-15}$. The corresponding upper limit for f given by (43) is 0·47, which is far beyond the error possible in the experiments; hence we may apply (43). On account of the very small value of F, the last factor of the right-hand member of the first equation is alone effective in determining the order of δ. Taking logarithms of both sides we find

$$\operatorname{Log}_{10}(1/\delta) > 10^{13}. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

13. Let us now consider the case where δ is negative. From (38) we see that $F(\tau-\delta)$ is greater than χ, so that the whole investigation of § 12 applies provided that the sign "less than" be replaced by "greater than." Thus (44) gives a lower limit for $-\delta$.

We may, however, obtain an upper limit for $-\delta$ by a different line of argument, based on the fact that according to (38) χ increases to a maximum as τ increases, and thereafter diminishes again. The maximum is given by $\tau=\log(-1/\delta)$ and is equal to $F\{\log(-1/\delta)-1-\delta\}$, and there is a corresponding maximum value of β, which is $\tanh F\{\log(-1/\delta)-1-\delta\}$. Experiment shows no trace of the existence of such a maximum, so that we may be sure that if it exists the velocities hitherto found for electrons lie very much below it. If therefore we calculate the value of $(-1/\delta)$ from the highest value of β found for a given value of F, this will certainly give us an upper limit for $-\delta$. In this way we find

$$-\delta\epsilon^{\delta} < \epsilon^{-1}\left(\frac{1-\beta}{1+\beta}\right)^{1/2F} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

With the same experimental data that we have used in § 13 we find

$$\mathrm{Log}_{10}(-1/\delta) > 10^{13},$$

practically the same limit as in the former case.

Hence we may assert as a result of Hupka's experiments that the deviation δ of the hypothesis of § 6 from the truth amounts to less than one part in the ten-million-millionth power of ten for a field of 20,000 volt/cm. This is the same thing as saying that for an electron moving with a velocity small compared with that of light in an electric field of the intensity stated, the acceleration differs from the mechanical force per unit mass by a fraction δ at most, in excess or defect.

It is possible that the deviation δ may depend upon the intensity of the electric field, but the experiments give no certain information on this point. The probable error seems to be rather smaller for a field of 5000 volt/cm. than for the stronger field, but the number of determinations is too small to afford a decisive result. Consequently it would be unsafe to draw any definite conclusion from the experiments respecting the dependence of δ on the field-intensity. Whatever this may be, it does not appear to be very considerable; hence it seems probable that our hypothesis may also be applied to variable fields of intensities of the same order of magnitude as those used in these experiments.

Since according to § 6 the hypothesis is equivalent to Newton's Second Law of Motion for slowly moving electrons, we have verified this law to a degree of accuracy far beyond that attained in astronomical investigations.

How far the law can be applied to electrons starting from rest in very intense fields such as those inside and close to the atom remains doubtful.