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Electromagnetic Mass Revisited

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Examples of uniformly moving charge distributions that possess conserved electromagnetic stress tensors are exhibited. These constitute stable systems with covariantly characterized electromagnetic mass. This note, on a topic to which Paul Dirac made a significant contribution in 1938, is dedicated to him for his 80th birthday.

It is the classical theory of electromagnetic mass that is reexamined here. And why, after all these years, and in view of its apparent irrelevance to the real world? Quite simply, because it still isn't right.

First, recall that there are two conventional meanings for electromagnetic mass. The electrostatic energy of a charge distribution at rest, divided by c^2, gives one mass, $m^{(1)}$. The electromagnetic momentum of the moving charge distribution defines another mass, $m^{(2)}$. They are not the same:

$$m^{(2)} = \frac{4}{3} m^{(1)} \tag{1}$$

It has been stated[1] that this apparent violation of relativistic invariance is to be blamed on the failure of the usual definition of electromagnetic momentum, a definition that refers only to electromagnetic fields and not to the state of motion of their sources.

Then, we recall that the charge distribution is considered to be unstable, in consequence of the Coulomb repulsion of its parts. Alternatively, one points to the nonvanishing integral of the stresses in the rest frame of the system. At this stage the nonelectromagnetic Poincaré stresses are usually introduced to produce stability.

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Underlying all of this is a bit of tunnel vision that I state as: In constructing mechanical properties consider only the stress tensor involving the fields,

$$t^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - g^{\mu\nu}\frac{1}{4}F^{\kappa\lambda}F_{\kappa\lambda}\right] \tag{2}$$

even though it is not divergenceless,

$$\partial_\nu t^{\mu\nu} = -F^{\mu\nu}\frac{1}{c}j_\nu \tag{3}$$

and therefore cannot of itself produce covariant results. In contrast, I now show that there is a class of fields and currents, associated with uniform motion, such that the right-hand side of (3) is the gradient of a scalar. That implies the existence of a conserved—divergenceless—electromagnetic stress tensor. Its use automatically guarantees stability, and the covariance of energy and momentum. The tensor is not unique, however. There is now the option to construct covariant theories in which the electromagnetic mass gives the field energy in the rest frame; and covariant theories in which the electromagnetic mass is inferred from the electromagnetic field momentum. The two invariant masses thereby associated with the same field-current distribution are related by (1). And, preconceptions aside, these mechanical systems are stable.

Any spherically symmetrical charge distribution of total charge e, at rest, is represented by the potentials

$$\phi = ef(r^2), \qquad \mathbf{A} = 0 \tag{4}$$

where, at sufficiently large distances,

$$f(r^2) \sim (r^2)^{-1/2} \tag{5}$$

The covariant description of this situation relative to any uniformly moving rest frame is [metric: $-1, 1, 1, 1$]

$$A^\mu(x) = \frac{e}{c}v^\mu f(\xi^2), \qquad \xi^\mu = x^\mu + \frac{1}{c}v^\mu\left(\frac{1}{c}vx\right) \tag{6}$$

where

$$v^2 \equiv v^\mu v_\mu = -c^2, \qquad v\xi = 0 \tag{7}$$

We also note that

$$\xi^2 = x^2 + \left(\frac{1}{c}\, vx\right)^2 \tag{8}$$

and

$$\frac{1}{2}\,\partial^\mu \xi^2 = \xi^\mu \tag{9}$$

The latter is used in evaluating

$$\partial^\nu A^\mu(x) = 2\,\frac{e}{c}\,\xi^\nu v^\mu f'(\xi^2) \tag{10}$$

with the consequences that

$$\partial_\mu A^\mu(x) = 0 \tag{11}$$

and

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = 2\,\frac{e}{c}\,(\xi^\mu v^\nu - \xi^\nu v^\mu)\, f'(\xi^2) \tag{12}$$

The current vector $j^\mu(x)$ then produced by

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}\, j^\mu \tag{13}$$

is

$$j^\mu(x) = \frac{e}{2\pi}\, v^\mu [-2\xi^2 f''(\xi^2) - 3f'(\xi^2)] \tag{14}$$

It is the covariant form of the rest frame charge density

$$\rho = \frac{e}{2\pi}\,[-2r^2 f''(r^2) - 3f'(r^2)] \tag{15}$$

and the total charge is computed as

$$\begin{aligned}\int_0^\infty dr\, r^2 4\pi\rho &= e\int_0^\infty dr^2 (r^2)^{1/2}(-2r^2 f'' - 3f') \\ &= e\int_0^\infty d[-2(r^2)^{3/2} f'(r^2)] = e \end{aligned} \tag{16}$$

according to the asymptotic form (5) and the assumption that f is sufficiently well behaved at the origin.

We illustrate these results with the simple example

$$f(\xi^2) = (\xi^2 + a^2)^{-1/2} \tag{17}$$

so that

$$F^{\mu\nu} = \frac{e}{c}\frac{v^\mu \xi^\nu - v^\nu \xi^\mu}{(\xi^2 + a^2)^{3/2}} \tag{18}$$

and

$$j^\mu = ev^\mu \frac{3}{4\pi}\frac{a^2}{(\xi^2 + a^2)^{5/2}} \tag{19}$$

In the limit $a^2 \to +0$, this current vector becomes that of a uniformly moving point charge.

The general form of the electromagnetic field stress tensor (2) is

$$t^{\mu\nu}(x) = \frac{e^2}{\pi}\left[-\xi^\mu \xi^\nu + \frac{1}{c^2} v^\mu v^\nu \xi^2 + \frac{1}{2} g^{\mu\nu}\xi^2\right](f'(\xi^2))^2 \tag{20}$$

Its divergence, as computed from (3), is given by

$$\partial_\nu t^{\mu\nu} = -\xi^\mu \frac{e^2}{\pi}[2\xi^2 f' f'' + 3f'^2] \tag{21}$$

And now, if we define a function $t(\xi^2)$ to satisfy

$$t'(\xi^2) = -\frac{e^2}{2\pi}[2\xi^2 f' f'' + 3f'^2] \tag{22}$$

the right-hand side of (21) becomes

$$\partial^\mu t(\xi^2) = \partial_\nu (g^{\mu\nu}\, t(\xi^2)) \tag{23}$$

and we have identified a conserved tensor:

$$T^{\mu\nu} = t^{\mu\nu} - g^{\mu\nu} t, \qquad \partial_\nu T^{\mu\nu} = 0 \tag{24}$$

In the example (17), the evaluation of $\partial_\nu t^{\mu\nu}$ is supplied by

$$-F^{\mu\nu}\frac{1}{c} j_\nu = -\xi^\mu \frac{3}{4\pi} e^2 \frac{a^2}{(\xi^2 + a^2)^4} \tag{25}$$

leading to the identification

$$t(\xi^2) = \frac{e^2}{8\pi} \frac{a^2}{(\xi^2 + a^2)^3} \tag{26}$$

which incorporates the boundary condition of vanishing $t(\xi^2)$ as $\xi^2 \to \infty$.

Another model should be mentioned here,

$$\xi^2 > a^2\colon\ f(\xi^2) = (\xi^2)^{-1/2}, \qquad \xi^2 < a^2\colon f(\xi^2) = (a^2)^{-1/2} \tag{27}$$

with the consequence

$$F^{\mu\nu}(x) = \frac{e}{c} \frac{v^\mu \xi^\nu - v^\nu \xi^\mu}{(\xi^2)^{3/2}} \eta(\xi^2 - a^2) \tag{28}$$

where the step function $\eta(x)$ is such that

$$\eta(x) = \begin{cases} x > 0 : 1 \\ x < 0 : 0 \end{cases} \tag{29}$$

In addition, we have

$$j^\mu(x) = ev^\mu \frac{1}{2\pi a} \delta(\xi^2 - a^2), \qquad \delta(x) = \eta'(x) \tag{30}$$

which is the covariant form of the rest frame surface charge distribution

$$\rho = \frac{e}{2\pi a} \delta(r^2 - a^2) = \frac{e}{4\pi a^2} \delta(r - a) \tag{31}$$

The counterpart of (25) is

$$\begin{aligned} -F^{\mu\nu} \frac{1}{c} j_c &= -\xi^\mu \frac{e^2}{2\pi a^4} \eta(\xi^2 - a^2) \frac{d}{d\xi^2} \eta(\xi^2 - a^2) \\ &= \xi^\mu \frac{e^2}{4\pi a^4} \frac{d}{d\xi^2} \eta(a^2 - \xi^2) \end{aligned} \tag{32}$$

which uses the relations

$$(\eta(x))^2 = \eta(x) = 1 - \eta(-x) \tag{33}$$

and we are led to

$$t(\xi^2) = \frac{e^2}{8\pi a^4} \eta(a^2 - \xi^2) \tag{34}$$

so that $t(\xi^2)$ vanishes for all $\xi^2 > a^2$.

The spatial components of $T_{\mu\nu}$ in the rest frame of this shell charge distribution are

$$T_{kl}=\frac{e^2}{4\pi r^6}\left(-x_k x_l+\frac{1}{2}\delta_{kl}r^2\right)\eta(r-a)-\delta_{kl}\frac{e^2}{8\pi a^4}\eta(a-r) \tag{35}$$

displaying the respective contributions of the field tensor, t_{kl}, and of the new term, $-\delta_{kl}t$. One easily verifies directly that $\partial_l T_{kl}=0$ holds everywhere. The radial component of the stress is

$$T_{rr}=-\frac{e^2}{8\pi}\frac{1}{r^4}\eta(r-a)-\frac{e^2}{8\pi}\frac{1}{a^4}\eta(a-r) \tag{36}$$

where the contribution of the first term,

$$r>a\colon\ T_{rr}=-\frac{e^2}{8\pi}\frac{1}{r^4} \tag{37}$$

evaluated at $r=a$, is the familiar outward traction on the charge shell produced by the electric field. But now the second term,

$$r<a\colon\ T_{rr}=-\frac{e^2}{8\pi}\frac{1}{a^4} \tag{38}$$

produces a precisely compensating inward traction at $r=a$; there is no net radial force on the shell.

While we're at it, let's calculate the volume integral of the diagonal sum of these spatial stress elements,

$$T_{kk}=\frac{e^2}{8\pi}\frac{1}{r^4}\eta(r-a)-3\frac{e^2}{8\pi a^4}\eta(a-r) \tag{39}$$

It is

$$\begin{aligned}\int(d\mathbf{r})\,T_{kk}&=\int_0^a dr\,r^2 4\pi\left[-3\frac{e^2}{8\pi a^4}\right]+\int_a^\infty dr\,r^2 4\pi\left[\frac{e^2}{8\pi r^4}\right]\\&=-\frac{e^2}{2a}+\frac{e^2}{2a}=0\end{aligned} \tag{40}$$

We have referred to an ambiguity in the conserved stress tensor. That expresses the freedom to add an additional divergenceless term, as exhibited by

$$\partial_\nu\left[\frac{1}{c}v^\mu\frac{1}{c}v^\nu t(\xi^2)\right]=\frac{2}{c^2}v^\mu(v\xi)\,t'(\xi^2)=0 \tag{41}$$

This possibility will be utilized in considering two different covariant versions of the concept of electromagnetic mass. The first one uses the stress tensor

(1) $$T^{\mu\nu} = t^{\mu\nu} - \left(g^{\mu\nu} + \frac{1}{c^2} v^\mu v^\nu\right) t \tag{42}$$

it is such that the energy density in the rest frame is entirely electrostatic field energy:

$$\mathbf{v} = 0, \qquad T^{00} = t^{00} = \frac{1}{8\pi} E^2 \tag{43}$$

The second choice adopts the stress tensor already presented in (24),

(2) $$T^{\mu\nu} = t^{\mu\nu} - g^{\mu\nu} t \tag{44}$$

this one is such that the momentum density (multiplied by c) is entirely produced by the electromagnetic field:

$$T^0{}_k = t^0{}_k = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B})_k \tag{45}$$

Perhaps it is worth repeating here that $t^{\mu\nu}$, which would seem to satisfy both requirements by itself, is not conserved—except in the total absence of charge. Then the mechanical system being described is a light pulse, traveling at speed c; there is no rest frame.

We now evaluate

$$E = \int (d\mathbf{r})\, T^{00}, \qquad \mathbf{p}_k = \int (d\mathbf{r}) \frac{1}{c} T^0{}_k \tag{46}$$

for the two types of models. To begin,

$$\begin{aligned} t^{00} &= \frac{e^2}{\pi} \left[-(\xi^0)^2 + \left(\left(\frac{1}{c} v^0 \right)^2 - 1 \right) \xi^2 + \frac{1}{2} \xi^2 \right] (f'(\xi^2))^2 \\ t^0{}_k &= \frac{e^2}{\pi} \left[-\xi^0 \xi_k + \frac{1}{c^2} v^0 v_k \xi^2 \right] (f'(\xi^2))^2 \end{aligned} \tag{47}$$

where, for motion at velocity $v = \beta c$ along the z-axis,

$$v_3 = \beta v^0 = \beta c (1 - \beta^2)^{-1/2} \tag{48}$$

and

$$\xi^0 = \beta\xi_3 = \frac{\beta}{1-\beta^2}(z - vt) \tag{49}$$

We also write

$$\xi_1{}^2 + \xi_2{}^2 = x^2 + y^2 = \rho^2 \tag{50}$$

and get

$$\xi^2 = \rho^2 + \frac{(z-vt)^2}{1-\beta^2} \tag{51}$$

The resulting expressions,

$$\begin{aligned} -(\xi^0)^2 + \left(\left(\frac{1}{c}v^0\right)^2 - 1\right)\xi^2 &= \frac{\beta^2}{1-\beta^2}\rho^2 \\ -\xi^0\xi_3 + \frac{1}{c^2}v^0 v_3 \xi^2 &= \frac{\beta}{1-\beta^2}\rho^2 \end{aligned} \tag{52}$$

lead to

$$\int (d\mathbf{r})\, t^0{}_3 = \frac{\beta}{1-\beta^2}\frac{e^2}{\pi}\int (d\mathbf{r})\, \rho^2 (f'(\xi^2))^2 \tag{53}$$

and

$$\int (d\mathbf{r})\, t^{00} = \beta\int (d\mathbf{r})\, t^0{}_3 + \frac{e^2}{\pi}\int (d\mathbf{r})\, \frac{1}{2}\xi^2(f'(\xi^2))^2 \tag{54}$$

Then we redefine the z-variable in these integrals,

$$\frac{z-vt}{(1-\beta^2)^{1/2}} \to z, \qquad \xi^2 \to \rho^2 + z^2 = r^2 \tag{55}$$

after which angular integration ($\rho^2 \to \frac{2}{3}r^2$) leaves one with a radial integral:

$$\begin{aligned} \int (d\mathbf{r})\, t^0{}_3 &= \frac{\beta}{(1-\beta^2)^{1/2}} m^{(2)}c^2 \\ \int (d\mathbf{r})\, t^{00} &= \frac{\beta^2}{(1-\beta^2)^{1/2}} m^{(2)}c^2 + (1-\beta^2)^{1/2}\frac{3}{4}m^{(2)}c^2 \\ &= \frac{1}{(1-\beta^2)^{1/2}} m^{(2)}c^2 - (1-\beta^2)^{1/2}\frac{1}{4}m^{(2)}c^2 \end{aligned} \tag{56}$$

where we have introduced

$$m^{(2)}c^2 = \frac{4}{3} e^2 \int_0^\infty dr^2 (r^2)^{3/2} (f'(r^2))^2 \tag{57}$$

The noncovariance of the theory that is based entirely on the electromagnetic field tensor is exhibited here. We also see the origin of the particular numerical factor [Eq. (1)] that connects $m^{(1)}c^2$, the rest frame value of the field energy,

$$m^{(1)}c^2 = \frac{3}{4} m^{(2)}c^2 \tag{58}$$

with the mass $m^{(2)}$ that measures electromagnetic field momentum.

Now we turn to the evaluation of

$$\begin{aligned} \int (d\mathbf{r})\, t(\xi^2) &= (1-\beta^2)^{1/2}\, 2\pi \int_0^\infty dr^2 (r^2)^{1/2}\, t(r^2) \\ &= -(1-\beta^2)^{1/2} \frac{4\pi}{3} \int_0^\infty dr^2 (r^2)^{3/2}\, t'(r^2) \end{aligned} \tag{59}$$

where, according to (22), the latter integral is

$$-\frac{e^2}{2\pi}\int_0^\infty dr^2 (r^2)^{3/2} \left[r^2 \frac{d}{dr^2}(f')^2 + 3(f')^2 \right] = -\frac{e^2}{4\pi}\int_0^\infty dr^2 (r^2)^{3/2} (f')^2 \tag{60}$$

That produces

$$\int (d\mathbf{r})\, t(\xi^2) = (1-\beta^2)^{1/2} \frac{1}{4} m^{(2)}c^2 \tag{61}$$

Finally, looking first at stress tensor (2), we arrive at the total energy and momentum

$$\begin{aligned} E &= \int (d\mathbf{r})(t^{00} + t) = \frac{1}{(1-\beta^2)^{1/2}} m^{(2)}c^2 \\ p &= \int (d\mathbf{r}) \frac{1}{c} t^0{}_3 = \frac{1}{(1-\beta^2)^{1/2}} m^{(2)} v \end{aligned} \tag{62}$$

the covariant forms based on rest mass $m^{(2)}$, while stress tensor (1) produces

$$\begin{aligned} E &= \frac{1}{(1-\beta^2)^{1/2}} m^{(2)}c^2 - \frac{1}{1-\beta^2}(1-\beta^2)^{1/2}\frac{1}{4} m^{(2)}c^2 = \frac{1}{(1-\beta^2)^{1/2}} m^{(1)}c^2 \\ p &= \frac{1}{(1-\beta^2)^{1/2}} m^{(2)} v - \frac{\beta}{1-\beta^2}(1-\beta^2)^{1/2}\frac{1}{4} m^{(2)}c = \frac{1}{(1-\beta^2)^{1/2}} m^{(1)} v \end{aligned} \tag{63}$$

the covariant forms based on rest mass $m^{(1)}$:

$$m^{(1)}c^2 = e^2 \int_0^\infty dr^2 (r^2)^{3/2} (f'(r^2))^2 \tag{64}$$

Using the example of the shell model, Eq. (27), one finds immediately that $m^{(1)}c^2 = e^2/2a$, which is indeed the electrostatic energy of the system at rest. Incidentally, although we have carried out this discussion in the context of systems with charge $e \neq 0$, it is clear that current distributions of zero net charge also possess electromagnetic mass. And the conceivable contribution of magnetic charge should not be overlooked.

We close this note with a glance at the action-principle approach to electromagnetic mass. The action of the electromagnetic field in interaction with electric current is

$$W = \frac{1}{c} \int (dx) \left[\frac{1}{c} j^\mu A_\mu - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \right] \tag{65}$$

or, in virtue of the respectively linear and quadratic field dependences of these two terms of a stationary expression,

$$W = -\frac{1}{c} \int (dx) \left(-\frac{1}{16\pi} \right) F^{\mu\nu} F_{\mu\nu} \tag{66}$$

On referring to the field construction (12), we get

$$-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{e^2}{2\pi} \xi^2 (f'(\xi^2))^2 \tag{67}$$

Then we write the invariant element of volume as the rest frame product of the proper time element, ds, with the element of spatial volume appropriate to spherical symmetry ($\xi^2 = r^2$):

$$\frac{1}{c} (dx) = ds\, dr^2 (r^2)^{1/2}\, 2\pi \tag{68}$$

That gives

$$\begin{aligned} W &= -\int ds\, e^2 \int_0^\infty dr^2 (r^2)^{3/2} (f'(r^2))^2 \\ &= -m^{(1)} c^2 \int ds \end{aligned} \tag{69}$$

according to (64). It is natural that the electrostatic energy definition should appear here; the rest frame interpretation of the field structure (67) is just the electrostatic energy density.

REFERENCE

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