Theory of Split-Anode Magnetrons by Sin-itiro Tomonaga

Koichi Shimoda

During the World War II theory of the magnetron was developed by S. Tomonaga in collaboration with M. Kotani. Assuming a parabolic potential between the cathode and the cylindrical anode of the magnetron, he calculated the electron orbits perturbed by the oscillating field of the anode. Secular change of the electron orbit results in the formation of rotating space charge inside the split-anode magnetron, which is found to enhance the oscillating field when a resonance condition is satisfied.

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1. INTRODUCTION

The magnetron is now widely used in microwave ovens. It was invented by A. W. Hull in 1921 [1]. It has a cylindrical anode with a co-axial cathode and operates under a magnetic field applied along the axis. In the absence of magnetic field, electrons emitted from the cathode take straight paths towards the anode. When a weak magnetic field is applied, the electron orbit will be curved. As the magnetic field is increased, the electron orbit will be more curved so that the electron orbit may become tangential to the anode at a magnetic field $H=H_c$ as shown in Fig. 1. If the magnetic field H is increased further, $H > H_c$, the electrons do not reach the anode, but return to the cathode.

The critical field for an anode radius r_a and anode potential U_a is calculated to be

$$H_c = \frac{c}{r_a}\sqrt{\frac{8mU_a}{e}} \tag{1}$$

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where c is the speed of light, m the electron mass and $-e$ the electron charge.

Assume that the anode potential is fixed, the anode current as a function of the applied magnetic field H will drop to zero at the critical field H_c, as shown by a dotted curve in Fig. 2. Instead of the sharp cut-off the experimentally observed currents do not show sharp cut-off as shown by a solid curve in Fig. 2 for example. Very often high frequency oscillations are observed near and slightly above the critical field H_c. A. W. Hull observed the magnetron oscillation at a frequency of a few hundred kilo-hertz, which was increased upto 10 megahertz in 1924.

Kinjiro Okabe in Osaka University introduced a split-anode magnetron in 1927 [2] so that he could generate micro-

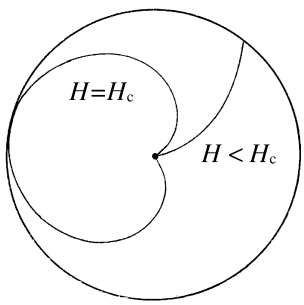


Fig. 1: Electron orbits in a magnetron.

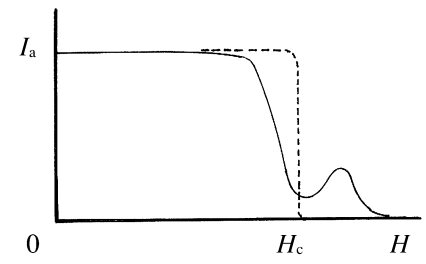


Fig. 2: Anode current of the magnetron as a function of the magnetic field.

waves of 12 cm wavelength or a frequency of 2.5 GHz. Extensive studies on split-anode magnetrons, generating powerful microwaves, have been carried out at the Technical Institute of the Japanese Navy since 1933. A variety of anode structures as shown in Fig. 3 has been investigated and developed.

Most types of these anode structures were called after the names of associated flowers. Type T in Fig. 3 was called Tachibana (Mandarine-orange flower); type Ya was called Yaguruma (Corn-flower); type C was called Kosumosu (Cosmos or Rising sun); type U was called Umebachi (Japanese apricot flower); type Ki was called Kiku (Chrysanthemum); type Yu was called Yuri (Lily); and type Ka was called Kago (Basket or cage). Type S in Fig. 3 represents a structure where every other segments are linked together. In all types of these anodes polyphase oscillations as well as push-pull oscillations were observed.

The characteristics of magnetrons with these anodes are too complicated to be described in short.

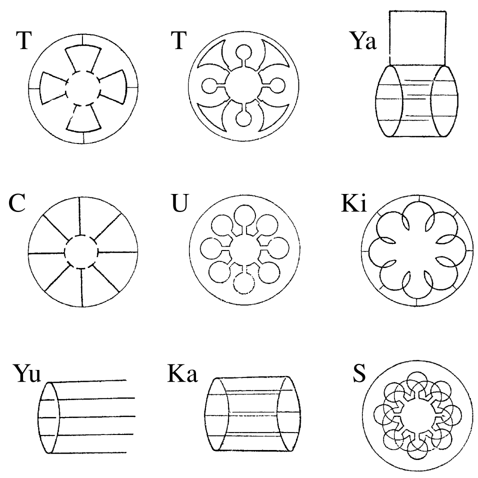


Fig. 3: A variety of anode structures.

2. WORKSHOP ON THE THEORY OF MAGNETRONS

During the World War II a workshop for theoretical studies of magnetrons was formed in September 1943 in the Department of Physics, Imperial University of Tokyo. Members of the workshop were Yusuke Hagihara (Professor of celestial mechanics), Masao Kotani (Professor of theoretical chemical physics), Tatuoki Miyazima (Assistant Professor of electromagnetic theory), Kiichiro Ochiai (Professor of relativity), Takuzo Sakai (Professor of statistical physics), Sin-itiro Tomonaga (Professor of field theory) in alphabetical order, and a few others. I was an observer in most of the meetings of the workshop.

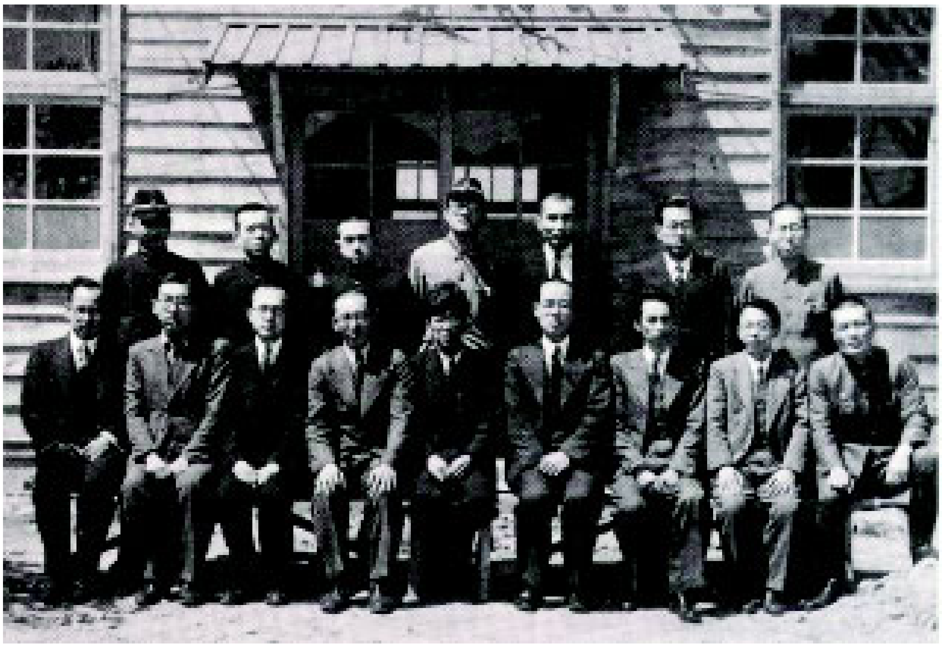
At the second meeting on September 10, 1943 K. Ochiai gave a theory based on a kinematical study of electrons, while T. Sakai presented a theory based on gas-dynamical approach. At the second and third meetings on September 10 and 16, 1943 T. Miyazima introduced and discussed papers by E. G. Linder published in 1938. At the fourth meeting on September 24, 1943 S. Tomonaga presented his theory on the assumption of a parabolic potential, as described in the next Section. At the fifth meeting on October 12, 1943 Dr. S. Mizuma of the Technical Institute of the Japanese Navy gave a review of experimental and theoretical works performed at the Institute since 1933. It was followed by reports of the development of the theory of magnetrons by S. Tomonaga and M. Kotani. At the sixth meeting on November 4, 1943 further studies by T. Miyazima, S. Tomonaga, M. Kotani and Y. Hagihara were reported respectively and discussed. Extensive work by S. Tomonaga was finally reported at the meeting on November 26, 1943.

Then whole contents of the theory of magnetrons by S. Tomonaga were presented in a larger but limited circle in January 1944. No journals of the Physico-Mathematical Soceity of Japan could be published at that time. The Physical Society of Japan was established after the War in April, 1946. Then English versions of his papers were published in 1948 [3], together with papers by Y. Hagihara [4] and M. Kotani [5]. It may be noted that S. Tomonaga and M. Kotani jointly received the Japan Academy Award of 1948 for their works on the theory of magnetrons.

3. THEORY OF SPLIT-ANODE MAGNETRONS BY S. TOMONAGA

A brief outline of the theory by S. Tomonaga is given below. The readers shall refer to his original papers [3] for details of the theory.

We shall find how electrons in the magnetron move, and how the electron energy is transferred to the electric oscillation taking place in the resonator attached to the split anode. It is neces-



At the Shimada Laboratory, Technical Institute of Japanese Navy, Shizuoka Prefecture in 1944 or 1945. Front row, from left: Yuzuru Watase, Tatuoki Miyazima, Takeo Nagamiya, Kodi Husimi, Yusuke Hagihara, Hideki Yukawa, Sin-itiro Tomonaga, Masao Kotani; Back row, from left: Minoru Oda (the third), Masaichiro Mizuma (the fifth). (Courtesy of Atsushi Tomonaga.)

sary for understanding the magnetron oscillation to know how the phases of electron motions, which are distributed over all values between 0 and 2π, when they leave the cathode, are confined to such a range that electrons may move in clusters and thus inducing electric oscillations in the anode structure.

In studying such motion of electrons two distinct methods had been used: a particle-mechanical method on the one hand and a hydrodynamical method on the other. Although the latter method seemed to be preferable since it had no difficulty of many-body problem, it was noted that the swarm of electrons in the magnetron was far too rare to allow hydrodynamical approach.

We were thus forced to use a particle-mechanical method. In this treatment the difficulty of the many-body problem can be overcome if the effect of the electron space charge is replaced by a suitably chosen electrostatic potential.

There are strong presumptions that this electrostatic force has the potential of the form

$$U(r) = U_a (r/r_a)^2, \quad (2)$$

where r is the distance from the center of the magnetron. If we assume that the potential of this form exists in the magnetron, it is comparatively easy to develop a theory of the magnetron oscillation, as given in the following.

3.1. Unperturbed Motion of Electrons

We investigate first the motion of an electron when no oscillation is taking place in the split anode. Then the Hamiltonian of the electron can be written as

$$\mathscr{H} = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + \omega_H p_\theta + \frac{m}{2}(\omega_H^2 - \omega_c^2)\, r^2 , \quad (3)$$

where the Larmor frequency and the critical frequency are respectively defined by

$$\omega_H = \frac{eH}{2mc} , \quad (4)$$

$$\omega_c = \sqrt{\frac{2eU_a}{mr_a}} . \quad (5)$$

The solution of the electron orbit $re^{i\theta} = x + iy$ can then be written in general in the form

$$x = R_1 \cos(\Omega_1 t + \alpha_1) + R_2 \cos(\Omega_2 t + \alpha_2), \quad (6a)$$
$$y = R_1 \sin(\Omega_1 t + \alpha_1) + R_2 \sin(\Omega_2 t + \alpha_2), \quad (6b)$$

where

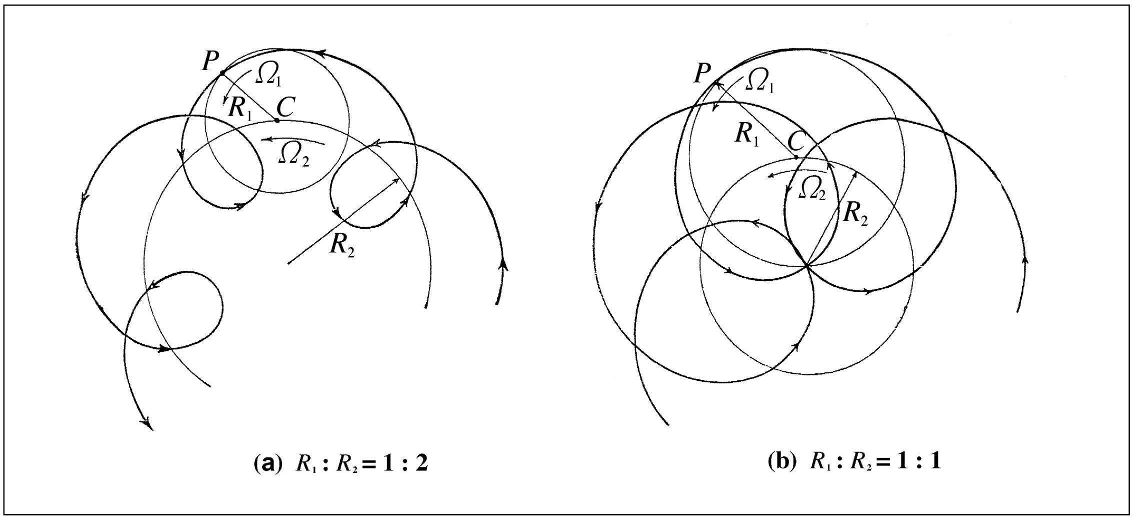


Fig. 4: Unperturbed electron orbits in the magnetron (a) for $R_2 = 2R_1$, and (b) for $R_1 = R_2$.

$$\Omega_1 = \omega_H + \sqrt{\omega_H^2 - \omega_c^2}\,, \qquad (7a)$$

$$\Omega_2 = \omega_H - \sqrt{\omega_H^2 - \omega_c^2}\,. \qquad (7b)$$

Thus the motion of the electron is a superposition of two uniform circular motions in the absence of oscillating field: one is the rolling motion with radius R_1 and frequency Ω_1 and the other is the revolutional motion with radius R_2 and frequency Ω_2.

Four constants of motion are given by the initial condition of the electron. The aphelion radius where the electron is most distant from the center is

$$r_{aph} = R_1 + R_2$$

and the perihelion radius where the electron is closest to the center is

$$r_{per} = |R_2 - R_1|\,.$$

When $R_2=2R_1$, for example, the electron orbit is shown in *Fig. 4 (a)*. For the electron leaving the cathode at the center, $r=0$, we have $R_1=R_2$ so that the orbit may become as shown in *Fig. 4 (b)*.

The energy of the electron in the magnetron can be calculated from Eqs. (6a) and (6b) in the form

$$E = E_1 + E_2\,,$$

where

$$E_1 = m\,\Omega_1 R_1^2 \sqrt{\omega_H^2 - \omega_c^2} = \frac{1}{2} m\,\Omega_1^2 R_1^2 - \frac{eU_a}{r_a^2} R_1^2 \qquad (8a)$$

represents the energy of the electron rolling at frequency Ω_1 and amplitude R_1, and

$$E_2 = -m\,\Omega_2 R_2^2 \sqrt{\omega_H^2 - \omega_c^2} = \frac{1}{2} m\,\Omega_2^2 R_2^2 - \frac{eU_a}{r_a^2} R_2^2 \qquad (8b)$$

represents the energy of the electron revolving around the center at frequency Ω_2 with amplitude R_2.

It may be noted that the revolving energy is negative since the revolving frequency is so low that the kinetic energy is smaller than the negative potential energy.

3.2. Motion of Electrons Perturbed by an Oscillating Field

When an electric oscillation is present in the magnetron, there exists an alternating field due to this oscillation. Perturbation of the electron motion will be significant if the frequency of electric oscillation is resonant with the proper frequency of the electron so that

$$\omega = \Omega_1 \;\text{ or }\; \omega = \Omega_2\,. \qquad (9)$$

If the oscillating field is assumed to be uniform inside the magnetron, we find that the electron energy increases when $\omega = \Omega_1$ and decreases when $\omega = \Omega_2$.

In the former case the electron energy increases by absorbing the energy of electric oscillation. In the latter case the electron delivers its energy to the electric oscillation so that the oscillation can build up. This is the case for self-sustained oscillation of the magnetron.

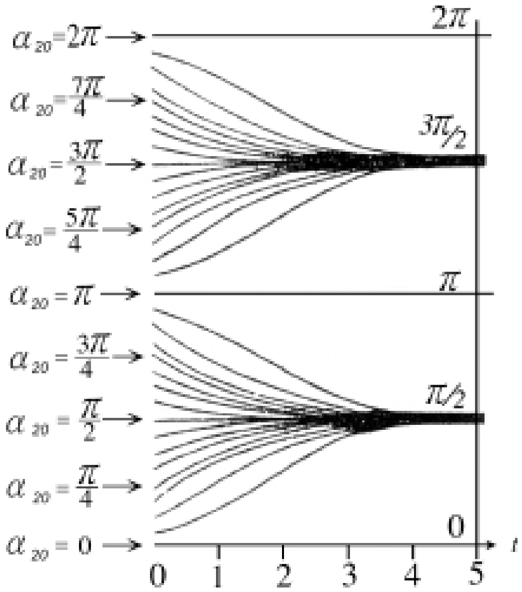


Fig. 5: Temporal changes of the phases α_2 of electrons for different initial phases α_{20}.

Then the oscillation frequency of the magnetron is given by substituting Eqs. (4) and (5) into Eq. (7b) to be

$$\omega = \frac{eH}{2mc} - \sqrt{\left(\frac{eH}{2mc}\right)^2 - \frac{2eU_a}{mr_a^2}} \ .$$

This may be rewritten by using Eq. (1) as

$$\omega = \frac{e}{2mc}\left(H - \sqrt{H^2 - H_c^2}\right) . \qquad (10)$$

3.3. Motion of Electrons in a Split-Anode Magnetron

The electric potential inside the anode due to oscillating potential of the anode segments can generally be expressed in the form of a Fourier expansion as

$$\Phi\ (r, \theta, t) = \mathrm{Re} \sum_{\sigma=1}^{\infty} (A_{\sigma}\, e^{i\omega t} + B_{\sigma}\, e^{i\omega t})\, (re^{i\theta})^{\sigma}. \qquad (11)$$

Here the Fourier coefficients are determined by the Laplace equation with the boundary condition given by the potentials of the anode segments.

Perturbation of this potential on the electron motion will be significant when the electron orbit suffers a secular change at a resonant frequency. The resonant frequency has been calculated in general to be

$$\omega = \sigma\Omega_2 + \tau(\Omega_1 - \Omega_2)\ , \qquad \tau = 1, 2, 3, \ldots\ , \sigma. \qquad (12)$$

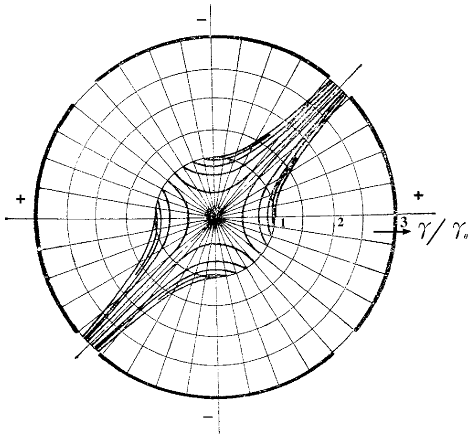


Fig. 6: Trajectories of the center of the rolling motion of electrons in a four-split-anode magnetron, where γ / γ_0 is the relative magnitude of R_2.

Polyphase oscillations may take place in a split anode magnetron, but we restrict our discussions to push-pull oscillations in this paper, because it is the most important.

When the number of the anode segments is even, a push-pull oscillation is possible, where the voltage of every other segments are equal and the voltage of other segments are opposite. For example, in a four-split push-pull magnetron the oscillating voltages of four segments are given by

$$\Phi\,(r_a, \theta, t) = V\cos\omega t, \text{ for } 0 < \theta < \pi/2, \text{ and } \ \pi < \theta < 3\pi/2$$

and

$$\Phi\,(r_a, \theta, t) = -V\cos\omega t\ , \text{ for } \ \pi/2 < \theta < \pi, \text{ and } \ 3\pi/2 < \theta < 2\pi\ .$$

Here we have assumed that the gaps between neighbouring segments are very narrow.

Because of the four-fold symmetry of the potential only terms for

$$\sigma = 2, 6, 10, 14, \ldots \qquad (13)$$

appear in the expansion of Eq. (11). Since higher order terms are usually small, we consider the lowest term in the first approximation.

When $\sigma = 2$, we have $\tau = 0$, 1, and 2. It is found that electrons cannot deliver their energy to the oscillating field in the cases of $\tau = 1$, and 2. Then we consider the case for $\tau = 0$ in the following.

The resonance frequency in this case is then given by $\omega = 2\Omega_2$ from Eq. (12). Calculation of the secular change of electron orbits shows that the aphelion radius either increases

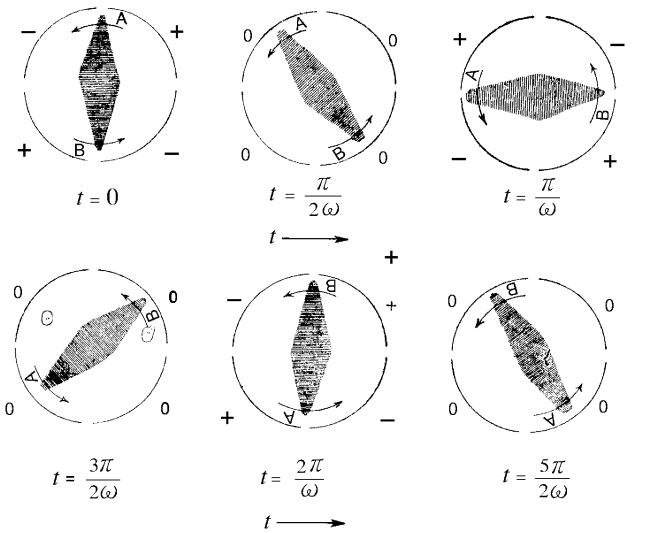


Fig. 7: Rotation of the electron cluster in the four-split-anode magnetron.

immediately, or decreases first, reaches a minimum and then increases. It is noted that this ultimate increase occurs independently of the initial phase of the electron motion. The electron energy decreases as the electron approaches the anode. This means that work is done by the electron on the alternating potential so as to supply energy to the oscillation.

The phases of electron motions, which were initially at random, are calculated to tend to either $\pi/2$ or $3\pi/2$ as shown in Fig. 5. The revolution of the center of the rolling motion of the electron, as given by amplitude R_2 and phase α_2 is found to vary as shown in Fig. 6.

Since electrons are revolving in the laboratory frame at frequency Ω_2, the electron cloud rotates inside the split-anode as shown in Fig. 7. Then the negative charges of the electron clusters induce positive charge in the anode segments whose potentials are rising; and they induce negative charge in the segments whose potentials are sinking. Thus the oscillation in the anode is enforced by the rotating clusters of electrons. This is the mechanism of self-sustained oscillation in the magnetron.

In the cases of the six-split-anode and the eight-split-anode the rotating clusters of electrons are formed as shown in Fig. 8, where the resonance conditions are $\omega = 3\Omega_2$ and $\omega = 4\Omega_2$ respectively.

It is obvious that this mechanism of oscillation by rotating clusters of electrons resembles very much the mechanism of the alternator. The alternating current is generated in the alternator by electromagnetic induction by rotating magnetic poles, while oscillation in the magnetron is excited by electrostatic induction by rotating clusters of electrons.

In the second paper by S. Tomonaga polyphase oscillations in split-anode magnetrons are discussed in general. Comparisons of the theory with experiments are given in the last section of the paper to show that the theoretical results agree satisfactorily with experimental evidences.

4. REFERENCES

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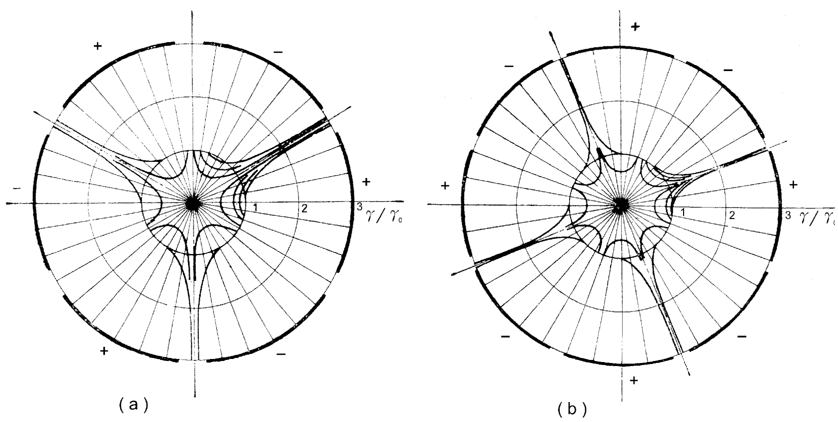

Fig. 8: Trajectories of the center of the rolling motion of electrons (a) in a six-split-anode magnetron, and (b) in a eight-split-anode magnetron.