Lines of Force in Electric and Magnetic Fields

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Assemblages of lines of force are widely used to represent electric and magnetic fields, the density of lines in the neighborhood of a point being proportional to the intensity of the vector at that point. It is not generally recognized, however, that the properties of continuity and individuality of such lines are irrelevant so far as observable electromagnetic phenomenon are concerned.

Contrary to widely held views, it is not possible generally to represent the magnetic field of a linear circuit carrying a steady current by such an assemblage of lines of force which are everywhere continuous. This is because the continuous magnetic lines of force do not generally form simple closed curves, which each link the circuit just once. The interlinkage which is significant is that between the circuit and arbitrarily chosen closed paths of integration of the magnetic vector.

THE electric and magnetic fields appear in Maxwell's equations as vector fields, that is, as vector functions of position in space which are also functions of time. Such a vector function assigns to each point in space at any moment in time, a magnitude and a direction, and that is all. Maxwell's theory is a successful "local action" theory in that a knowledge of these vector fields within a given local region is sufficient to account for or describe adequately all observable macroscopic electromagnetic phenomena within that region. If the region is surrounded by a closed geometric surface, then a knowledge of the field vectors at all points of the surface (and for all time), is sufficient for determining all "outside" influence upon the electromagnetic phenomena within the surface. Further detailed information as to the fields or electromagnetic goings-on outside the surface is unnecessary.

A proper geometric representation of a vector field would be the obvious one of placing an arrow at each point in space with the appropriate direction and length. Such a picture assigns to each space-point a direction and a magnitude and nothing more, and therefore introduces no irrelevant elements. However, another geometric picture is widely used, which, while it does generally properly assign the appropriate direction and magnitude to each point of space, nevertheless may induce the less critical user to introduce irrelevant elements which correspond to no observable electromagnetic phenomena. This is the method of using assemblages of continuous curves, lines of force, spaced so as to give a density at each point of space equal to the magnitude, there, of the vector being represented.

Failure to recognize the irrelevance of certain features of this picture leads many students to incorrect expectations as to the nature of the electric and magnetic fields. Particularly among electrical engineers, Faraday's law of induction is frequently presented in terms of this picture with an associated incorrect portrayal of the magnetic field. Meaningless, or at least not uniquely meaningful terms, such as common and leakage flux of two circuits, are found widely in engineering literature. It is the purpose of this paper to point out the irrelevant elements of the lines of force representation of electric and magnetic fields and some of the incorrect expectations aroused by these irrelevant elements.[1]

Line of Force Representation of a Vector Field

A line of force of an electric or magnetic field is usually defined as a directed continuous curve which at each of its points has the direction of the electric or magnetic vector there. At any point where the vector magnitude is not zero, we may uniquely draw such a line of force. An assemblage of such lines of force is a satisfyingly vivid means

[1] More detailed discussion of these points will be found in the Electrical Essays of the author, Elec. Eng. **66**, 872 (1947); **67**, 58 (1948); **67**, 530 (1948); **67**, 564 (1948); **68**, 449 (1949); **68**, 519 (1949); **68**, 518 (1949); **68**, 615 (1949); **68**, 613 (1949); **68**, 678 (1949); **68**, 677 (1949); **68**, 763–4 (1949); **68**, 762 (1949); **68**, 878 (1949); **68**, 984–5 (1949); **68**, 1081–2 (1949).

for portraying the directions of the vectors of the vector field at the various points of space.

In the immediate neighborhood of a point where the vector intensity is not zero, we may choose the density of lines of force in the assemblage as proportional to the vector magnitude. Then if the divergence of the vector field is zero there, we may prolong the same continuous lines of force away from the region, and the assemblage of these prolonged lines of force will continue by its density to represent the magnitude of the vector field, the lines of force spreading where the vector field weakens, and converging where the vector field strengthens.

If the divergence of the vector field is not zero, then the density of the prolonged lines of force will not continue to correspond to the vector intensity. We may, however, now keep up this correspondence by ending some of the lines or starting new lines as the need arises. Then the density of lines stays proportional to the vector intensity, but lines end or new lines begin with a volume density depending on the sign, and proportional to the magnitude, of the divergence of the vector field. Throughout the region where this assemblage of lines of force can be so constructed, we get a vivid portrayal of the intensity as well as the direction of the vector field with the line endings and beginnings portraying the divergence.

The construction of the assemblage of lines of force as described above may be stopped in two ways. We may arrive at a point where the field intensity is zero. Or the lines of force may sweep around and return to a region where we have already selected and drawn lines of force. It is the latter case which we shall discuss in this paper.

Irrelevance of Individuality and Continuity of a Line of Force

Since it is only the density and direction of the lines of force of the assemblage which are related to the electric vector or magnetic field, it should be quite evident that the individuality and continuity of individual lines of force of the assemblage are entirely irrelevant properties, so far as any observable electromagnetic phenomenon is concerned, postulating as we do that the vector fields are entirely sufficient for describing such phenomena. As we prolong the lines of force, we may at any point end all the lines and start new ones with the same density, without in any way impairing the correspondence between the assemblage of lines of force and the vector field. The assemblage of broken lines serves just as well as the assemblage of continuous lines of force.

No observable electromagnetic phenomenon can exist which involves two points in space, and which depends upon there being a *continuous* line of force joining the points. Such a phenomenon would contradict our postulate of the complete sufficiency of the local *vector* fields for describing local phenomena.

Magnetic Lines of Force Are Generally Not Closed Curves

The complete permissibility of arbitrary breaks in the continuity and individuality of the lines of force in an assemblage representing a vector field comes as a shock to many people, particularly for the case where the divergence of the vector field is zero. It is still more of a shock when they learn that the complete representation of such a divergence-free field by an assemblage of lines of force without such arbitrary discontinuities is, in general, not possible.

Consider the steady magnetic field produced by a current-carrying loop of wire, where the loop lies almost but not completely in one plane. We start an assemblage of lines of force from a limited region and follow them continuously around the wire. After circling the wire, the lines of force return to the region from which they started, but since the loop is not plane, the individual lines do not return to the individual identical points where they originated. The lines may not then be continued on into the initial region, since that would alter the density of lines there. The lines can only be stopped. The density of lines which are thus stopped in this region is just equal to the density of lines which were begun there, so the condition for zero divergence is satisfied. Thus we may construct a satisfactory lines of force representation of the magnetic field of the loop, but the lines of force cannot be completely continuous curves, but must be broken at least once on the average for each circumscription of the wire, if they are to have the proper density.

The author has inquired among a large number of trained physicists and engineers. Much more than half of them believed that the lines of force of the magnetic field of a current loop necessarily were closed curves each linking the loop once. This belief seems to arise from the belief in the universal possibility of the representation of the vector field by an assemblage of *continuous individual* lines of force. Actually, however, the lines of magnetic force will form closed curves each linking the wire loop once, only if the loop lies wholly in a plane.

Interlinkage of Magnetic Lines and Electric Circuits

The interlinkage of two closed curves in space has an intuitive meaning to most of us. It is a topologically invariant property, in that no continuous deformation of the closed curves with neither cutting the other, can alter that mutual property. The interlinkage may be quantitatively defined as the algebraic total number of intersections of either curve with any simply connected two-sided surface bounded completely by the other curve. In this definition, an arbitrary direction along each curve is taken as positive. From this, by a right-hand rule, a positive side is selected for the bounded surface, and thence one arrives at an intersection being called positive if it is made by the intersecting curve passing through the surface from the negative side to the positive, and negative if it passes through the surface in the inverse direction. The interlinkage thus quantitatively defined has for value a positive or a negative integer, or zero.[2]

If one or both of the two curves is open, or not closed, then interlinkage loses its sharp geometrical meaning. It is quite evident that in this case continuous deformation of the curves can always completely undo any degree of apparent, intuitively conceived, interlinkage. The quantitative definition given above also fails, since the open curve no longer completely bounds a simply connected two-sided surface. Since, as has been indicated in the previous section, the lines of magnetic force in general are not closed continuous curves, it has no meaning to speak of the interlinkage of a line of magnetic force with a circuit.

It is quite evident from his writings that Faraday had positive views as the physical significance of continuous lines of force which were quite different from those of this author. It seems highly probable, however, that Faraday also believed that magnetic lines of force always formed closed curves, and this last is demonstrably not true. That Faraday's apparently closed magnetic lines linked electric circuits undoubtedly gave him satisfaction as it apparently gave the magnetic field an intimate and unbreakable association with electric current similar or parallel to the association of the electric field with the electric charge upon which its open electric lines of force terminate.

Maxwell attempted, and successfully, to reduce Faraday's ideas to mathematical form. But in so doing, in his equations at least, he was obliged to strip the irrelevancies from Faraday's ideas. Gone from the equations were the continuous lines of force, reaching from plus charge to minus charge, or forming closed curves linking circuits. There remained only the vital essence, the vector electric and magnetic fields, each giving at each point of space a magnitude and a direction and that is all.

However, Maxwell's equations still assert relations involving interlinkages between magnetic fields and current circuits, but these relations are somewhat subtler than those suggested by Faraday. Applying Stokes' theorem to the appropriate Maxwell's equation we deduce the following for the case of a single electric circuit, which is of course closed. Consider any closed curve in space. Then the integral of the vector magnetic field around that closed curve is proportional to the product of the current magnitude and the number of interlinkages with the electric circuit of the closed curve of integration. The interlinkages then are not of the circuit with closed magnetic lines of force, but of the circuit with arbitrary closed paths of integration.

Likewise, applying Stokes' theorem to that Maxwell's equation which expresses Faraday's law of induction, we find that the integral of the vector electric field around any closed curve is proportional to the time rate of change of the

[2] Alexandroff and Hopf, *Topologie* (Verlag. Julius Springer, Berlin, 1935), pp. 410-417.

integral over any simply connected two-sided surface bounded by that curve of the normal component of the vector magnetic flux density. In the special case that the magnetic flux density lines do all form closed curves, which they generally do not, then an assemblage of them can be constructed whose density is everywhere proportional to the magnetic flux density magnitude, and then the surface integral just referred to will be proportional to the sum of the interlinkages with the closed electric circuit of all the individual closed flux line curves of the assemblage. It is this property of the surface integral for a very special case which causes it to be known even in the general case as the "flux linkage of the circuit."

Maxwell's equations and their integrals discussed above are widely known and correctly used by physicists and engineers. However, also widely accepted among these physicists and engineers are the erroneous elements of Faraday's anticipatory ideas, namely, that there is physical significance in the continuity and individuality of electric and magnetic lines of force, and that the lines of force of the magnetic field form simple closed curves.

The Solution of Differential Equations by Electrical Analog Computers

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Electrical systems for solving total differential equations are classified as high speed or low speed with citations of existing commercial equipment. A symbolism is proposed for the physical components which perform the mathematical operations without elaborating upon the technical details of their design.

The solutions of simultaneous linear algebraic equations, simultaneous differential equations, and differential equations with variable coefficients are discussed and illustrated. The article contains fourteen illustrations of symbols and system wiring schemes.

BECAUSE of the rapid growth of general purpose analog computing equipment, during recent years many engineers and physicists have been unaware of the vast problem-solving facilities now available. Many problems have never been formulated rigorously because of the man-hours required to obtain a solution, assuming a solution to be possible with existing mathematical knowledge. As an illustration, A. C. Hall[1] has pointed out that equations characterizing aircraft motions are commonly of order between twenty and thirty. The existing commercially available office-size analog computing equipment will now enable a scientist to obtain a numerical solution of practically any total differential equation or group of simultaneous equations to a precision of approximately 0.1 percent.

The purpose of this presentation is not to discuss the technical details involved in the construction of a particular computer but to point out the philosophy of an analog computing system. It is intended to give an over-all view of what is happening in the problem-solving process and arouse in the reader a creative interest in analog computers.

Broadly speaking, an analog computing system is one which substitutes a convenient physical system for an inconvenient physical system, the convenient physical system being described by differential equations which are numerically identical to those describing the inconvenient system. In other words, a convenient system is set up to simulate an inconvenient system. For example it might be required to study the motion of a locomotive described by the equation

$$F=Kv+Mdv/dt,$$

where v and t denote velocity and time, respectively.

A convenient physical system which is mathematically identical would be an electric current i

[1] A. C. Hall, Elec. Eng. **69**, No. 5, 433–436 (1950).