ELECTROMAGNETIC PONDEROMOTIVE FORCES WITHIN MATERIAL BODIES

BY JOSEPH SLEPIAN

WESTINGHOUSE ELECTRIC CORPORATION, EAST PITTSBURGH

Read before the Academy, April 26, 1950

Synopsis.—In classical electromagnetism the electrical force acting on a charged probe or small body is operationally defined as that force which must be added to the assumedly completely observable mechanical forces to restore otherwise failing classical particle mechanics. For a large body, rigid or non-rigid, lying in empty space, we may also define a total electromagnetic ponderomotive force as that force which added to the presumably observable total mechanical force restores otherwise failing mechanics. For such a body the total electromagnetic force thus operationally defined, may be determined by applying Maxwell's stress tensor in the empty space surrounding the body.

Very generally it is believed that specifically electromagnetic ponderomotive forces, volume and surface, exist within material bodies. Presumably these electromagnetic forces are to be defined by balancing properly with the mechanical stress tensor within the body.

One may attempt to define this mechanical stress tensor through the mechanical force required to keep the strain unchanged on making a cut along an element of surface within the body. However, in an electromagnetic field the force so obtained is not derivable from a tensor.

We may define as a possible electromagnetic stress tensor any tensor whose components are functions of the field vectors, **E**, **D**, **H** and **B**, and the charge and current densities ρ and **i**, and which in empty space, i.e., where $\mathbf{E} = \mathbf{D}$, $\mathbf{H} = \mathbf{B}$, $\rho = 0$, and $\mathbf{i} = 0$ reduces to Maxwell's electromagnetic stress tensor. Then we define the associated mechanical stress tensor through the vector difference between the calculated electrical surface force for the sides of the cut, and the mechanical force observed there, this difference being derivable from a tensor.

These two largely arbitrary tensors meet all the requirements of mechanics and electromagnetism and no experiment can distinguish between the validities of the various sets of such possible tensors. There is then no physically significant uniquely definable volume and surface electromagnetic ponderomotive force within a material body.

For the interior of a material body in an electric field, a net force tensor may be defined which gives the observable net volume forces, and net forces on surfaces of discontinuity. This net force tensor may be calculated from the volume energy density when such volume energy density exists.

I. Definition of Total Electromagnetic Ponderomotive Force for Particles and Isolated Bodies.—Classical electromagnetism, the electromagnetism of macroscopic bodies, begins with the theory of electrical ponderomotive forces acting upon particles. It is found by experiment that under certain conditions the classical mechanics of these small macroscopic bodies fails. This classical mechanics asserts that for such a small body we must have

$$m\mathbf{a} = \mathbf{F}_m \tag{1}$$

where m is the mass, $\mathbf{a}$ the acceleration, relative to a suitable frame of reference and $\mathbf{F}_m$ the resultant "mechanical" force acting on the particle. $\mathbf{F}_m$ is asserted to be completely determinable from observations on the state of contiguous macroscopic bodies, as for example the twist of the suspension string, and the strain in the supporting bar of a Coulomb experiment, and from the gravitational action of remoter bodies. When electrical phenomena appear, equation (1) fails, and we *define* $\mathbf{F}_e$, the electromagnetic ponderomotive force on the particle, as that vector which restores (1) giving

$$m\mathbf{a} = \mathbf{F}_m + \mathbf{F}_e. \tag{2}$$

Experience then shows that $\mathbf{F}_e$ thus defined can, in a wide variety of cases, be expressed at any point in space empty except for the particle and the means for impressing the mechanical force, $\mathbf{F}_m$, by the equation

$$\mathbf{F}_e = q\left(\mathbf{E} + \frac{1}{c}[\mathbf{v} \times \mathbf{H}]\right) \tag{3}$$

where the scalar q depends on the previous preparation of the particle, and $\mathbf{E}$ and $\mathbf{H}$ are vectors in space independent of q and $\mathbf{v}$, the velocity of the particle. q, $\mathbf{E}$ and $\mathbf{H}$ are thus *defined* by equation (3) except for a multiplicative constant which may then be fixed by some arbitrarily chosen operational definition of the unit of q.

Thus these definitions of $\mathbf{F}_e$, q, $\mathbf{E}$ and $\mathbf{H}$ through the failure of mechanics, rest on the assumption that the mechanical forces which act on the particle are completely known, and that the means for exerting these mechanical forces may have no influence on the electromagnetic field, that is that $\mathbf{F}_e$, q, $\mathbf{E}$ and $\mathbf{H}$ will be independent of the particular means for effecting $\mathbf{F}_m$.

The wide variety of cases referred to above can be described generally as occurring when the dimensions of the small body or particle are small enough compared to the distances from other bodies, excluding the means for impressing the force $\mathbf{F}_m$, and when the small body is prepared so as to make q small enough.

For a large macroscopic body, rigid or non-rigid, mechanics also makes a general assertion, namely that

$$\frac{d}{dt}\mathbf{M} = \mathbf{F}_m \tag{4}$$

where $\mathbf{M}$ is the total momentum of the body, and $\mathbf{F}_m$ is the total impressed mechanical force, which is determinable from the observations of the strains in contacting bodies, and the gravitational action of remoter bodies.

Again in the electromagnetic field (4) fails. Again, we may define a total electromagnetic ponderomotive force, $\mathbf{F}_e$, as that which restores (4) so that

$$\frac{d}{dt}\mathbf{M} = \mathbf{F}_m + \mathbf{F}_e, \tag{5}$$

and again, for this definition to have meaning with content, we must assume that means are available for impressing $\mathbf{F}_m$ which do not affect the electromagnetic field, i.e., which give the same $\mathbf{F}_e$ independent of the means used for effecting the $\mathbf{F}_m$.

II. Limitation of This Paper to Steady Electric Fields.—While the author believes that the conclusions given in this paper are completely general, for the sake of brevity he will limit their exposition to the case of steady electric fields, and therefore to the case of bodies at rest, without electric currents.

III. Maxwell's Stress Tensor and Isolated Bodies.—Given a body, at rest, lying in empty space and in an electric field. The body is subjected to mechanical forces of the type referred to in Section I, which do not influence the field, and which give a total mechanical force $\mathbf{F}_m$ which may not be zero. $\mathbf{F}_e$ for this case, is then defined as $-\mathbf{F}_m$.

Surround the body by a closed surface S. $\mathbf{E}$ defined in Section I, is known at all points of S. Then, according to Maxwell (1), we have

$$\int_S\int \frac{1}{8\pi}(2\mathbf{E}\mathbf{E}\cdot\mathbf{dS} - \mathbf{E}^2\,\mathbf{dS}) = \mathbf{F}_e. \tag{6}$$

We shall not take the space here to establish Maxwell's equation (6). A way of doing so would be to postulate certain additional properties of charged small bodies or macroscopic particles as given by experiment, and to show then that (6) holds if the only matter within the surface S is a finite number, though possibly very large, of such charged particles at rest. We then postulate that the Maxwell field theory is a successful "local action" theory, and that therefore (6) must hold irrespective of the actual nature of the material system within S.

We may not pass from the case of the system of charged particles to that of a continuous body by asserting that such a body may be regarded as being in any meaningful sense a very dense system of charged macroscopic particles. If we proceed to cut a macroscopic larger body into smaller and smaller macroscopic parts, we do not get particles in the sense used in previous paragraphs, because the dimensions of these individual smaller parts do not become small compared to the distance to the neighboring smaller parts. It is only when we reach the microscopic world of electrons and nuclei that this aspect of particles is reached. But at this point macroscopic mechanics loses its meaning, and with it also classical electromagnetism. Certainly, according to our present ideas of the quantum mechanics which governs the microscopic particles, electrons and nuclei, there is no "mechanical force" acting on a microscopic particle, and therefore there is no electrical ponderomotive force as defined in Section I.

The integrand in (6) is a *linear* vector function of the vector element of surface $\mathbf{dS}$. We may speak of it then as a force $\mathbf{f}_e$ acting on $|\mathbf{dS}|$ which is derivable from a tensor, Maxwell's symmetric stress tensor for empty space.

IV. The Mechanical Stress Tensor?—How shall we now define the electrical ponderomotive force within continuous matter? It would seem that again the definition should be through the failure of ordinary mechanics. Presumably, in an electric field, the completely recognizable mechanical forces acting on any arbitrarily chosen continuous volume within the body will not balance according to the mechanics of continuous bodies, and we must invoke a volume electrical force $\mathbf{F}_e$ which we thereby define to restore this balance.

But what are these completely recognizable mechanical forces? In the mechanics of continuous bodies in the absence of electromagnetic fields, it is asserted that in addition to gravitational or inertial volume forces, there is acting on each element of the bounding surface, $\mathbf{dS}$, of the volume, a force, $-\mathbf{f}_m|\mathbf{dS}|$ impressed by the contiguous matter,[1] and that this system of forces is derivable from a mechanical stress tensor, which is a function of the strains in the material. From the way this stress tensor is used in deriving the equations of mechanics, we may conclude that a meaningful operational definition of these mechanical forces is as follows.

Make a physical cut in the material along an element of surface. Introduce means for keeping the strains in the material on both sides of the cut the same as they were before the cut was made. Then the force introduced by these means is the force $-\mathbf{f}_m|\mathbf{dS}|$.

It is not assumed that the cut and the introduced means do not disturb the microstructure and micromechanics of the material. For example, in the case of a fluid the cut and means would cause molecules to be reflected which would otherwise pass through the geometric element of surface $\mathbf{dS}$. It is assumed, however, that in spite of the change in the micromechanics, there is no change in the observable macromechanics.

Now in the case that there is an electric field, let us tentatively continue to define the mechanical forces acting on a volume within matter in the same way. Now, however, just as for the particles and isolated bodies of Section I, we must limit the means used for keeping the strains on the two sides of the cut unchanged by the cut, to such as will also leave the electric fields on the two sides of the cut unchanged. Thus, for a dielectric fluid we may use as means, piston walls or heads made of very thin sheets of perfectly insulating material. But now, as we proceed to show, we run into the dilemma that the mechanical forces $-\mathbf{f}_m|\mathbf{dS}|$ thus found are not derivable from a tensor.

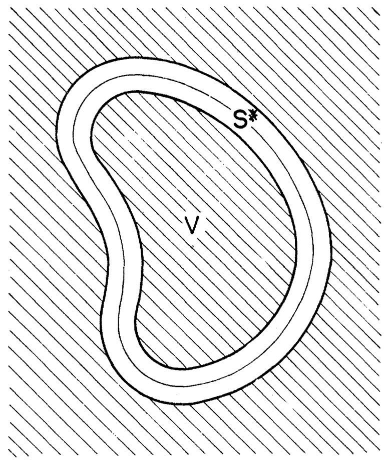

FIGURE 1

Consider a volume V, in a material, surrounded by a closed surface S^*. We now make a thin cut all along S^*, leaving a thin shell of empty space within which there will be an electric field $\mathbf{E}^*$, figure 1. If the means used for impressing the forces $-\mathbf{f}_m|\mathbf{dS}|$ do not introduce charges, then $\mathbf{E}^*$ is related to the fields $\mathbf{E}$ and $\mathbf{D}$ within the material at $\mathbf{S}^*$ by the relations

$$\mathbf{E}^*\cdot\mathbf{dS}^* = \mathbf{D}\cdot\mathbf{dS}^* \tag{7}$$

$$[\mathbf{E}^* \times \mathbf{dS}^*] = [\mathbf{E} \times \mathbf{dS}^*]. \tag{8}$$

Now the volume V, bounded by the shell of empty space along S^* is an isolated body acted on by the purely mechanical external forces $-\mathbf{f}_m|\mathbf{dS}^*|$ in the sense of Section III, and we may apply the equation (6) giving

$$\iint_{S^*} \frac{1}{8\pi}\,(2\mathbf{E}^*\mathbf{E}^*\cdot\mathbf{dS}^* - \mathbf{E}^{*2}\,\mathbf{dS}^*) = -\iint_{S^*} -\mathbf{f}_m|\mathbf{dS}^*| - \mathbf{F}_m \tag{9}$$

where presumably the right-hand member of (9) is the total electrical force acting on V, and $\mathbf{F}_m$ is the total volume distributed impressed mechanical force, or

$$-\mathbf{F}_m - \iint_{S^*} -\mathbf{f}_m|\mathbf{dS}^*| = \iiint_V \mathbf{F}_e\,dV \tag{10}$$

where $\mathbf{F}_e$ is the presumably uniquely determinable ponderomotive electrical force per unit volume within V.

Now, if **dS**, unrelated to **dS***, is an arbitrarily oriented element of surface within the fixed and unchanging shell of empty space, $\frac{1}{8\pi}(2\mathbf{E}^*\mathbf{E}^*\cdot\mathbf{dS} - \mathbf{E}^{*2}\,\mathbf{dS})$ is a linear vector function of **dS**. However, for the vector function of **dS***, $\mathbf{f}_e|\mathbf{dS}^*| = \frac{1}{8\pi}(2\mathbf{E}^*\mathbf{E}^*\cdot\mathbf{dS}^* - \mathbf{E}^{*2}\,\mathbf{dS}^*)$, as the orientation of **dS*** is changed, the bounding walls of the cut must also change, and we may not conclude that $\mathbf{f}_e|\mathbf{dS}^*|$ is a *linear* vector function of **dS***. In fact, applying equations (7) and (8) we see that $\mathbf{f}_e|\mathbf{dS}^*|$ is a cubic function of **n** the unit vector normal to **dS***, and **dS***, if **D** is not equal to **E**.

Hence, $\mathbf{f}_e|\mathbf{dS}^*|$ is not derivable from a tensor and $\iint_{S^*}\mathbf{f}_e|\mathbf{dS}^*|$ is not equal to $\iiint_V \mathbf{F}_e\,dV$, where $\mathbf{F}_e$ is a vector independent of the shape of the volume V. Hence, by (9) $-\mathbf{f}_m|\mathbf{dS}^*|$ is also not derivable from a tensor, and our attempt to define the mechanical stress tensor through the forces which keep the strain unchanged on making a cut, has failed.

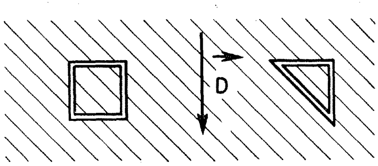

FIGURE 2

We may illustrate the foregoing development by a simple example. Consider a homogeneous dielectric material, with dielectric constant, k, very large, in a uniform field $\mathbf{D} = k\mathbf{F}$, figure 2. Consider a volume which is a unit cube with four faces parallel to **D**. Surround this cube with a cut making an empty shell space. In the two faces of the cut perpendicular to **D**, $\mathbf{E}^* = \mathbf{D}$. For these two faces the contribution to the component parallel to **D** of the Maxwell stress integral of (6) will be, respectively, $-\frac{1}{8\pi}\mathbf{D}^2$ and $+\frac{1}{8\pi}\mathbf{D}^2$. In the remaining four faces $\mathbf{E}^* = \mathbf{E} = \frac{1}{k}\mathbf{D}$ and is nearly zero. The component parallel to **D** of the Maxwell stress integral (9) is then zero, and therefore so also must be that component of the integral of $-\mathbf{f}_m|\mathbf{dS}^*|$ of (9) and (10).

But now consider the volume consisting of the half cube shown in figure 2 with one square face perpendicular to **D**, and the other square face and the two triangular faces parallel to **D**. In the cut along the square face perpendicular to **D**, $\mathbf{E}^* = \mathbf{D}$, and the contribution to the component parallel to **D** of the Maxwell stress integral of (6) is $-\frac{1}{8\pi}\mathbf{D}^2$. For the three faces parallel to **D**, **E*** is nearly zero and the contribution to the Maxwell stress integral is zero. For the remaining diagonal face, **E*** will be nearly perpendicular to the face and of magnitude $|\mathbf{D}|/\sqrt{2}$. The contribution to the component parallel to **D** of the Maxwell stress integral is $+\frac{1}{8\pi}\mathbf{D}^2/2$. The total Maxwell stress integral will then have a component parallel to $\mathbf{D}$ of magnitude $-\frac{1}{8\pi}\mathbf{D}^2/2$, and is not zero. Thus we see that the Maxwell stress integral, and that therefore also the integral of $-\mathbf{f}_m|\mathbf{dS}^*|$ depends on the shape and orientation of the volume considered, and that therefore the forces $\mathbf{f}_m|\mathbf{dS}^*|$ are not derivable from a tensor.

V. The Surface Electrical Ponderomotive Force.—The various formulae which have been proposed for the electrical ponderomotive force in matter, in the literature of classical electromagnetism, give surface forces at the bounding surfaces of material bodies, as well as volume forces within the bodies. This is because the proposed formulae for the volume force involve space derivatives of functions of the field vectors **D** and **E**, and material parameters such as density and dielectric constant. At the boundaries of bodies the field vectors **D** and **E** are generally discontinuous, and also the material parameters as density, etc. In the mathematical application of the volume ponderomotive force formulae, the bounding surface is replaced by a thin layer in which **D**, **E**, and the material parameters vary continuously from their values within the body to their different values just outside the boundary. Application of the volume force formulae to this thin layer then leads to a surface force formula.

The appearance of these surface electrical forces suggests that there may be a way out of the dilemma presented by the fact that the mechanical forces which must be introduced in a cut to keep the strain unchanged, are not derivable from a tensor. We may say that on making the cut, we introduce new electrical ponderomotive forces, namely the surface forces which are related to and calculable from the volume electrical forces. We may say then that the mechanical forces $-\mathbf{f}_m|\mathbf{dS}^*|$ which are introduced into the cut must compensate for the surface electrical forces as well as take the place of the contiguous material in keeping the strain what it was before the cut was made. We might expect that after allowing for the surface electrical force, $\mathbf{f}_e|\mathbf{dS}^*|$, the remaining mechanical force, $(-\mathbf{f}_m + \mathbf{f}_e)|\mathbf{dS}^*|$, will be derivable from a tensor.

We are led then to a tentative circular kind of definition of the mechanical stress tensor and volume electrical ponderomotive force. The mechanical forces $-\mathbf{f}_m|\mathbf{dS}^*|$ in a cut are observed. Then the volume electrical force, $\mathbf{F}_e$ is such as leads to surface forces $\mathbf{f}_e|\mathbf{dS}^*|$ which make $(-\mathbf{f}_m + \mathbf{f}_e)|\mathbf{dS}^*|$ derivable from a tensor, and such that for any continuous volume, and bounding surface S^*,

$$\iiint_V \mathbf{F}_e\,dV + \iint_{S^*} \mathbf{f}_e|\mathbf{dS}^*| = -\iint_{S^*} -\mathbf{f}_m|\mathbf{dS}^*| - \mathbf{F}_m. \quad (11)$$

However, this does not lead to a unique electrical ponderomotive force and mechanical stress tensor.

VI. Non-uniqueness of the Electrical Ponderomotive Force and Mechanical Stress Tensor.—We may readily see that infinitely many volume electrical forces, $\mathbf{F}_e$, with related surface force $\mathbf{f}_e|\mathbf{dS}|$ may be found which will satisfy (11) and which will make $(-\mathbf{f}_m + \mathbf{f}_e)|dS^*|$ derivable from a tensor. For this we may take any tensor whose components are functions of the field vectors, $\mathbf{E}$, $\mathbf{D}$, and the scalar charge density, ρ, and material parameters, such as density or dielectric constant, and which in empty space, where $\mathbf{E} = \mathbf{D}$, and $\rho = 0$, and the material parameters reduce to the appropriate values, becomes identical with Maxwell's electrical stress tensor. Then the divergence of this tensor gives a valid electrical volume force.

Maxwell's stress tensor itself, for empty space, by equation (6) has components,

$$T^M_{xx} = \frac{1}{8\pi}(\mathbf{E}_x^2 - \mathbf{E}_y^2 - \mathbf{E}_z^2)$$

$$T^M_{xy} = \frac{1}{8\pi}(2\mathbf{E}_x\mathbf{E}_y) \qquad (12)$$

etc.

Now by (9) for any volume surrounded by an empty shell which follows the bounding surface S^*,

$$\iint_{S^*} T^M_{ij}\, \mathbf{dS}_j^* = -\iint_{S^*} -\mathbf{f}_{mi}|\mathbf{dS}^*| - \mathbf{F}_m \qquad (13)$$

where T^M_{ij} refers to the empty space outside S^*, and the usual summation convention for repeated subscripts is implied.

We now consider any other tensor T_{ij} which reduces to T^M_{ij} in empty space. Then of course

$$\iint_{S^*} T^{(1)}_{ij}\, \mathbf{dS}_j^* = \iint_{S^*} T^M_{ij}\, \mathbf{dS}_j^* = -\iint_{S^*} \mathbf{f}_{mi}|\mathbf{dS}^*| - \mathbf{F}_m \quad (14)$$

where $T^{(1)}_{ij}$ refers to the values of T_{ij} outside S^*. We now apply Gauss's theorem to the first integral of (14) but take into account the fact that T_{ij} is discontinuous at S^*. We therefore have

$$\iint_{S^*} T^{(1)}_{ij}\, dS_j = \iint_{S^*} (T^{(1)}_{ij} - T^{(2)}_{ij})\, dS_j^* + \iint_{S^*} T^{(2)}_{ij}\, dS_j^*$$

$$= \iint_{S^*} (T^{(1)}_{ij} - T^{(2)}_{ij})\, dS_j^* + \iiint_V \frac{\partial T_{ij}}{\partial x_j}\, dV$$

$$= -\iint_{S^*} -\mathbf{f}_{mi}|dS^*| - \mathbf{F}_m \qquad (15)$$

where $T^{(2)}_{ij}$ refers to values of T_{ij} inside S^*.

Since T_{ij} is a tensor, $\frac{\partial T_{ij}}{\partial x_j}$ is the x_i component of a vector, which we shall now call the volume electric force, $\mathbf{F}_e$. $(T_{ij}^{(1)} - T_{ij}^{(2)})\, dS_j^*$ is the x_i component of a vector, if $\mathbf{dS}^*$ is fixed, and we call it the x_i component of $\mathbf{f}_e|dS^*|$, the surface electrical force related to $\mathbf{F}_e$. Then according to (15) we do have the necessary condition (11).

Furthermore, since V is an arbitrary volume, it follows that $(-\mathbf{f}_m + \mathbf{f}_e)|\mathbf{dS}^*|$ is derivable from a tensor which we call the mechanical stress tensor, M_{ij}.

The non-uniqueness of the electrical and mechanical stress tensors, T_{ij} and M_{ij}, is evident from the method of their derivation. We may see directly, however, that they are not unique in their validity as follows. Let R_{ij} be any tensor function of the electrical field variables, the strain variables, and the electrical and mechanical parameters which we may choose to characterize the matter being examined. Then we may take as also valid electrical and mechanical stress tensors, $T'_{ij} = T_{ij} + R_{ij}$ and $M'_{ij} = M_{ij} - R_{ij}$. Since any experiment which can be performed can only determine the net force on a total body, or the net force applied to an external surface, and since these net forces are completely determinable from the sum of the two tensors, $T_{ij} + M_{ij}$, and since $T'_{ij} + M'_{ij} = T_{ij} + M_{ij}$, no experiment can distinguish between the validities of T'_{ij}, M'_{ij}, and T_{ij}, M_{ij}.

Throughout the literature of the subject, the electrical volume ponderomotive force is assumed to have unique meaning, although a meaningful operational definition is not given. One universal formula offered[2] is

$$\mathbf{F}_e = E(\rho + \rho') = \mathbf{E}\left(\frac{1}{4\pi}\,\mathrm{div}\,\mathbf{D} - \mathrm{div}\,\mathrm{P}\right) = \frac{1}{4\pi}\,\mathbf{E}\,\mathrm{div}\,\mathbf{E}, \quad (16)$$

where $\mathbf{P}$ is the polarization vector

Since

$$\iiint_V \frac{1}{4\pi}\,\mathbf{E}\,\mathrm{div}\,\mathbf{E}\,dV = \frac{1}{8\pi}\iint_S (2\mathbf{E}\mathbf{E}\cdot\mathbf{dS} - \mathbf{E}^2\,\mathbf{dS}) \quad (17)$$

we see that we have here one of the possible but not uniquely valid formulae as described in Section VI.

Another universal formula[3] is

$$\iiint_V \mathbf{F}_e\,dV = \iiint_V \mathbf{P}\cdot\nabla\mathbf{E}\,dV = \frac{1}{8\pi}\iint_S 2\mathbf{E}\mathbf{D}\cdot\mathbf{dS} - \mathbf{E}^2\,\mathbf{dS} \quad (18)$$

again giving a tensor which agrees with Maxwell's in empty space.

For the special case of a material which has a dielectric constant, k, which is a function only of the density, τ, the formula

$$\mathbf{F}_e = \frac{1}{8\pi}\left(-\mathbf{E}^2 \text{ grad } k + \text{grad } \mathbf{E}^2\tau\frac{dk}{d\tau}\right) \tag{19}$$

is frequently given.[4] Again we have that (19) reduces to

$$\iiint_V \mathbf{F}_e\, dV = \frac{1}{8\pi}\iint_S 2\mathbf{ED}\cdot d\mathbf{S} - \left(\mathbf{E}\cdot\mathbf{D} - \mathbf{E}^2\tau\frac{dk}{d\tau}\right) d\mathbf{S} \tag{20}$$

and again on the right of (20) we have a tensor which reduces to Maxwell's stress tensor in empty space where $\mathbf{D} = \mathbf{E}$, and $\tau = 0$.

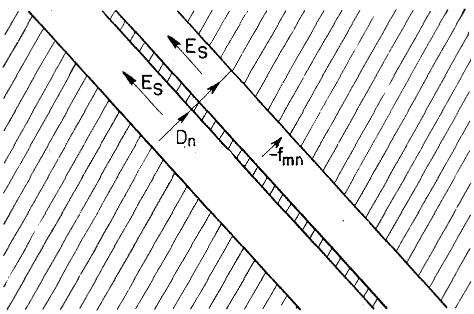

FIGURE 3

Expressions such as (19) are often derived[4] by assuming the existence of an energy density function, u, and equating the increase in the energy of a system, to the work done by applied force in a change of strain. It is clear, however, from the foregoing, that only net forces can be so determined, and that the designation of electrical parts and mechanical parts of such net forces is without meaning.

VII. The Net Stress Tensor and Energy Density.—It is clear that all the information which can be verified concerning the ponderomotive forces on a medium is contained in a knowledge of the field vectors **E** and **D**, and of the net force $-\mathbf{f}_m|dS|$ at a cut, for all orientations of the cut, and at all points in the medium.

From equation (10) modified to include the application of net externally applied volume forces **F**, we have

$$\iint_{S^*} -\frac{1}{8\pi}(2\mathbf{E}^*\mathbf{E}^*\cdot d\mathbf{S}^* - \mathbf{E}^{*2}\, d\mathbf{S}^*) + \mathbf{f}_m|dS^*| = \iiint_V \mathbf{F}\, dV. \tag{21}$$

It follows then that the integrand on the left of (21) defines a tensor which we call the net stress tensor.

If the matter in question has an electromechanical energy density function, u, then $\mathbf{f}_m$ and the net stress tensor may be determined therefrom, but they also have meaning independently of the existence of such a function.

As an example, take the case where there is an energy density u, which is a function of the electric field, **E**, or **D**, and the material density, τ, only.

Referring to figure 3, consider a cut, and in the cut, place a thin slab of material, subject to the same set of surface forces $-\mathbf{f}_m|dS|$ as the sides of the cut. Then in this slab, the field vectors, **E** and **D**, are the same as in the neighboring material.

Now give the slab a shear, slipping the one surface of the slab parallel to the other. Since the volume of the slab, and therefore also its density, and since also the field vectors are unchanged, then the energy of the slab is unchanged. The surface forces $\mathbf{f}_m$, therefore, do no work, during this shear, and therefore the force component parallel to the surface, $\mathbf{f}_{ms}$, is zero.

$$\mathbf{f}_{ms} = 0. \tag{22}$$

Now let the slab be expanded, by motion of its sides in the normal direction so that its volume per unit area is increased by ΔV. The increase in energy within the slab per unit area will be,

$$\begin{aligned} \Delta(uV) = V\Delta u + u\Delta V &= V\left(\frac{\partial u}{\partial \tau}\right)_{E_s,\,D_n} \frac{\Delta\tau}{\Delta V}\,\Delta V + u\Delta V \\ &= \left(-\tau\left(\frac{\partial u}{\partial \tau}\right)_{E_s,\,D_n} + u\right)\Delta V \end{aligned} \tag{23}$$

where the subscripts E_s, D_n, indicate that E_s and D_n are to be kept constant during the differentiation of u with respect to τ.

The increase of energy per unit area in the empty space of the cut will be $-\frac{1}{8\pi}\,(E_s^2 + D_n^2)\;\Delta V.$

$\mathbf{D}_s$ in the slab will be changed by the change in τ, and there will be energy per unit area of the cut supplied through the electric field given by

$$\frac{1}{4\pi}\,\mathbf{E}_s\cdot\left(V\Delta\mathbf{D}_s + [\mathbf{D}_s - \mathbf{E}_s]\right)\Delta V = \frac{1}{4\pi}\left(-\tau\left[\frac{\partial D_s}{\partial \tau}\right]_{E_s,\,D_n} + \mathbf{D}_s - \mathbf{E}_s\right)\Delta V \tag{24}$$

where of course, $\mathbf{D}_s$ is related to $\mathbf{E}_s$, $\mathbf{D}_n$, and τ by

$$\mathbf{E}_s = 4\pi\,\frac{\partial}{\partial D_s}\,(u[\mathbf{D}_s,\,\mathbf{D}_n,\,\tau]). \tag{25}$$

Equating the increase of slab energy (23) minus the loss of energy in the empty space, $\frac{1}{8\pi}\,(\mathbf{E}_s^2 + \mathbf{D}_n^2)\;\Delta V$ to the sum of the energy supplied electrically (24) and the work done mechanically, or $-\mathbf{f}_{mn}\Delta V$, we have

$$\left(-\mathbf{f}_{mn} - \frac{1}{4\pi}\tau\mathbf{E}_s\left[\frac{\partial \mathbf{D}_s}{\partial \tau}\right]_{E_s,\,D_n} + \frac{1}{4\pi}\mathbf{E}_s\mathbf{D}_s - \frac{1}{4\pi}\mathbf{E}_s^2\right)\Delta V = \left[\left(-\tau\left[\frac{\partial u}{\partial \tau}\right]_{E_s,\,D_n} + u\right) - \frac{1}{8\pi}(\mathbf{E}_s^2 + \mathbf{D}_n^2)\right]\Delta V. \quad (26)$$

This, then, gives us

$$\mathbf{f}_{mn} = +\ \tau\left(\frac{\partial u}{\partial \tau}\right)_{E_s,\,D_n} - u - \frac{1}{4\pi}\tau\mathbf{E}_s\left[\frac{\partial \mathbf{D}_s}{\partial \tau}\right]_{E_s,\,D_n} + \frac{1}{8\pi}(\mathbf{D}_n^2 - \mathbf{E}_s^2 + 2\mathbf{E}_s\mathbf{D}_n). \quad (27)$$

To get the *net* mechanical volume force, **F**, which must be introduced to hold a given volume, V, within the medium in equilibrium, we then have (21) which we rewrite as

$$\iiint_V \mathbf{F}\,dV = -\iint_{S^*} \frac{1}{4\pi}\,\mathbf{ED}\cdot\mathbf{dS}^* + \frac{1}{8\pi}\,(\mathbf{D}_n^2 - \mathbf{E}_s^2 - 2\mathbf{E}_n\mathbf{D}_n)\mathbf{dS}^* - \mathbf{f}_{mn}\mathbf{dS}^* \quad (28)$$

Let us further specialize u to be of the form

$$u = \frac{1}{8\pi}k(\tau)\mathbf{E}^2 + W(\tau) = \frac{1}{8\pi}\left(k\mathbf{E}_s^2 + \frac{1}{k}\mathbf{D}_n^2\right) + W = \frac{1}{8\pi}\mathbf{E}\cdot\mathbf{D} + W. \quad (29)$$

Then

$$\tau\left(\frac{\partial u}{\partial \tau}\right)_{E_s,\,D_n} = \frac{1}{8\pi}\left(\mathbf{E}_s^2 - \frac{1}{k^2}\mathbf{D}_n^2\right)\tau\frac{dk}{d\tau} + \tau\frac{dW}{d\tau} = \frac{1}{8\pi}(\mathbf{E}_s^2 - \mathbf{E}_n^2)\,\tau\frac{dk}{d\tau} + \tau\frac{dW}{d\tau} \quad (30)$$

and

$$\tau\left(\frac{\partial \mathbf{D}_s}{\partial \tau}\right)_{D_n,\,E_s} = \tau\frac{\partial}{\partial \tau}(k\mathbf{E}_s) = \mathbf{E}_s\,\tau\frac{dk}{d\tau}. \quad (31)$$

Substituting (31) and (30) into (27)

$$\mathbf{f}_{mn} = -\frac{1}{8\pi}\mathbf{E}^2\,\tau\frac{dk}{d\tau} + \tau\frac{dW}{d\tau} - W + \frac{1}{8\pi}(\mathbf{E}_s\mathbf{D}_s - \mathbf{E}_n\mathbf{D}_n - \mathbf{E}_s^2 + \mathbf{D}_n^2). \quad (32)$$

Substituting into (28)

$$-\iiint_V \mathbf{F}\,dV = \iint_{S^*} \frac{1}{8\pi}\,(2\mathbf{ED}\cdot d\mathbf{S}^* - \mathbf{E}\cdot\mathbf{D}\,d\mathbf{S}^*) + \left(\frac{1}{8\pi}\mathbf{E}^2\,\tau\frac{dk}{d\tau} - \tau\frac{dW}{d\tau} + W\right)d\mathbf{S}^*. \quad (33)$$

The integrand on the right of (33) is now a linear vector function of $\mathbf{dS}^*$, and defines a tensor, the negative of the net stress tensor. If we apply Gauss's theorem, we get

$$-\mathbf{F} = \rho\mathbf{E} - \frac{1}{8\pi}\,\mathbf{E}^2 \text{ grad } k + \text{grad } \frac{1}{8\pi}\,\mathbf{E}^2\,\tau\frac{dk}{d\tau} + \text{grad}\left(W - \tau\frac{dW}{d\tau}\right) \quad (34)$$

where $\rho = \dfrac{1}{4\pi}\text{ div }\mathbf{D}$.

Equation (34) gives the negative of the net volume force which is impressed on the material to hold it in equilibrium. In various places, the first term, the sum of the first two terms and the sum of the first three terms are designated, respectively, as the electric volume force.

[1] In subsequent integrations over a closed surface enclosing matter, the direction of **dS** will be that of the outer normal. This is the reason for the negative sign used here.

[2] Richardson, *The Electron Theory of Matter*, Cambridge University Press, p. 206, 1914.

[3] Page and Adams, *Principles of Electricity*, D. Van Nostrand, New York, 15th Printing, pp. 45–49.

[4] Stratton's *Electromagnetic Theory*, McGraw-Hill, 1941, p. 139. A number of early references are given on pp. 145 and 150.

INFRA-RED BANDS IN THE SPECTRUM OF NH_3

By Darwin L. Wood, Ely E. Bell and Harald H. Nielsen

Mendenhall Laboratory of Physics, The Ohio State University, Columbus

Communicated by R. S. Mulliken, July 25, 1950

Measurements made in this laboratory attempting to establish whether or not ammonia was a component of the earth's atmosphere required the remeasurement in the laboratory of the fundamental bands in the ammonia spectrum. It was found, notably in the bands ν_2 and ν_4, that the spectra were appreciably better resolved than in earlier attempts and it has therefore seemed of interest to look at these again with some care. Measure-