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Charged Right Circular Cylinder

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A paper with the above title appeared in this Journal in 1956 giving the charge distribution on, and capacitance of, an electrified right circular solid conducting cylinder for length-to-diameter ratios of 0.25 to 4. Much more accurate values, calculated for the range 0.125 to 8 on a digital computer, are tabulated in the present paper. Over this range the capacitance is given to 0.2% accuracy by the formula

$$C=[0.708+0.615(b/a)^{0.76}]\times 10^{-10}a \text{ farads},$$

where $2b$ is the length and a the radius. A spherical harmonic expansion for the potential outside the circumscribed sphere is given.

1. INTRODUCTION

A PAPER[1] with the above title which appeared in 1956 gave the capacitances and charge distributions on solid right circular conducting cylinders with length-to-diameter ratios ranging from 0.25 to 4. The labor of solving simultaneous equations and summing series on a desk computer restricted the results to one or two coefficient combinations and made error estimates uncertain. A digital computer calculation with the same formulas, using many coefficient combinations yielded the improved results recorded here.

2. THEORY

The original paper[1] should be consulted for detailed formulas. The method used assumes that the charge densities σ_s on the sides and σ_e on the ends of a cylinder bounded by $z=\pm b$ and $\rho=a$ can be expanded in the form

$$\sigma_s=\sum_{n=0}^{N} A_n(1-b^{-2}z^2)^{n-\frac{1}{3}},\quad \sigma_e=\sum_{m=0}^{M} B_m(1-a^{-2}\rho^2)^{m-\frac{1}{3}}. \quad (1)$$

This A_n is that of the original paper multiplied by $b^{2n-\frac{2}{3}}$ so that both A_n and B_m now have the dimensions of charge density and are of convenient size. Near the corners σ_s and σ_e become infinite properly and match if

$$b^{\frac{1}{3}}A_0=B_0. \quad (2)$$

The potential may be expanded at the origin in the form

$$V(z,\rho)=\sum_{p=0}^{\infty}(-\tfrac{1}{4}\rho^2)^p(p!)^{-2}\partial^{2p}V(z,0)/\partial z^{2p} \quad (3)$$

and it must be constant inside the cylinder so that

$$\partial^{2p}V(z,0)/\partial z^{2p}=\delta_p{}^0V_0, \quad (4)$$

where $\delta_p{}^0$ is one if $p=0$ and zero if $p\neq 0$.

[1] W. R. Smythe, J. Appl. Phys. **27**, 917 (1956).

The potential and its p even derivatives are calculated from (1). The solution of the simultaneous equations (4) together with (2) then yield $p+2=N+M+2$ of the lowest-order coefficients in (1). The number of significant digits carried determines the optimum values of N and M for a given b/a ratio. The best choice is that for which A_n and B_m give potentials nearest V_0 at pole and equator. The check point nearest the origin always gives V_0 to six or seven places and is a less sensitive indicator of the optimum value than the more remote check point. If M or N is too large, then the contributions of the individual terms, which alternate in sign, becomes much larger than their sum, greatly reducing the accuracy of the latter. Most terms in (3) were found by summing the hypergeometric series and verified by the recursion formulas.

It should be emphasized that *all* coefficients must be used in any field calculation for the omission of any term may give large errors. There are usually from three to five combinations of A_n and B_m in the optimum range. The one using the fewest coefficients is the one tabulated and is nearly as accurate as the best combination. For rough values where b/a is near one, the results in the original paper may be used. A calculation of more than eight place accuracy would be needed to improve the results in Table I. In many cases the last digit is not significant because of roundoff errors. The potentials at pole and equator for the tabulated A_n and B_m are given as well as the corresponding proportional displacement $\Delta b/b$ and $\Delta a/a$ of the actual unit potential surface from that of the cylinder.

3. FIELD OF CHARGED CYLINDER

The potential outside the charged cylinder can be found by integration of $\sigma dS/(4\pi\epsilon R)$ over the surface of the cylinder, where R is the distance of dS from the field point. In the region adjacent to the walls it appears that numerical integration must be used for the side terms. The potential of the end terms can be expressed as a series of oblate spheroidal harmonics valid everywhere.

TABLE I. Charge density coefficients for unit potential

$b=a/8$	
$A_0=1.5224294$	$\epsilon V_p=1.0000001$
$B_0=0.76121470$	$\epsilon V_e=0.9921134$
$B_1=-0.38086366$	
$B_2=0.29232287$	$\Delta b=0.0000016b$
$B_3=-0.09010631$	$\Delta a=-0.005a$
$b=a/4$	
$A_0=1.0685084$	$B_5=-0.03877087$
$A_1=0.1384610$	
$B_0=0.67311809$	$\epsilon V_p=1.0000001$
$B_1=-0.30192902$	$\epsilon V_e=1.0007242$
$B_2=0.41105811$	
$B_3=-0.37639902$	$\Delta b=-0.00000008b$
$B_4=0.18777757$	$\Delta a=-0.0006a$
$b=a/2$	
$A_0=0.77698601$	$B_6=13.756435$
$A_1=0.19084155$	$B_7=-7.7502333$
$A_2=-0.16151002$	$B_8=2.5138393$
$A_3=0.06343914$	$B_9=-0.3580848$
$B_0=0.61669421$	
$B_1=-0.43141315$	$\epsilon V_p=0.9999999$
$B_2=1.8073652$	$\epsilon V_e=1.0000026$
$B_3=-5.6072409$	
$B_4=11.453745$	$\Delta b=0.0000004b$
$B_5=-15.478182$	$\Delta a=-0.000003a$
$b=a$	
$A_0=0.55941519$	$B_3=-3.9209295$
$A_1=0.24032463$	$B_4=6.2012014$
$A_2=-0.46123818$	$B_5=-5.6962826$
$A_3=0.71795706$	$B_6=2.8184609$
$A_4=-0.67534061$	$B_7=-0.5816914$
$A_5=0.34357563$	
$A_6=-0.07271528$	$\epsilon V_p=1.0000002$
$B_0=0.55941519$	$\epsilon V_e=1.0000001$
$B_1=-0.35462716$	
$B_2=1.4624910$	$\Delta b=-0.0000004b$
	$\Delta a=-0.0000002a$
$b=2a$	
$A_0=0.40842489$	$B_1=-0.41788413$
$A_1=0.23612315$	$B_2=1.6823388$
$A_2=-0.49824155$	$B_3=-3.2845929$
$A_3=1.0437740$	$B_4=2.9374045$
$A_4=-1.6104248$	$B_5=-0.9796062$
$A_5=1.7462732$	
$A_6=-1.2940218$	$\epsilon V_p=0.9999970$
$A_7=0.62430746$	$\epsilon V_e=0.9999998$
$A_8=-0.17688137$	
$A_9=0.02234862$	$\Delta b=0.000003b$
$B_0=0.51458312$	$\Delta a=0.0000004a$
$b=4a$	
$A_0=0.30232821$	$B_1=-0.09459217$
$A_1=0.19496655$	
$A_2=-0.20714722$	$\epsilon V_p=0.9954321$
$A_3=0.17065210$	$\epsilon V_e=0.9999999$
$A_4=-0.07932248$	
$A_5=0.01551192$	$\Delta b=0.004b$
$B_0=0.47991610$	$\Delta a=0.0000003a$
$b=8a$	
$A_0=0.22262099$	$\epsilon V_p=1.0234341$
$A_1=0.20155115$	$\epsilon V_e=1.0000000$
$A_2=-0.17770411$	
$A_3=0.09907634$	$\Delta b=-0.007b$
$A_4=-0.02279702$	$\Delta a=-0.0000000a$
$B_0=0.44524199$	

Outside the circumscribed sphere the potential may be written as a double series of spherical harmonics.

$$V=\frac{a}{2\epsilon}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p(C_e+C_s)(2p+2s)!\,a^{2p}b^{2s}}{4^p(p!)^2(2s)!\,r^{2s+2p+1}}\times P_{2s+2p}(\cos\theta),\quad (5)$$

where

$$C_e=a\sum_{m=0}^{M}B(p+1,\,m+\tfrac{2}{3})B_m,$$

$$C_s=b\sum_{n=0}^{N}B(s+\tfrac{1}{2},\,n+\tfrac{2}{3})A_n,\quad (6)$$

and $B(x,y)$ is a beta function.

TABLE II. Capacitance of right circular cylinder in farads for various length to diameter ratios, lengths in meters.

b/a	Capacitance	Var	Num	Eq. (7)
0	$0.708347\times10^{-10}a$	±0	...	$0.708\times10^{-10}a$
$\frac{1}{8}$	$0.8312\times10^{-10}a$	±8	5	$0.833\times10^{-10}a$
$\frac{1}{4}$	$0.9214\times10^{-10}a$	±3	5	$0.923\times10^{-10}a$
$\frac{1}{2}$	$1.07251\times10^{-10}a$	±6	3	$1.072\times10^{-10}a$
1	$1.32576\times10^{-10}a$	±2	3	$1.323\times10^{-10}a$
2	$1.75036\times10^{-10}a$	±3	2	$1.750\times10^{-10}a$
4	$2.467\times10^{-10}a$	±1	4	$2.472\times10^{-10}a$
8	$3.700\times10^{-10}a$	±6	3	$3.696\times10^{-10}a$

4. CAPACITANCE

At a great distance the potential has the form $q/(4\pi\epsilon r)$ so that the charge and hence the capacitance may be found from the first term ($p=0$, $s=0$) of (5). This was done for many combinations giving a range of values wider than that anticipated in the original paper. The values given in Table II are the means of those given by all combinations in the optimum range. The digit in the adjacent column is the amount by which the last digit must be varied to cover all these combinations including that in Table I. The next column gives the number averaged. The $b=8$ capacitance is exactly $8\epsilon a$. The following very simple formula gives the capacitance with an accuracy of 0.2% or better over the range from $b=0$ to $b=8a$.

$$C=[0.708+0.615(b/a)^{0.76}]\times10^{-10}\,a\text{ farads.}\quad (7)$$

The values from this formula appear in the last column of Table II.