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Current distributions for a two-phase material with chequer-board geometry

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Abstract. The current distribution is investigated for a two-phase material with chequer-board geometry and a low conductivity ratio σ_2/σ_1 ($\equiv s^2$). The current density at small distances r from the corners where the phases meet varies as r^{-1-s}, in the limit of small s. The current lines form circle arcs in the poorly conducting phase and straight lines in the well conducting phase. For small s, the Joule heat is concentrated to the phase boundaries. As a consequence, the effective conductivity of alternately packed cubes is $2(\sigma_1\sigma_2)^{1/2}$.

1. Introduction

In a study of composite materials or percolation phenomena, one is often forced, by the complexity of the problem, to investigate simple geometries of the two constituent phases. In one such simple system, one considers the conductivity transverse to the axis of prisms with square cross section, alternately arranged as in a regular chequer-board (e.g. Milton *et al* 1981, Fogelholm and Grimvall 1983). It is well known (e.g. Bruggeman 1935, Dykhne 1970) that the effective conductivity is $\sigma_e = (\sigma_1\sigma_2)^{1/2}$ irrespective of the ratio of the conductivities σ_1 and σ_2. The current distribution, however, depends strongly on the ratio σ_2/σ_1 ($\equiv s^2$). Consider a geometry such as in figure 1. As s becomes small, it is intuitively clear, and also supported by numerical calculations (Fogelholm and Grimvall 1983) that the current density becomes very high at the edges of the cylinders. It is the purpose of this paper to investigate this singular behaviour of the current density when s tends to 0.

We shall deal separately with the current distributions in the poorly and well conducting phases. This is done in §§ 2 and 3 respectively. In § 4, the analytic results are used to draw conclusions about related systems.

2. The current distribution in the poorly conducting phase (σ_2)

Let $j(r, \varphi)$ be the current density in phase 2, with polar coordinates as in figure 2. The total current through a specimen of width W, height H (along the external field) and prism length D, with the phases arranged as in figure 1, is

$$I = 2(W/L)\int_0^{L/2} j_z(r, 0)D\ \mathrm{d}r. \tag{1}$$

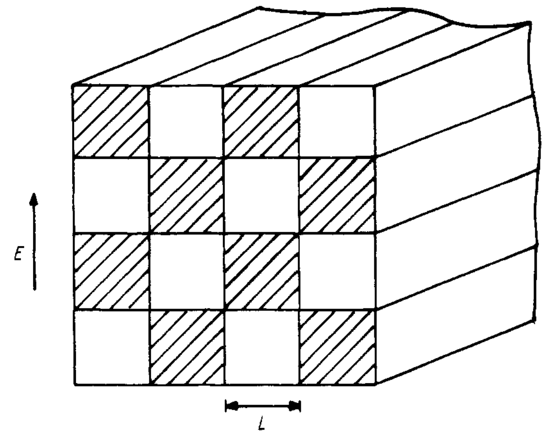

Figure 1. A piece of a material which effectively forms a two-dimensional two-phase system. The external field is applied along the z axis.

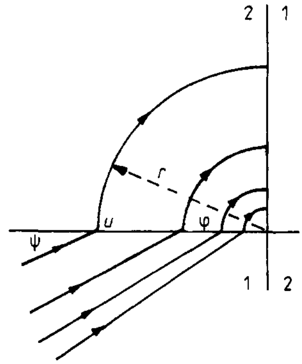

Figure 2. In the poorly conducting phase, σ_2, the current lines for small conductivity ratio σ_2/σ_1 and near the corner of the phase form circle arcs. In the well conducting phase, σ_1, the current lines in the same limit form straight lines.

Dykhne (1970) has shown that the Joule heat is equal in the two phases, for any s (see also Fogelholm and Grimvall 1983). Equating the macroscopic and the microscopic expressions for the Joule heat leads to

$$\frac{I^2}{\sigma_e}\left(\frac{H}{WD}\right)\left(\frac{L^2}{WH}\right) = 8\int_0^{L/(2\cos\varphi)}\int_0^{\pi/4}\frac{j^2(r,\varphi)}{\sigma_2}Dr\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{2}$$

In spite of a considerable effort, we have not been able to derive an explicit analytic result for j, valid in the entire specimen and for any s. However, equations (1) and (2) suffice to determine the singular behaviour of $j(r, \varphi)$ for small r and s. We make the *ansatz* for j, in polar coordinates,

$$j(r,\varphi) = r^{-(1-\alpha)}[A(r,\varphi)\hat{e}_r + B(r,\varphi)\hat{e}_\varphi], \tag{3}$$

where $A(r, \varphi)$ and $B(r, \varphi)$ are bounded in the integration domain of equation (2), and α is to be determined.

In the limit of small s, when the Joule heat is generated mainly close to the edges (small r), it is a good approximation to replace the integration limit $L/(2\cos\varphi)$ in relation (2) by $L/2$. Then, equations (1)–(3) yield

$$\frac{\pi}{4}\frac{\alpha}{s} = \frac{\alpha}{2}\left(\int_0^{L/2} B(r,0)r^{-1+\alpha}\,\mathrm{d}r\right)^2 \bigg/ \left(\frac{4}{\pi}\int_0^{L/2}\int_0^{\pi/4}[A^2(r,\varphi)+B^2(r,\varphi)]r^{-1+2\alpha}\,\mathrm{d}r\,\mathrm{d}\varphi\right). \tag{4}$$

The right-hand side of relation (4) is of the order 1, and hence $\alpha = \beta s$, where $\beta \sim 1$. We now require that $\mathrm{div}\, \boldsymbol{j} = 0$ and $\mathrm{rot}\, \boldsymbol{E} = \sigma_2^{-1}\, \mathrm{rot}\, \boldsymbol{j} = 0$, with $\boldsymbol{j}$ from equation (3). Taking out the most singular contribution when $r \to 0$, we get

$$\alpha A + \partial B/\partial\varphi = 0, \qquad \alpha B - \partial A/\partial\varphi = 0, \tag{5}$$

or $\partial^2 A/\partial\varphi^2 + \alpha^2 A = 0$ and $\partial^2 B/\partial\varphi^2 + \alpha^2 B = 0$. Therefore, when s is small, $A(r, \varphi)$ and $B(r, \varphi)$ vary slowly with φ. At a phase boundary between two media with very different conductivities, the current in the poorly conducting phase is almost perpendicular to the boundary. Hence, $A \ll B$ in equation (3); in fact, $|A/B| \simeq s^2$ (see below).

If we neglect A in relation (4) and approximate $B(r, \varphi)$ by its limit $j_0 B_0(s)$ when $r \rightarrow 0$, we may use relation (1), with $I = j_0 WD$ (j_0 is the average current density) to get

$$j(r, \varphi) \simeq j_0 \frac{B_0(s)}{r^{1-\alpha}} \hat{e}_\varphi \simeq j_0 \frac{s}{r^{1-s}} \left(\frac{L}{2}\right)^{1-s} \hat{e}_\varphi \tag{6}$$

for small r and s. Then the current lines are circle segments.

3. The well conducting phase (σ_1)

To get the current distribution in the well conducting phase it is convenient to use the following result, implicit in Dykhne (1970); see also Fogelholm and Grimvall (1983). The power density at a particular point remains unaltered if phases 1 and 2 are interchanged and the external applied field is rotated 90°. Since the power density in the poorly conducting phase is approximately independent of the angle φ, for small r, this should be the case also in the well conducting phase. Then, the singular part of j is given by straight lines as in figure 2. The angle $\psi(u)$ varies with the distance u from the corner as

$$\tfrac{1}{4}\pi - \psi(u) \simeq (u/\tfrac{1}{2}L)^s. \tag{7}$$

The continuity equation is satisfied if the current density, close to the phase boundary and for small u, is

$$|j| = (1/\sin\psi)\,|j(u, 0)| \tag{8}$$

with $j(u, 0)$ from equation (6). The condition for the tangential components of the electric field is satisfied by relations (3), (7) and (8) if $|A/B| \simeq s^2$.

4. Discussion and conclusions

4.1. *The concentration of the Joule heat to corners*

Consider the expression (3) for j in phase 2. Let p be the ratio between the Joule heats generated within the areas with $r \leqslant R$ and $r \leqslant L/2$ respectively. One immediately finds that

$$R/(\tfrac{1}{2}L) = (p)^{1/2s}. \tag{9}$$

As an example, let $p = 0.5$ and $s = 0.1$. Then $R/(\tfrac{1}{2}L) = 1/32$, i.e. half of the total Joule heat in the specimen is generated within approximately 0.1% of its volume.

4.2. *Almost touching corners*

Let the side of the square for one of the phases in figure 1 shrink so that a gap, filled with the other phase, is created at the edges where the phases meet. The effective conductivity σ_e changes radically, for small s, even with a very narrow gap. Numerical calculations by Milton *et al* (1981), for the analogous problem of the effective dielectric constant, show that with $\sigma_1 = 100$ and $\sigma_2 = 1$ (i.e. $s = 0.1$), a gap of width $(2\sqrt{2}/100)\,(L/2)$ yields $\sigma_e = 5.15$, to be compared with $\sigma_e = 10$ when the corners touch. This result is consistent with the qualitative conclusion in § 4.1 that a substantial fraction of the Joule heat, for touching corners and $s = 0.1$, is generated within a distance $\simeq 0.03(L/2)$ from the corner.

At distances $r = R'$ much larger than the width of the gap but still with $R' \ll \frac{1}{2}L$, the current distribution must be essentially unaltered. Points of equal Joule heat per area then form approximate circles. This is in agreement with numerical results by Fogelholm and Grimvall (1983).

4.3. *The conductivity of regularly arranged cubes*

Consider cubes with conductivities σ_1 and σ_2, regularly arranged to form the three-dimensional generalisation of the chequer-board. The external field $\boldsymbol{E}$ is parallel to a cube edge. From the results above it follows that the Joule heat is predominantly generated in the immediate vicinity of the cube edges (perpendicular to $\boldsymbol{E}$), when s is small. Corner effects are negligible, since a non-uniform current density along the cube edges (for a given total current) would increase the entropy production and therefore not correspond to the true current density. Let I be the total current through the sample, and σ_e be the effective (macroscopic) conductivity. Divide I equally between the edges and take the Joule heat at each edge as given in § 2. The total Joule heat is $\propto I^2/\sigma_e$. It follows that

$$\sigma_e = (\sigma_1 \sigma_2)^{1/2} f(s) \tag{11}$$

where $f(s)$ is a smoothly varying function of σ_2/σ_1 with $f(0) = f(\infty) = 2$ and $f(1) = 1$. Since σ_e has cubic symmetry, it is isotropic.

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