$T<T_c$ takes the system through the coexistence curve and M changes discontinuously to $M>0$. The dashed curve below the T axis is the coexistence curve for the point C having $M>0$.

V. SUMMARY AND DISCUSSION

We have demonstrated that a mechanical model can simulate both first- and second-order phase transitions for certain values of the parameters. This was shown by direct analysis of the equations of motion and by examination of the minima of the effective potential energy. It was then demonstrated that the latter method is similar to the Landau theory of continuous phase transitions. The energy-position graphs in Figs. 3–6 are of the same general shape as those in Ref. 15 for a ferromagnet and Ref. 16 for a general free energy.

The analogy between the phase diagrams of the mechanical model and the ferromagnet is also very close if the hysteresis properties of the ferromagnet are included. Equivalently, if the mass m could tunnel from the higher energy local minimum to the lower in the metastable state, the mechanical system would simulate an ideal ferromagnet. The quantum mechanical version of this model ($\alpha=0$) is considered in Ref. 17.

Finally, it would be interesting to build a working model of this system. Although finding a method for making quantitative measurements may require some imagination, the qualitative aspects of the phase transitions would be easy to observe.

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Magnetic dipole oscillations and radiation damping

Daniel R. Stump and Gerald L. Pollack

Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824

(Received 23 October 1995; accepted 8 April 1996)

We consider the problem of radiation damping for a magnetic dipole oscillating in a magnetic field. An equation for the radiation reaction torque is derived, and the damping of the oscillations is described. Also discussed are runaway solutions for a rotating magnetic dipole moving under the influence of the reaction torque, with no external torque.

I. INTRODUCTION

When a compass needle is put in the earth's magnetic field, it oscillates about its equilibrium position for a few seconds before coming to rest. The energy of oscillation has been dissipated by friction. Even if there were no friction, however, the oscillations would still be damped, although very slowly, because a compass needle is an oscillating magnetic dipole and radiates electromagnetic energy.

Figure 1 shows a magnetic dipole, with magnetic moment $\mathbf{m}$, which is free to rotate in the x,y plane about a pivot fixed at the origin. The magnetic moment $\mathbf{m}(t)$ is

$$\mathbf{m}(t)=m_0(\cos\phi(t)\hat{\mathbf{i}}+\sin\phi(t)\hat{\mathbf{j}}). \tag{1}$$

The equation of motion is that the torque equals the rate of change of angular momentum, $\mathbf{N}=d\mathbf{L}/dt$. The torque is $\mathbf{N}=\mathbf{m}\times\mathbf{B}=-m_0B\sin\phi\,\hat{\mathbf{k}}$ for the magnetic field $\mathbf{B}=B\hat{\mathbf{i}}$; the angular momentum is $\mathbf{L}=I\dot{\phi}\hat{\mathbf{k}}$, where I is the moment of

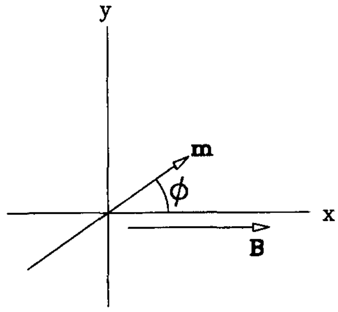

Fig. 1. The magnetic dipole **m** is free to rotate in the x,y plane. The magnetic field $\mathbf{B}=B\hat{\mathbf{i}}$ is constant. The displacement angle ϕ is the angle between **m** and **B**.

inertia of the magnetic dipole about the pivot axis. The equation of motion is therefore

$$\ddot{\phi}=-\frac{m_0 B}{I}\sin\phi. \tag{2}$$

This equation has the same form as the equation of motion for a pendulum, so ϕ oscillates about 0.

II. RADIATION AND RADIATION REACTION

Any magnetic dipole that varies with time radiates electromagnetic waves. One interesting case, which has practical applications for antennas and is often considered in textbooks, is a circular loop of wire in the x,y plane which carries an harmonic current $i(t)=i_0\cos\omega t$ in the azimuthal direction. For that case, the magnetic moment is always parallel to the z axis, but its magnitude varies with time. In contrast, for our problem the magnitude of the dipole moment is constant in time, but its x- and y components vary.

A. Fields of an oscillating magnetic dipole

The electric and magnetic fields are $\mathbf{E}=-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\nabla\times\mathbf{A}$, where $\mathbf{A}(\mathbf{r},t)$ is the retarded vector potential.[1,2] The equations for the fields in the radiation zone are[3,4]

$$\mathbf{E}(\mathbf{r},t)=\frac{\mu_0}{4\pi c r}\hat{\mathbf{r}}\times\ddot{\mathbf{m}}(t_r) \tag{3}$$

and

$$\mathbf{B}(\mathbf{r},t)=-\frac{\mu_0}{4\pi c^2 r}[\ddot{\mathbf{m}}(t_r)-\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\ddot{\mathbf{m}}(t_r)]. \tag{4}$$

In Eqs. (3) and (4) $t_r=t-r/c$ is the retarded time.

Poynting's vector **S**, the energy radiated per unit area per unit time, is given by $\mathbf{S}=\mathbf{E}\times\mathbf{B}/\mu_0$. For Eqs. (3) and (4), we obtain

$$\mathbf{S}(\mathbf{r},t)=\frac{\mu_0}{16\pi^2 c^3 r^2}[\ddot{\mathbf{m}}^2(t_r)-(\ddot{\mathbf{m}}(t_r)\cdot\hat{\mathbf{r}})^2]\hat{\mathbf{r}}. \tag{5}$$

B. Radiated power and the radiation reaction

In order to find the total instantaneous radiated power, $P(t)$, for this problem, we integrate $\mathbf{S}(\mathbf{r},t)\cdot\hat{\mathbf{r}}$ over the surface of a sphere, centered at the origin with the surface in the radiation zone. Integration of the first term in Eq. (5) is straightforward because it is proportional to $\ddot{\mathbf{m}}^2(t_r)$, a quantity independent of direction. Integration of the second term, however, involves $(\ddot{\mathbf{m}}(t_r)\cdot\hat{\mathbf{r}})^2$ which depends on polar and azimuthal angles. The final result is

$$P(t)=\oint\mathbf{S}(\mathbf{r},t)\cdot\hat{\mathbf{r}}dA=\frac{\mu_0}{6\pi c^3}\ddot{\mathbf{m}}^2(t_r)=\frac{1}{6\pi\epsilon_0 c^5}\ddot{\mathbf{m}}^2(t_r). \tag{6}$$

For comparison, the time-averaged power radiated by the current-loop dipole described in the first paragraph of Sec. II, for which $\mathbf{m}_{\text{loop}}=(m_0\cos\omega t)\mathbf{k}$, may be obtained from Eq. (6): $\langle P_{\text{loop}}\rangle=m_0^2\omega^4/12\pi\epsilon_0 c^5$ (Ref. 6).

Because the electromagnetic radiation carries off energy, the kinetic energy of the dipole decreases as the dipole radiates. Therefore, in the dynamical equation $\mathbf{N}=d\mathbf{L}/dt$, we must understand **N** to include a radiation reaction torque, in addition to the torque exerted by the external magnetic field. Work done by the radiation reaction torque reduces the energy of the dipole, as an equal amount of energy appears in the electromagnetic radiation. We will derive an equation for $\mathbf{N}_{\text{rad}}$, the effective radiation reaction torque on **m**, by making some approximations that are valid because the rate of energy radiation is very small.

We write the equation of motion of the dipole as

$$\mathbf{N}_{\text{ext}}+\mathbf{N}_{\text{rad}}=\frac{d\mathbf{L}}{dt}, \tag{7}$$

where $\mathbf{N}_{\text{ext}}=\mathbf{m}\times\mathbf{B}$ is the external torque. Because $\mathbf{N}_{\text{ext}}$ and **L** are in the z direction, so also is $\mathbf{N}_{\text{rad}}=N_{\text{rad}}\mathbf{k}$. We obtain $\mathbf{N}_{\text{rad}}$, in the customary way,[1,2] by requiring that the work done by $\mathbf{N}_{\text{rad}}$ on **m** during a time interval (t_1,t_2) is equal but opposite to the energy radiated during that time interval. That is, since the rate of work is $N_{\text{rad}}\dot{\phi}$, we require

$$\int_{t_1}^{t_2}N_{\text{rad}}\dot{\phi}(t')dt'=\frac{-1}{6\pi\epsilon_0 c^5}\int_{t_1}^{t_2}\ddot{\mathbf{m}}^2(t')dt'. \tag{8}$$

[There is a subtle point here: The time t' in Eq. (8) is not t, previously used in Eq. (6) to denote the time when the radiation passes a sphere of radius r, but rather t_r, the time when the radiation is emitted. In the equations that follow, we drop the prime on t'.] Equation (8) may be written, by recasting $\ddot{\mathbf{m}}^2$, as

$$\int_{t_1}^{t_2}N_{\text{rad}}\dot{\phi}dt=\frac{-1}{6\pi\epsilon_0 c^5}\int_{t_1}^{t_2}\left[\frac{d(\dot{\mathbf{m}}\cdot\ddot{\mathbf{m}})}{dt}-\dot{\mathbf{m}}\cdot\dddot{\mathbf{m}}\right]dt. \tag{9}$$

How should we choose the time interval (t_1,t_2)? If we let it be infinitesimal, then Eq. (9) would mean that the instantaneous rate of work done by $\mathbf{N}_{\text{rad}}$ is equal but opposite to the power radiated. Instead, we choose (t_1,t_2) to be one cycle, or a few cycles, of the dipole oscillation. This choice is useful because the motion may be approximated as periodic over a few cycles, i.e., the slight decay may be neglected, because the power radiated is very small. In this approximation the first term in square brackets in Eq. (9), which is an exact derivative, integrates to 0 because $\mathbf{m}(t)$ is periodic. Then, equating the integrands of the other terms in Eq. (9) gives

$$N_{\text{rad}}\dot{\phi}=\frac{1}{6\pi\epsilon_0 c^5}\dot{\mathbf{m}}\cdot\dddot{\mathbf{m}}. \tag{10}$$

Finally, using Eq. (1) to express $\dot{\mathbf{m}}\cdot\dddot{\mathbf{m}}$ in terms of $\phi(t)$, and cancelling factors of $\dot{\phi}$, we find

$$N_{\rm rad}=\frac{m_0^2}{6\pi\epsilon_0 c^5}(\dddot{\phi}-\dot{\phi}^3). \qquad (11)$$

Equation (11) is the radiation reaction torque for a special case of the motion of a magnetic dipole, in which the dipole moment is always in the x,y plane, as in Fig. 1 and Eq. (1). A general formula for the radiation reaction torque, in which the dipole moment $\mathbf{m}$ moves with arbitrary orientation in three dimensions is, $\mathbf{N}_{\rm rad}=\mathbf{m}\times\dddot{\mathbf{m}}/6\pi\epsilon_0 c^5$.

It is interesting to compare the radiation reaction torque $N_{\rm rad}$ in Eq. (11) and the radiation reaction force on a moving charge q, derived by Abraham and Lorentz, which is $\mathbf{F}_{\rm rad}=q^2\ddot{\mathbf{v}}/6\pi\epsilon_0 c^3$. Both $N_{\rm rad}$ and $\mathbf{F}_{\rm rad}$ depend on the third derivative of the dynamical coordinate, ϕ and $\dddot{\mathbf{x}}$, respectively. But $N_{\rm rad}$ also depends on $\dot{\phi}$, through the nonlinear term $\propto\dot{\phi}^3$. Because of the nonlinear term, the reaction torque is *nonzero for constant angular velocity*. We may understand this physically: A uniformly rotating dipole radiates, so there must be a reaction torque if $\dot{\phi}$ is constant. In contrast, a uniformly moving charge does not radiate, so the reaction force on it is zero for constant velocity.

Another interesting difference between these two related problems of radiation reaction is that in our dipole case we are considering radiation reaction in the rest frame of the radiator. In comparison, the radiation reaction for an accelerated charge is inevitably associated with a moving frame.

III. MOTION WITH RADIATION DAMPING

We now modify Eq. (2) for the motion of the magnetic dipole of Fig. 1, by including the radiation reaction torque. The result, which is just Eq. (7) in terms of $\phi(t)$, is the nonlinear, third-order differential equation

$$I\ddot{\phi}=-m_0 B\sin\phi+\frac{m_0^2}{6\pi\epsilon_0 c^5}(\dddot{\phi}-\dot{\phi}^3). \qquad (12)$$

Equation (12) may be called the Abraham–Lorentz equation of motion for this problem.

A complete solution of Eq. (12) requires finding $\phi(t)$ for arbitrary initial angular position $\phi(0)$, velocity $\dot{\phi}(0)$, and acceleration $\ddot{\phi}(0)$, because it is a third-order equation. That includes motions for which the initial kinetic energy of the oscillator is large enough that it spins around through many revolutions until it loses enough energy, by radiation, to settle down to weakly damped oscillatory motion. The complete solution of Eq. (12) is unknown. In this section we discuss approximations to its oscillatory solutions, using the fact that the radiation damping term is small.

A. Radiation reaction is a small, damping effect

In order to examine the effect of the radiation reaction torque on the motion, we first show that the effect is weak, by making numerical estimates for three model magnetic dipole systems. To start, we rewrite Eq. (12) as

$$\left(\frac{\ddot{\phi}}{\omega_0^2}\right)=-(\sin\phi)+\frac{m_0^2\omega_0}{6\pi\epsilon_0 c^5 I}\left(\frac{\dddot{\phi}-\dot{\phi}^3}{\omega_0^3}\right), \qquad (13)$$

in which $\omega_0=(m_0B/I)^{1/2}$ is the frequency of small-amplitude oscillations from Eq. (2). The three terms in parentheses are dimensionless, and of order unity for oscillatory motion. Therefore, the magnitude of the dimensionless coefficient $\Delta\equiv m_0^2\omega_0/(6\pi\epsilon_0 c^5 I)$ is a measure of the effect of radiation reaction.

1. Numerical examples

(1) Consider first a magnetized iron needle in the shape of a cylindrical bar, 1 mm in diameter and 1 cm long, oscillating at room temperature in a magnetic field. If the bar is at the saturation magnetization of Fe, about 1.7×10^6 A/m, then $m_0=1.3\times10^{-2}$ A m^2 and $I\approx5\times10^{-10}$ kg m^2. In a field $B=0.1$ T we have ω_0 (Fe needle) $\approx2\times10^3$ rad/s, and $\Delta=1\times10^{-24}$. (In the Earth's magnetic field of 5×10^{-5} T, this magnetic needle would oscillate at about 6 Hz.)

(2) Next consider a superconducting current loop of radius 1 cm, made up of, say, 100 turns of 10 mil, Nb_3Sn wire, oscillating at 4 K in a magnetic field. If the current in the wire is about 500 A, corresponding to the critical current density of 10^6 A/cm^2, and the loop is free to oscillate about an axis through the center and in its plane, then $I\approx1.3\times10^{-7}$ kg m^2. In a field $B=0.1$ T we have ω_0 (superconducting loop)$\approx3\times10^3$ rad/s, and $\Delta=2\times10^{-20}$. We have assumed here that the current is constant, although current changes would be induced in the superconducting loop as it oscillates in the external magnetic field. These changes are negligibly small if, as is the case for this loop, the external field is much smaller than the field produced in the loop by the original current.

(3) As the final system consider a neutronlike oscillator. An actual neutron does not oscillate, but rather precesses, in a magnetic field. We consider instead a "classical neutron," modeled as a solid sphere having the mass, radius, and magnetic moment of a neutron, $m_0=-0.966\times10^{-26}$ A m^2, oscillating about an axis through its center. For this sphere $I\approx5\times10^{-58}$ kg m^2, so that in a field $B=0.1$ T we have ω_0 ("classical neutron") $\approx1\times10^{15}$ rad/s and $\Delta=6\times10^{-13}$.

For all three of these systems the coefficient Δ is much smaller than unity, so we conclude that radiation reaction generally has a small effect on the motion.

2. Damped oscillations

Next we analyze the solution of Eq. (13) including radiation reaction. For large-amplitude oscillations the nonlinearity of the equation is important, and analysis is difficult: Even the undamped oscillation involves Jacobian elliptic functions. We postpone treatment of large-amplitude oscillations until Sec. IV. However, for small oscillations the problem may be linearized. In that case the undamped oscillations satisfy $\ddot{\phi}=-\omega_0^2\phi$. Because the damping is weak, we may approximate $\dddot{\phi}$ by $-\omega_0^2\dot{\phi}$ in the radiation reaction torque, which becomes $N_{\rm rad}=-(m_0^2/6\pi\epsilon_0 c^5)(\omega_0^2+\dot{\phi}^2)\dot{\phi}$, where we have (somewhat inconsistently) kept the nonlinear term in $N_{\rm rad}$. This result shows that for small oscillations $N_{\rm rad}$ produces damping: Both the $\dddot{\phi}$ and $-\dot{\phi}^3$ terms act in the direction opposite to the angular velocity. The amplitude of oscillation decreases exponentially in the linear approximation, with the time constant $\tau_{\rm decay}=12\pi\epsilon_0 c^5 I/m_0^2\omega_0^2$.

B. Computer solution of the Abraham–Lorentz equation

One of the difficulties in finding computer solutions to Eq. (13) is that the radiation reaction torque is so small. If we introduce the nondimensional time variable $\xi=\omega_0 t$, we may write Eq. (13) as

$$\phi''=-\sin\phi+\Delta(\phi'''-\phi'^3), \qquad (14)$$

in which primes denote differentiation with respect to ξ, and Δ is the dimensionless damping coefficient defined after Eq.

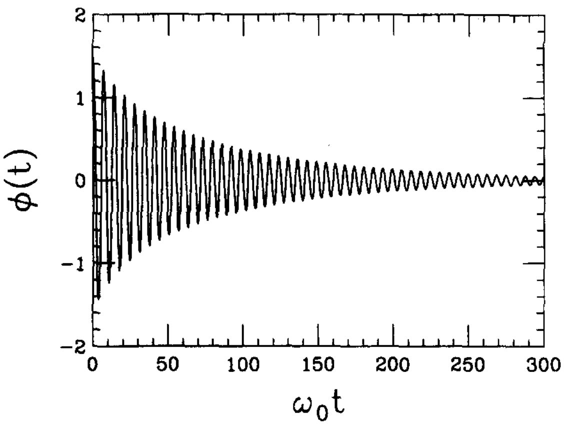

Fig. 2. A computer-generated solution of Eqs. (12) and (14), the Abraham–Lorentz equation of motion for an oscillating magnetic dipole, with damping coefficient Δ equal to 0.02. The abscissa is $\omega_0 t$ where ω_0 is the angular frequency for small oscillations, and the ordinate is the displacement angle $\phi(t)$. The initial conditions for this figure are $\phi(0)=\pi/2$ and $\dot{\phi}(0)=0$.

(13). For physical values of Δ, which are, as we have seen, less than 10^{-12}, it is difficult to construct and display computer solutions to this equation. However, for larger values of Δ one can do this readily. Figure 2 is such a plot of the solution $\phi(t)$ vs $\omega_0 t$ with $\Delta=0.02$. The initial amplitude in Fig. 2 is $\phi(0)=\pi/2$, not a small amplitude. This result shows by direct numerical calculation that N_{rad} produces damping.

C. Self-acceleration of a magnetic dipole with no external torque

In our derivation of the radiation reaction torque N_{rad}, we assumed that the dipole oscillation is approximately periodic. This approximation is valid for a dipole in a magnetic field, because the undamped motion is periodic and the radiation damping is small. However, there must also be radiation reaction for nonperiodic motion, and presumably Eq. (11) also describes the reaction torque in that case.[7] Therefore, to study further the meaning of N_{rad}, we now suppose that Eq. (11) is generally valid, and consider the case of zero external field. Neglecting radiation reaction the equation of motion would just be $\ddot{\phi}=0$, with solution $\phi(t)=\phi(0)+\dot{\phi}(0)t$, i.e., constant angular velocity. But, if radiation reaction is included, the equation of motion has unphysical solutions that describe runaway self-acceleration.

For zero external field, Eq. (12) for the motion of the dipole becomes

$$I\ddot{\phi}=\frac{m_0^2}{6\pi\epsilon_0 c^5}\,(\dddot{\phi}-\dot{\phi}^3). \tag{15}$$

We will later analyze Eq. (15) completely, but as a first step we consider the linearized equation, obtained by dropping the $\dot{\phi}^3$ term. The linearized equation is a good approximation if $\dot{\phi}$ is small, but we are more interested in the linearized equation as a first step toward understanding Eq. (15) rather than as an approximation to Eq. (15).

The linearized equation may be written as

$$\dot{\alpha}(t)=\frac{1}{\tau}\,\alpha(t), \tag{16}$$

where $\alpha(t)=\ddot{\phi}(t)$ is the angular acceleration, and $\tau=m_0^2/(6\pi\epsilon_0 c^5 I)$ is a parameter of the magnetic dipole with units of time. The general solution of Eq. (16) is

$$\alpha(t)=\alpha(0)e^{t/\tau}. \tag{17}$$

This is a runaway solution if $\alpha(0)\neq 0$, because then the acceleration increases without bound with t on a time scale of the order of τ, a time too small for physical measurements. For our three model dipoles: $\tau(\text{Fe needle})=9\times 10^{-28}$ s, τ (superconducting loop)$=5\times 10^{-24}$ s, and τ(“classical neutron”)$=5\times 10^{-28}$ s. The angle $\phi(t)$ corresponding to Eq. (17) is

$$\phi(t)=c_0+c_1 t+\tau^2\alpha(0)e^{t/\tau}, \tag{18}$$

where c_0 and c_1 are constants. Because there is no external field, and hence no external torque, the radiation reaction itself produces the acceleration, which we call *self-acceleration.* The unphysical runaway solutions of the *linearized* equation are mathematically equivalent to the runaway solutions in the classical theory of radiation reaction on an accelerating charge.[1,2]

The linearized version of Eq. (15), as well as Eq. (15) itself, is a third-order differential equation. To determine the motion from initial conditions we must specify three initial values, $\phi(0)$, $\dot{\phi}(0)$, and $\ddot{\phi}(0)$. In order to exclude the runaway solutions (of the linearized equation) we must require that the initial acceleration $\alpha(0)$ is zero. For that choice, $\phi(t)=c_0+c_1 t$ is just a linear function of time, i.e., the dipole has constant angular velocity. However, this is not a solution of the complete *nonlinear* equation. Physically we know that a rotating dipole will be damped because of its radiation; the nonlinear term $\dot{\phi}^3$, in the expression for N_{rad}, ensures that constant angular velocity is not a solution of the equation of motion.

We now return to the complete nonlinear equation of motion, Eq. (15), and ask whether it has runaway solutions. We already know that the linear approximation to Eq. (15) does have runaway solutions, but the linear approximation breaks down because $\dot{\phi}(t)$ increases without bound for a runaway solution, making the nonlinear term $\dot{\phi}^3$ large. Thus one interesting question is whether the nonlinear term somehow prevents the solution from running away. The answer is that the nonlinear term does not kill the runaway solution, but rather *increases* the runaway self-acceleration.

Equation (15) may be written as the two coupled equations

$$\dot{\omega}=\alpha \quad \text{and} \quad \dot{\alpha}=\frac{1}{\tau}\,\alpha+\omega^3, \tag{19}$$

in which ω is the angular velocity. From Eqs. (19) one can easily see how the nonlinear term ($\propto\omega^3$) increases the runaway self-acceleration: If α and ω are positive then the right-hand sides in Eqs. (19) are both positive, so α and ω are increasing with t; in this case the ω^3 term makes a *positive* contribution to the change of α, and so increases the self-acceleration. This analysis shows that there exist runaway solutions of Eq. (15), but a more thorough analysis is instructive.

1. Phase-plane trajectories

To describe the solutions of Eqs. (19) we plot the phase-plane trajectories. We choose the phase-plane coordinates to be $x=\tau\omega$ and $y=\tau^2\alpha$, which are dimensionless, and seek the trajectories which are solutions to Eqs. (19). The slope of any trajectory is

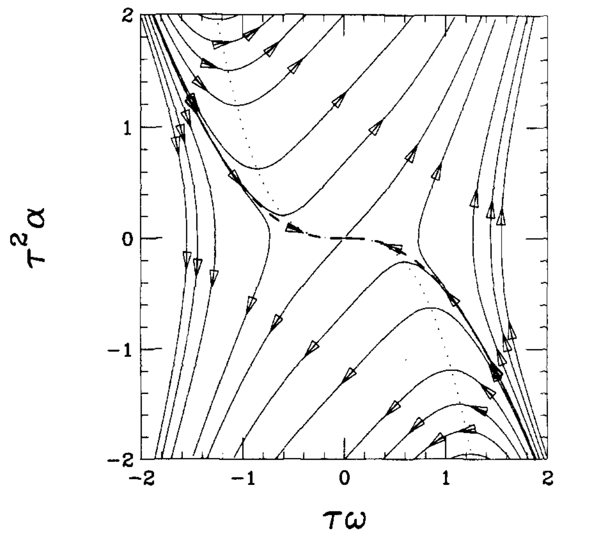

Fig. 3. Phase-plane trajectories for a dipole subject to the radiation reaction torque by Eq. (15). These trajectories were generated from Eq. (20). The ordinate is $y=\tau^2\ddot{\phi}$, and the abscissa is $x=\tau\dot{\phi}$. Solid curves are typical runaway trajectories with arrowheads showing the direction of motion. The two trajectories shown as dashed curves, which converge to the fixed point at the origin, are the only trajectories that remain finite.

$$\frac{dy}{dx}=\frac{\tau^2\dot{\alpha}}{\tau\dot{\omega}}=1+\frac{x^3}{y}. \tag{20}$$

The phase-plane portrait is shown in Fig. 3, which was generated by computer from Eq. (20). Every point in the plane of Fig. 3 represents a dipole with the corresponding angular velocity $\omega=x/\tau$ and angular acceleration $\alpha=y/\tau^2$. One and only one trajectory curve goes through each point in the plane, except for the fixed point at the origin. Typical trajectories are shown in Fig. 3, with arrowheads indicating the direction of motion. The qualitative features of Fig. 3 can be understood by considering the slope in various regions. Note, for example, that the slope at any point on the y axis ($\omega=0$) is 1, and the slope at any point on the x axis ($\alpha=0$) is infinity. The slope anywhere on the curve $y=-x^3$, which is shown as a dotted curve and is not itself a trajectory, is zero.

In the phase-plane portrait there is just one fixed point, namely, $(\omega,\alpha)=(0,0)$, and it corresponds to the dipole at rest. This solution is an unstable equilibrium, in that almost all perturbations away from (0,0) are runaway solutions. There are, however, two *finite trajectories*, and only two, shown as dashed curves, that converge to the fixed point. They correspond to damping of the rotation, as the dipole comes to rest at (0,0), so they are physically acceptable solutions. Therefore for any initial angular velocity $\omega(0)$ there does exist a unique angular acceleration $\alpha(0)$ for which the subsequent trajectory remains finite. For this to happen, the point $(\omega(0),\alpha(0))$ lies on one of the two dashed curves, so that the trajectory does not run away but rather approaches the fixed point as $t\to\infty$. [Asymptotically, as $t\to\infty$, the finite trajectories have $\omega(t)\sim\pm(2\tau t)^{-1/2}$.] The finite trajectories separate trajectories that run away to $\alpha=+\infty$ and $\alpha=-\infty$. Any other initial values of ω and α, except those which lie on the finite trajectories, result in a runaway solution.

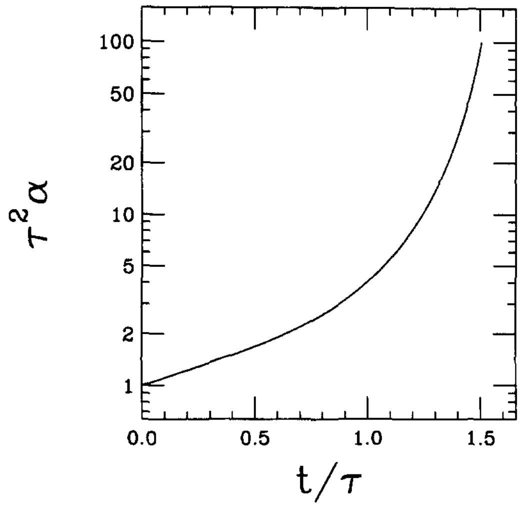

Fig. 4. A computer-generated solution to Eq. (19), showing a typical runaway solution in terms of the dimensionless angular acceleration $\tau^2\alpha(t)$ and the dimensionless time t/τ. The initial values for this solution are $\tau^2\alpha(0)=1$ and $\omega(0)=0$.

2. An analog mechanical problem

The phase-plane portrait in Fig. 3 is mathematically identical to the following, fictional, mechanics problem. Consider a particle of unit mass, moving along the x axis, with coordinate x and velocity $\dot{x}=y$, under the influence of a conservative force F_c with potential energy $V(x)=-x^4/4$, which is unbounded below, plus an additional force $F_v=+y$. One can picture F_v as a "negative viscous force" because it is proportional to the velocity but acts *in the direction of motion.* For this analog problem, the slope of the phase-plane curve at (x,y) is $dy/dx=\dot{y}/\dot{x}=(+y+x^3)/y$, which is identical to Eq. (20). Both forces F_c and F_v drive trajectories to infinity in the phase plane.

Figure 4 shows a computer solution of Eqs. (19) plotting $\tau^2\alpha(t)$ vs t/τ for the initial values $\tau^2\alpha(0)=1$ and $\omega(0)=0$, which yields a runaway solution. The solution runs away even faster than exponentially, reaching $\alpha=\infty$ in a finite time![8]

In the case of an accelerating charge it is well known that excluding the runaway solutions implies the phenomenon of *acausal preacceleration.*[1,2] The same is true for a radiating dipole with zero external field. In order to exclude a runaway solution for $t\geq 0$ we must specify just the right initial acceleration $\alpha(0)$, depending on $\omega(0)$, to lie on the finite trajectory which returns to the fixed point. For example, consider a dipole disturbed by an external impulsive torque $N_{\text{ext}}=K\delta(t)$ at time $t=0$.[9] Because the equation of motion includes $\dddot{\phi}$, the impulse produces a discontinuity in α

$$\alpha(0+)-\alpha(0-)=-\frac{K}{\tau I}. \tag{21}$$

To avoid a runaway solution for $t>0$, the phase-plane point $(\omega(0),\alpha(0+))$ must lie on the finite trajectory. Thus the dipole must have undergone a preacceleration during $t<0$, such that just before the impulse its phase-plane point is $(\omega(0),\alpha(0-))$. Thus the classical theory of radiation reaction for a magnetic dipole has the same unphysical dilemma—either a runaway solution or acausal preacceleration—as the theory for a radiating charge.

IV. DECAY OF OSCILLATIONS

We have seen that oscillations of a magnetic dipole will be slowly damped as it radiates energy. In this section, we derive and solve an equation which describes the time dependence of the damping. Runaway solutions are not relevant to this problem, because there is a restoring torque due to **B**. This calculation is interesting because we keep the full nonlinearity of the problem. We carry out the calculation using perturbation theory.

The idea we use is that the energy of the oscillating dipole may be expressed in terms of ϕ_m, the amplitude of the oscillation. If we neglect damping, then the energy of the oscillator is

$$\frac{1}{2} I\dot{\phi}^2(t) - m_0 B \cos\phi(t) = -m_0 B \cos\phi_m, \tag{22}$$

where ϕ_m is constant. However, the real oscillator loses energy as it radiates power at the rate $P(t)$ given in Eq. (6). Therefore, the amplitude, which we will call $\phi_m(t)$ to emphasize its time dependence, slowly decreases. We will express the total energy of the oscillator in terms of $\phi_m(t)$ and equate the rate of loss of energy to $P(t)$. The calculation requires comparatively unfamiliar operations on Jacobian elliptic functions.[10,11] We proceed in steps, first finding a suitable expression for $P(t)$, then for $E(t)$, and, finally, setting $dE(t)/dt = -P(t)$, which is the equation we seek to solve.

Equation (6) gives the power $P(t)$ in terms of $\ddot{\mathbf{m}}^2$. Using Eq. (1) we rewrite $P(t)$ in terms of ϕ

$$P(t) = \frac{m_0^2}{6\pi\epsilon_0 c^5} [\ddot{\phi}^2(t) + \dot{\phi}^4(t)], \tag{23}$$

in which t is the time when the radiation is emitted.

We now introduce the notation

$$a(t) = \sin\frac{\phi_m(t)}{2}. \tag{24}$$

The solution to Eq. (2), for oscillations of arbitrary amplitude, with the initial conditions $\phi_0=0$ and $\dot{\phi}_0>0$, may be written as

$$\sin\frac{\phi(t)}{2} = \sin\frac{\phi_m}{2}\,\mathrm{sn}\left(\omega_0 t, \sin\frac{\phi_m}{2}\right) = a\ \mathrm{sn}(\omega_0 t, a). \tag{25}$$

The function $\mathrm{sn}(\omega_0 t, a)$ is a Jacobian elliptic function, which is periodic with period $T = 4K(a)/\omega_0$, where $K(a)$ is the complete elliptic integral. Some properties of these functions are given in the Appendix.

Because the rate of radiation is very small, we may approximate $a(t)$ as a constant over the period of one or a few cycles, at least for the purpose of calculating $P(t)$. Differentiating Eq. (25) [approximating $a(t)$ as a constant], and using the Appendix, we obtain $\dot{\phi}^2(t) = 4a^2\omega_0^2\,\mathrm{cn}^2(\omega_0 t, a)$, and $\ddot{\phi}^2(t) = 4a^2\omega_0^4\,\mathrm{sn}^2(\omega_0 t, a)\mathrm{dn}^2(\omega_0 t, a)$. Thus the power is

$$P(t) = \frac{2m_0^2 a^2\omega_0^4}{3\pi\epsilon_0 c^5} [\mathrm{sn}^2(\omega_0 t, a)\mathrm{dn}^2(\omega_0 t, a) + 4a^2\,\mathrm{cn}^4(\omega_0 t, a)]. \tag{26}$$

All of the Jacobian elliptic functions in Eq. (26) are periodic with period T, so $P(t)$ is also periodic. We may replace $P(t)$ by its average over a quarter period, $K(a)/\omega_0$, because a significant decrease of energy only occurs after many cycles and so the rate of decrease may be approximated by the average over a cycle. The average of the quantity in square brackets in Eq. (26) is

$$[\ ]_{\mathrm{av}} = \frac{1}{K(a)} \int_0^{K(a)} \{\mathrm{sn}^2(u,a)\mathrm{dn}^2(u,a) + 4a^2\,\mathrm{cn}^4(u,a)\} du \equiv \Pi(a^2), \tag{27}$$

where the variable of integration is $u = \omega_0 t$. By the further change of integration variables from u to x where $x = \mathrm{sn}(u,a)$, the function $\Pi(a^2)$ may be re-expressed as

$$\Pi(a^2) = \frac{1}{K(a)} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}} [x^2(1-a^2x^2) + 4a^2(1-x^2)^2]. \tag{28}$$

Note that $\Pi(a^2)$ is a dimensionless function independent of physical parameters. The final form for the average power radiated by an oscillating magnetic dipole is

$$P_{\mathrm{av}} = \frac{2m_0^2\omega_0^4}{3\pi\epsilon_0 c^5} a^2 \Pi(a^2). \tag{29}$$

In obtaining Eq. (29) we treated $a(t)$ as a constant.

We are now ready to describe the decay of $a(t)$. The energy of the oscillation at time t is, by Eq. (22),

$$E(t) = -m_0 B \cos\phi_m(t) = m_0 B[2a^2(t) - 1]. \tag{30}$$

Therefore, the relation $dE(t)/dt = -P_{\mathrm{av}}(t)$ may be written as an integrodifferential equation for $a^2(t)$,

$$\frac{da^2(t)}{dt} = -\frac{m_0\omega_0^4}{3\pi\epsilon_0 c^5 B} a^2(t)\Pi(a^2(t)). \tag{31}$$

If $\Pi(a^2)$ is constant, then Eq. (31) implies that a^2, which is a measure of the amplitude of the motion and also of the energy, decays exponentially with time. For small oscillations, $\Pi(a^2)$ is nearly constant. For large oscillations, however, because of the nonlinearity of the problem, $\Pi(a^2)$ does depend on a^2. Finally, we may rewrite the equation in terms of a dimensionless time variable Θ as

$$\frac{da^2}{d\Theta} = -a^2\Pi(a^2), \tag{32}$$

where $\Theta = (m_0\omega_0^4/3\pi\epsilon_0 c^5 B)t$. Figure 5 shows a computer-generated solution of Eq. (32).

For small amplitudes ($a \ll 1$) we have $K(a) \to \pi/2$ and $\Pi(a^2) \to 1/2$. In that limit $a(t) \propto \exp(-\Theta/4)$, i.e., the oscillations decay exponentially with time constant $\tau_{\mathrm{decay}} = 4t/\Theta = 12\pi\epsilon_0 c^5 B/m_0\omega_0^4$. We can see how slow this decay is by evaluating it for our three modeled dipoles. The results for $B = 0.1$ T are τ_{decay}(Fe needle)$= 9\times 10^{20}$ s, τ_{decay}(superconducting loop)$= 4\times 10^{16}$ s, and τ_{decay}("classical neutron")$= 2\times 10^{-3}$ s. The value of τ_{decay} for our "classical neutron" model is surprisingly short, but its physical significance is limited because the behavior of neutrons in a magnetic field must be treated by quantum mechanics.

V. CONCLUSIONS

It is interesting to note the range of the two times that are built into the classical electrodynamics of oscillating magnetic dipoles, namely, the self-acceleration time τ of Sec. III C, which is very short, and the decay time τ_{decay} of Sec. IV, which is very long. The product of these times is something simple, viz., $\tau\tau_{\mathrm{decay}} = 2/\omega_0^2$, where $\omega_0 = (m_0 B/I)^{1/2}$ is the

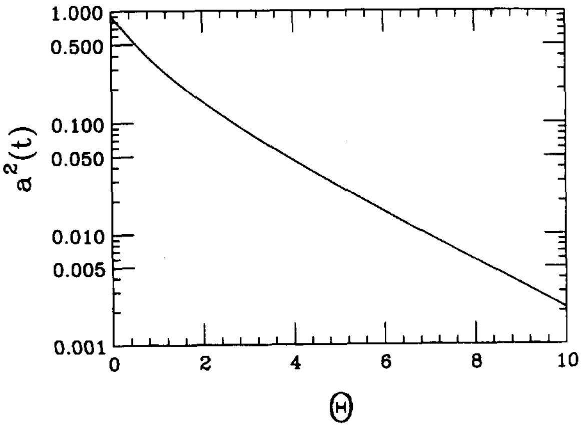

Fig. 5. A computer-generated solution of Eq. (32), the integrodifferential equation for decay of oscillations of a magnetic dipole in a magnetic field. The ordinate is $a^2(t)=\sin^2(\phi_m(t)/2)$, where $\phi_m(t)$ is the amplitude of oscillations at time t, and the abscissa is the dimensionless variable Θ, proportional to time, defined below Eq. (32). We note that when $a^2(t)$ is small it decays exponentially in time, giving the straight-line portion of the curve at the right. The initial condition for this figure is $a^2(0)=0.9$, which corresponds to $\phi_m(0)=143°$.

angular frequency for small oscillations, which appeared in Eq. (2) at the very beginning of this problem.

APPENDIX

The Jacobian elliptic function $\mathrm{sn}(u,a)$ is defined implicitly by

$$u=\int_0^{\mathrm{sn}(u,a)} \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}}. \tag{A1}$$

The function $\mathrm{sn}(u,a)$ varies between $+1$ and -1, and is periodic in u with period $4K(a)$, where $K(a)$ is the complete elliptic integral

$$K(a)=\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}}. \tag{A2}$$

It is conventional to define also the functions $\mathrm{cn}(u,a)$ and $\mathrm{dn}(u,a)$ by

$$\mathrm{cn}^2(u,a)=1-\mathrm{sn}^2(u,a), \quad \text{and} \quad \mathrm{dn}^2(u,a)=1-a^2\mathrm{sn}^2(u,a). \tag{A3}$$

We used the derivative identities

$$\frac{d\,\mathrm{sn}}{du}=\mathrm{cn}\;\mathrm{dn}, \quad \text{and} \quad \frac{d\,\mathrm{cn}}{du}=-\mathrm{sn}\;\mathrm{dn}. \tag{A4}$$

[1]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.

[2]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, 1989), 2nd ed.

[3]H. C. Ohanian, *Classical Electrodynamics* (Allyn and Bacon, Boston, 1988), Eqs. (57) and (58) of Chap. 14.

[4]Our problem is closely related to the problem of an electric dipole oscillating in an electric field. The principal difference is that the radiation rate is larger for the electric dipole. One can obtain the electric and magnetic radiation-zone fields for an electric dipole, whose electric-dipole moment is **p**, by making the replacements $\mathbf{m}\to c\mathbf{p}, \mathbf{E}\to -c\mathbf{B}$, and $\mathbf{B}\to\mathbf{E}/c$, in Eqs. (3) and (4) (Ref. 5).

[5]M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, (Saunders, Fort Worth, 1995), Eqs. (9.11) and (9.12), 3rd ed.

[6]Reference 2, Eq. (9.53).

[7]It is interesting to ask whether $N_{\rm rad}$ can be derived from the interaction of **m** with its own radiation field, in the manner that Abraham and Lorentz, and later Dirac, derived the reaction force on an accelerating charge from the interaction of the charge with its own radiation field. However, this would be the subject of another paper.

[8]It can be shown that a particle moving classically in the potential $-Cx^4$ reaches $x=\pm\infty$ in a finite time.

[9]Dirac analyzed the equivalent problem for a radiating charge. P. A. M. Dirac, "Classical theory of radiating electrons," Proc. R. Soc. London Ser. A **167**, 148–169 (1938).

[10]M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions* (U. S. Government Printing Office, Washington, DC, 1965).

[11]E. Jahnke and F. Emde, *Tables of Functions* (Dover, New York, 1945), 4th ed.

PARITY CONSERVATION

One of the symmetry principles, the symmetry between the left and the right, is as old as human civilization. The question whether Nature exhibits such symmetry was debated at length by philosophers of the past. Of course, in daily life, left and right are quite distinct from each other. Our hearts, for example, are on our left sides. The language that people use both in the orient and the occident, carries even a connotation that right is good and left is evil. However, the laws of physics have always shown complete symmetry between the left and the right, the asymmetry in daily life being attributed to the accidental asymmetry of the environment, or initial conditions in organic life. To illustrate the point, we mention that if there existed a mirror-image man with his heart on his right side, his internal organs reversed compared to ours, and in fact his body molecules, for example sugar molecules, the mirror image of ours, and if he ate the mirror image of the food that we eat, then according to the laws of physics, he should function as well as we do.

Chen Ning Yang, "The law of parity conservation and other symmetry laws of physics," (Nobel Lecture, December 11, 1957, reprinted in *Nobel Lectures, Physics*, Vol. 3, 1942–1962, Elsevier Amsterdam, 1964).