RADIATION EMITTED BY UNIFORMLY MOVING ELECTRONS

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If the velocity of an electron traversing a ponderable medium exceeds the velocity of light in the medium the electron emits a coherent radiation even if it moves with a constant velocity. Frank and Tamm have shown in 1937 that the peculiar visible radiation emitted by liquids and solids bombarded by fast electrons, which was discovered by Čerenkov in 1934, is of this nature.

The present paper is an extension of the work of Frank and Tamm and contains a more detailed theory of the radiation of an uniformly moving electron, as well as a discussion of conditions, under which the theory can be applied to the visible radiation of electrons, which in their passage through the medium are deflected by collisions, gradually lose energy by ionization, etc.

§ 1. It is a common knowledge that an uniformly moving electric charge does not radiate. It is far less known, that there are exceptions to the rule. Namely a charge does radiate light (and electromagnetic waves in general) even if it is moving uniformly, provided that its velocity is greater than the (phase) velocity of light.

Of course, in vacuum this condition can never be satisfied, but if an electron moves in a ponderable medium its velocity can very well exceed the velocity of light in the medium.

The theory of this radiation was developed some time ago by I. Frank and the present author (1) in an attempt to explain a peculiar phenomenon discovered by P. Čerenkov (2) in 1934. All liquids and solids exposed to γ-rays do emit a peculiar visible radiation quite different from the eventual ordinary fluorescence. This radiation is partially polarized, the electric vector being *parallel* to the direction of γ-rays, and its intensity can be reduced neither by temperature nor by addition to the liquid exposed of quenching substances. It was suggested by S. Vavilov (3), that this radiation is connected with the «Bremsung» of the Compton electrons produced by the γ-rays. Further work of Čerenkov showed, that the radiation is, indeed, produced by fast electrons, so that it is in many respects preferable to exicite the radiation not by γ-rays, but by β-electrons. The clue to the understanding of this phenomenon was provided when Čerenkov discovered the highly pronounced asymmetry of the radiation: the light is emitted practically only under acute angles to the direction of motion of the electrons, but not backwards. This asymmetry is a conclusive evidence of the coherence of the radiation produced by an electron along a path of at least half a wave length of the visible light.

It follows that the phenomenon in question does not depend pracitically on the atomistic structure of the substance bombarded and can be treated with the help of the macroscopic electromagnetic theory. Frank and Tamm (1) have shown, that the macroscopic theory does indeed provide an explanation of the Čerenkov radiation, which is produced by electrons having velocities greater than the velocity of light in the medium traversed.

There is a close analogy between the Čerenkov radiation and some well khown acoustical and hydrodynamical phenomena. Thus, so long as the velocity of a ship is greater than the minimum velocity of waves on the surface of water (23 cm/sec. = =0.83 km/hour), the ship continually generates wave trains even if its velocity remains constant. This is the cause of the wave resistance of ships. A wave resistance of the same kind is also encountered by projectiles moving through air with velocities greater than the velocity of sound. These phenomena are, however, more complicated than the one studied in this paper on account of the non-linearity of the hydrodynamical equations.

Numerous carefull experiments of Čerenkov * have confirmed the theory of Frank and Tamm in every detail (absolute intensity, polarization and angular distribution of the radiation and its dependence on the refraction index and the density of the substances bombarded and on the velocity of the electrons). Recently Collins and Reiling ([5]) have also investigated the Čerenkov radiation using electrons of 2 MeV energy from an electrostatic generator; their results are also in complete agreement with the theory of Frank and Tamm **.

The present paper is an extension of the work of Frank and Tamm. Principal contents of the paper of Frank and Tamm are reproduced in § 2 for convenience of the reader. In § 3 we give a new and more rigourous proof of the fundamental formula of the theory, which determines the intensity and the spectral distribution of the radiation. Simple expressions for the field of the electron are deduced in § 4 under neglect of dispersion. The field in a dispersive medium is discussed in § 5. In § 6 the problem is considered from the point of view of an observer moving with the electron (electron at rest, medium moving). A generalized expression for the relativistic electromagnetic stress-energy tensor is discussed and applied to the determination of forces acting on the electron in its rest system of reference.

Finally in § 7 the notion of the eternal uniform motion of the electron is discarded with and the electron is supposed to move with constant velocity during a final interval of time only. The investigation of this case enables one to establish the conditions under which the influence of the acceleration of the electron on its coherent radiation is negligible.

From the point of view of the microscopic theory the radiation in question is not directly emitted by the electron, but is due to the excitation by the electron of coherent electrical vibration in the molecules of the medium. We will, however, not enter here in the microscopic treatment of the problem.

§ 2. Let us consider an electron moving with a constant velocity v along the z axis through a medium characterized by its index of refraction n. The conditions under which the electron will radiate can be determined by a very simple argument.

The field of the electron may be considered as a superposition of spherical waves of retarded potentials, which are being continually emitted by the moving electron and are propagated with the velocity $\frac{c}{n}$. All the consecutive waves will be in phase along a direction making the angle θ with the axis of motion z, if and only if v, n and θ do satisfy the condition:

$$\frac{c}{n} = v \cos\theta; \quad \cos\theta = \frac{1}{\beta n} \qquad (2,1)$$

where $\beta = \frac{v}{c}$. Under this condition there will be a radiation emitted in the direction θ, whereas the interference of waves will prevent radiation in any other direction.

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* See ([4]), where further references can be found.

** It should be noted, that Collins and Reiling have quite erroneously interpreted the physical implications of this theory. Giving an account of the theory of Frank and Tamm they say: «The electron in its passage through the medium gradually loses its energy through ionization and excitation processes and the resulting acceleration is responsible for the Čerenkov radiation». This is exactly the opposite to the main point of the theory, according to which the acceleration of the electron has only a secondary influence on the Čerenkov radiation [cf. e. g. equation (7, 11)].

The condition (2, 1) can be satisfied only in $\beta n > 1$, i. e. only in the case of fast electrons in a medium of an index of refraction n larger than 1 (for the frequencies in question). If, e. g. $n=1.33$ (water, $\lambda = 5900$ Å) the energy of the electron must be not smaller than 260 kV.

We proceed to develop a more detailed theory. So far as we are interested in visible radiation (or in the radiation of greater wave lengths) we can treat the medium macroscopically, applying to it the usual equations of the electromagnetic theory of light. It is indispensable to take in account the dispersion of light in the medium, since otherwise an electron with a velocity $v > \frac{c}{n}$ would radiate waves of all frequencies up to $\omega = \infty$, and its total radiation would be infinite. However, very short waves, e. g. X-rays, can never be radiated by an uniformly moving electron since for these rays $n \leq 1$.

To obtain a connection between the electric intensity **E** and the electric induction **D** in a dispersive medium one has to expand **E** and **D** in Fourier series:

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_\omega e^{i\omega t} d\omega, \quad \mathbf{D} = \int_{-\infty}^{+\infty} \mathbf{D}_\omega e^{i\omega t} d\omega; \quad (2, 2)$$

the connection in question is:

$$\mathbf{D}_\omega = n^2(\omega)\, \mathbf{E}_\omega, \quad (2, 3)$$

where $n(\omega)$ is the index of refraction of the medium for the frequency ω. Neglecting the absorption of light in the medium we shall assume n to be real and the conductivity of the medium (as well as its magnetic susceptibility) to be zero.

Expanding all the field variables in Fourier series of the type (2, 2) one can easily reduce the Maxwell equations to the following set:

$$\left.\begin{aligned} \mathbf{H}_\omega = \operatorname{rot} \mathbf{A}_\omega,\ \mathbf{E}_\omega = -\nabla \varphi_\omega - \frac{1}{c}\frac{\partial \mathbf{A}_\omega}{\partial t}, \\ \nabla^2 \varphi_\omega + \frac{\omega^2 n^2}{c^2}\varphi_\omega = -\frac{4\pi}{n^2}\rho_\omega; \end{aligned}\right\} \quad (2, 4)$$

$$\nabla^2 \mathbf{A}_\omega + \frac{\omega^2 n^2}{c^2}\mathbf{A}_\omega = -\frac{4\pi}{c} j_\omega, \quad (2, 5)$$

$$\operatorname{div} \mathbf{A}_\omega + \frac{i\omega}{c} n^2 \varphi_\omega = 0. \quad (2, 6)$$

In so far as one is not interested in the static part of φ and **E**. one can express φ_ω in terms of $\mathbf{A}_\omega$ with the help of the relation (2, 6) between the scalar and the vector potentials; the expression for $\mathbf{E}_\omega$ becomes then:

$$\mathbf{E}_\omega = -\frac{ic}{\omega n^2}\nabla \operatorname{div} \mathbf{A}_\omega - \frac{i\omega}{c}\mathbf{A}_\omega. \quad (2, 7)$$

If an electron e is moving along the axis z with a constant velocity v the corresponding current density **j** is equal to

$$j_x = j_y = 0,$$
$$j_z = ev\delta(x)\,\delta(y)\,\delta(z - vt), \quad (2,8)$$

where δ denotes the Dirac's function. Expanding j_z one gets:

$$j_z(\omega) = \frac{e}{2\pi} \cdot e^{-\frac{i\omega z}{v}}\delta(x)\,\delta(y),$$

or, introducing cylindrical coordinates ρ, φ, z,

$$j_z(\omega) = \frac{e}{2\pi^2\rho}\, e^{-\frac{i\omega z}{v}}\,\delta(\rho)\ *.$$

Inserting this expression in (2,5) and putting

$$A_\rho = A_\varphi = 0,\ A_z(\omega) = \frac{e}{2c} a(\rho)\, e^{-\frac{i\omega z}{v}}, \quad (2,9)$$

one obtains:

$$\frac{\partial^2 a}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial a}{\partial \rho} + s^2 a = -\frac{4}{\pi\rho}\delta(\rho), \quad (2,10)$$

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* The corresponding equation in the paper of Frank and Tamm differs from the present one by a factor $\frac{1}{2}$ on the right hand side, since it was assumed by Frank and Tamm that $\int_0^{\rho>0} \delta(\rho)\, d\rho = 1$. However, since the singular point of the function $\delta(\rho)$ coincides with one of the limits of integration, it seems more correct to define the function $\delta(\rho)$ in such a way as to make this integral equal to $\frac{1}{2}$. Anyhow all final formulae [beginning with (2,13)] are quite independent from this distinction in the definition of $\delta(\rho)$.

where

$$s^2 = \frac{\omega^2}{v^2}(\beta^2 n^2 - 1) = -\sigma^2. \qquad (2,11)$$

Thus a is a cylinder function satisfying the Bessel equation

$$\frac{\partial^2 a}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial a}{\partial \rho} + s^2 a = 0 \qquad (2,12)$$

everywhere with the exception of the pole $\rho = 0$. To find the condition to be satisfied by a at $\rho = 0$ we first replace the right hand side of (2,10) by f:

$$f = -\frac{4}{\pi\rho_0^2} \quad \text{if} \quad \rho < \rho_0; \; f = 0 \quad \text{if} \quad \rho > \rho_0;$$

integrate then this equation over the surface of the circle of the radius ρ_0, and finally go over to the limit $\rho_0 \to 0$. In this way we obtain:

$$\lim_{\rho \to 0}\left(\rho \frac{\partial a}{\partial \rho}\right) = -\frac{2}{\pi}. \qquad (2,13)$$

The general solution of (2,12) can be written as follows:

$$a = c_1 \cdot H_0^{(1)}(s\rho) + c_2 \cdot H_0^{(2)}(s\rho) = \\ = c_1 \cdot H_0^{(1)}(i\sigma\rho) + c_2 \cdot H_0^{(2)}(i\sigma\rho), \qquad (2,14)$$

where c_1 and c_2 are constants and $H_0^{(1)}$ and $H_0^{(2)}$ denote Hankel functions of the first and the second kind.

In the case of small velocities $\beta n < 1$, $s^2 < 0$ and σ is real; we will assume σ to be positive $\left(\sigma = \frac{|\omega|}{v}\sqrt{1 - \beta^2 n^2}\right)$. Since $H_0^{(1)}(i\sigma\rho)$ tends to infinity as $\sigma\rho$ increases, we must put $c_2 = 0$. Determining c_1 from the condition (2, 13) one obtains the real solution

$$a = iH_0^{(1)}(i\sigma\rho). \qquad (2,15)$$

The asymptotic value of a for $\sigma\rho \gg 1$ is

$$a = \sqrt{\frac{2}{\pi\sigma\rho}}\, e^{-\sigma\rho}.$$

Thus in the case of small velocities ($\beta n < 1$) the Fourier coefficients of the field decrease exponentially with ρ, so that there is no radiation at all.

If, however, the velocity of the electron is so large that within a certain frequency range $\beta n > 1$, then the parameter s is real within this range and the functions $H_0^{(1)}(s\rho)$ and $H_0^{(2)}(s\rho)$ represent in infinity cylindrical waves. One of this waves is outgoing (i. e. is propagated from the axis $\rho = 0$ towards $\rho = \infty$), the other ingoing. One has obviously to retain in (2, 14) only the outgoing wave. This condition along with the condition (2,13) fixes the solution completely and one obtains:

$$\left.\begin{array}{lll} a = -iH_0^{(2)}(s\rho) & \text{if} & \omega > 0, \\ a = iH_0^{(1)}(s\rho) & \text{if} & \omega < 0, \end{array}\right\} \qquad (2,16)$$

where s is assumed to be positive:

$$s = \frac{|\omega|}{v}\sqrt{\beta^2 n^2 - 1}. \qquad (2,17)$$

If $s\rho \gg 1$ one can insert for H_0 its asymptotic value and obtains from (2,16) and (2,9):

$$A_z(\omega)\, e^{i\omega t} = \frac{-ie}{c\sqrt{2\pi s\rho}}\, e^{i\omega\left(t - \frac{z}{v}\right) - is\rho + i\frac{\pi}{4}}, \quad \omega > 0 \qquad (2,18)$$

and a complex conjugate expression for $\omega < 0$. Transforming the exponent with the help of (2,17) one obtains:

$$A_z(\omega)\, e^{i\omega t} = -\frac{ie}{c\sqrt{2\pi s\rho}}\, e^{i\omega\left(t - \frac{z\cos\theta + \rho\sin\theta}{u}\right) + \frac{i\pi}{4}}, \qquad (2,19)$$

where the angle θ is defined by (2,1) and $u = \frac{s}{n}$. Thus, if $\beta n > 1$, a wave is propagated in infinity under the angle θ to the z axis.

The expressions for the field vectors in terms of the function $a(\rho, \omega)$ follow from (2,9), (2,4) and (2,6):

$$A_z = \frac{e}{2c}\int_{-\infty}^{+\infty} e^{i\omega\left(t-\frac{z}{v}\right)} a(\rho, \omega)\, d\omega, \tag{2,20}$$

$$H_\varphi = -\frac{\partial A_z}{\partial \rho},\ E_\rho = -\frac{e}{2\beta c}\int_{-\infty}^{+\infty} e^{i\omega\left(t-\frac{z}{v}\right)} \frac{1}{n^2}\frac{\partial a}{\partial \rho}\, d\omega,\quad E_z = \frac{ie}{2c^2}\int_{-\infty}^{+\infty} e^{i\omega\left(t-\frac{z}{v}\right)}\left(\frac{1}{\beta^2 n^2}-1\right)\omega a\, d\omega; \tag{2,21}$$

all other components of **A**, **H** and **E** vanish.

§ 3. In calculating the energy radiated by an electron per unit path Frank and Tamm have made use of the asymptotic expansions of the Hankel functions for large values of the argument. However, the arguments $\sigma\rho$ and $s\rho$ tend to zero for any fixed value ρ if ω tends to zero. It seems therefore desirable to give an other proof of the formula deduced by Frank and Tamm, which does not involve asymptotic expansions at all.

The radial component of the Poynting vector **S** is equal to

$$S_\rho = -\frac{c}{4\pi}E_z H_\varphi = \frac{ie^2}{16\pi c^2}\int_{-\infty}^{+\infty} e^{i\omega\left(t-\frac{z}{v}\right)}\left(\frac{1}{\beta^2 n^2}-1\right)\omega\cdot a(\omega)\cdot d\omega \int_{-\infty}^{+\infty} e^{i\omega'\left(t-\frac{z}{v}\right)}\frac{\partial a(\omega')}{\partial \rho}\, d\omega'.$$

The total energy radiated by the electron through the surface of a cylinder of length dl, the axis of which coincides with the line of motion of the electron, is equal to

$$dW = 2\pi\rho\, dl \int_{-\infty}^{+\infty} S_\rho\, dt. \tag{3,1}$$

With the help of the formula

$$\int_{-\infty}^{+\infty} e^{i(\omega+\omega')t}\, dt = 2\pi\,\delta(\omega+\omega') \tag{3,2}$$

one obtains

$$\frac{dW}{dl} = \frac{i\pi e^2\rho}{4c^2}K, \tag{3,3}$$

where

$$K = \int_{-\infty}^{+\infty}\left(\frac{1}{\beta^2 n^2}-1\right) a(\omega)\frac{\partial a(-\omega)}{\partial \rho}\omega\, d\omega = \int_0^{\infty}\left(\frac{1}{\beta^2 n^2}-1\right)\left[a(\omega)\frac{\partial a(-\omega)}{\partial \rho} - a(-\omega)\frac{\partial a(\omega)}{\partial \rho}\right]\omega\, d\omega. \tag{3,4}$$

If $\beta n < 1$, then according to (2,15) $a(-\omega) = a(\omega)$ and the integrand vanishes. Thus the integration in (3,4) can be re stricted to the frequency range defined by $\beta n(\omega) > 1$. Within this range a is defined by (2,16) and (2,17) and one obtains:

$$X = a(\omega)\frac{\partial a(-\omega)}{\partial \rho} - a(-\omega)\frac{\partial a(\omega)}{\partial \rho} = H^{(2)}(s\rho)\frac{\partial H_0^{(1)}(s\rho)}{\partial \rho} - H_0^{(1)}(s\rho)\frac{\partial H_0^{(2)}(s\rho)}{\partial \rho}.$$

Since

$$H_0^{(1)} = J_0 + iN_0,\quad H_0^{(2)} = J_0 - iN_0 \tag{3,5}$$

the expression is equal to

$$X = 2i\left\{J_0(s\rho)\frac{\partial N_0(s\rho)}{\partial \rho} - N_0(s\rho)\frac{\partial J_0(s\rho)}{\partial \rho}\right\} = -2is\{J_0(s\rho)N_1(s\rho) - N_0(s\rho)J_1(s\rho)\}.$$

The expression in brackets is equal [6] to $-\frac{2}{\pi s\rho}$, so that $X = \frac{4i}{\pi\rho}$.

Inserting this in (3,4) and (3,3) one obtains the fundamental formula of the theory:

$$\frac{dW}{dl} = \frac{e^2}{c^2}\int_{\beta n>1}\omega\, d\omega\left(1-\frac{1}{\beta^2 n^2}\right), \tag{3,6}$$

which is identical with the formula (15) of Frank and Tamm. This formula determines not only the total energy radiated by the electron per unit path, but obviously also the spectral distribution of this radiation.

According to an estimate of Frank and Tamm the total loss of energy of an electron in liquids and solids due to the Čerenkov radiation is of the order of several kilovolts per centimeter path and is thus quite negligible in comparison to the losses of energy by other causes.

§ 4. In this section we will neglect the dispersion of light and will assume n to be independent of ω. This will enable us to express the field of the electron in a very simple form.

First let $\beta n < 1$. Introducing the notations

$$\tau = t - \frac{z}{v}, \quad r = \frac{\rho}{v}\sqrt{1 - \beta^2 n^2}, \quad (4,1)$$

one obtains from (2,20) and (2,15):

$$A_z = \frac{ie}{2c}\int_0^\infty d\omega\,(e^{i\omega\tau} + e^{-i\omega\tau})\,H_0^{(1)}(i\omega r).$$

Expressing $H_0^{(1)}$ with the help of Heine's integral

$$H_0^{(1)}(i\omega r) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} e^{-\omega r \operatorname{ch} u} du$$

and changing the order of integration one obtains:

$$A_z = \frac{e}{2\pi c}\int_{-\infty}^{+\infty} du \int_0^\infty d\omega\,(e^{i\omega\tau} + e^{-i\omega\tau})\,e^{-\omega r \operatorname{ch} u} = \frac{e}{\pi c}\int_{-\infty}^{+\infty}\frac{r \operatorname{ch} u\,du}{\tau^2 + r^2 \operatorname{ch}^2 u}.$$

Making the substitution $r \operatorname{sh} u = \xi$ one finds

$$A_z = \frac{e}{\pi c}\int_{-\infty}^{+\infty}\frac{d\xi}{\tau^2 + r^2 + \xi^2} = \frac{e}{c\sqrt{\tau^2 + r^2}},$$

or, according to (4,1):

$$A_z = \frac{ev}{c\sqrt{(z - vt)^2 + \rho^2(1 - \beta^2 n^2)}}. \quad (4,2)$$

This is identical with the well known Lienard-Wiechert's retarted potential of a moving point charge:

$$A_z = \frac{ev}{c\left[R\left(1 - \frac{nv_R}{c}\right)\right]}, \quad (4,3)$$

where $\frac{c}{n}$ stands for the velocity of light.

Let us now consider the case $\beta n > 1$. Introducing this time the notations

$$\tau = t - \frac{z}{v}, \quad r = \frac{\rho}{v}\sqrt{\beta^2 n^2 - 1}, \quad (4,4)$$

one obtains from (2,20), (2,16) and (2,17):

$$A_z = -\frac{ie}{2c}\int_0^\infty d\omega\,\{e^{i\omega\tau}H_0^{(2)}(\omega r) - e^{-i\omega\tau}H_0^{(1)}(\omega r)\}.$$

This expression can be transformed with the help of (3,5) into

$$A_z = \frac{e}{c}\int_0^\infty d\omega\,\{J_0(\omega r)\cdot\sin\omega\tau - N_0(\omega r)\cdot\cos\omega\tau\},$$

or, if one introduces the notations $\xi = \frac{|\tau|}{r}$, $\eta = \omega r$, into

$$A_z = \frac{e}{cr}\int_0^\infty d\eta\{\pm J_0(\eta)\sin(\xi\eta) - N_0(\eta)\cos(\xi\eta)\},$$

where the sign of the first term in brackets is positive if $\tau > 0$ and negative if $\tau < 0$. Making use of the formulae

$$\int_0^\infty J_0(\eta)\sin(\xi\eta)\,d\eta = -\int_0^\infty N_0(\eta)\cos(\xi\eta)\,d\eta = \begin{cases} 0 & \text{if } 0 < \xi < 1, \\ \frac{1}{\sqrt{\xi^2 - 1}} & \text{if } \xi > 1, \end{cases}$$

known from the theory of Bessel functions ([7]), one obtains finally:

$$\left.\begin{aligned} A_z &= \frac{2e}{cr\sqrt{\xi^2-1}} = \\ &= \frac{2ev}{c\sqrt{(z-vt)^2-\rho^2(\beta^2n^2-1)}} \\ &\quad \text{if } vt-z>\rho\sqrt{\beta^2n^2-1}, \\ A_z &= 0 \\ &\quad \text{if } vt-z<\rho\sqrt{\beta^2n^2-1}. \end{aligned}\right\} \quad (4,5)$$

Thus the field, being stationary with respect to the electron, is discontinuous and is bounded by the cone:

$$z = vt - \rho\sqrt{\beta^2n^2-1}, \qquad (4,6)$$

the vertex of which $\rho=0$, $z=vt$ coincides with the momentary position of the electron. In front of the cone the field vanishes; on the surface of the cone **A** as well as **E** and **H** are infinite and gradually diminish as one moves backwards from the surface of the cone. Thus a conical wave of discontinuity is propagated along the z axis with the velocity of the electron; the normal to this wave makes with the axis z an angle $\theta = \arc\cos\frac{1}{\beta n}$.

Noticing that according to (2,4) and (2,5)

$$\varphi = \frac{c}{vn^2}A_z, \qquad (4,7)$$

one can easily show that behind the cone (4,6)

$$\left.\begin{aligned} H_\varphi &= -\frac{q\beta\rho}{R^3}, \\ E_\rho = -\frac{q\rho}{n^2R^3}, &\quad E_z = \frac{q(vt-z)}{n^2R^3}, \end{aligned}\right\} \quad (4,8)$$

where

$$\left.\begin{aligned} R &= \sqrt{(z-vt)^2-\rho^2(\beta^2n^2-1)}, \\ q &= 2e(\beta^2n^2-1); \end{aligned}\right\} \quad (4,9)$$

all other components of **H** and **E** vanish. Thus, the magnetic lines of force are circles with centres lying on the z axis, while the electric lines of force at any moment of time are straight lines diverging from the point $\rho=0$, $z=vt$, occupied at this moment by the electron. The lines of the Poynting vector lie in the meridian planes and are arcs of concentric circles, the common centre of all the circles coinciding with the position of the electron; these vector lines are directed from the z axis towards the limiting cone (4,6).

Discontinuous waves of this kind are well known in ballistics. If the velocity of a projectile is greater than the velocity of sound in air the projectile excites a rather narrow conical acoustical wave, which is stationary in respect to the projectile (Mach's wave). The normal to this wave is inclined to the line of motion of the projectile at an angle $\theta = \arc\cos\frac{c'}{v}$, where c' denotes the velocity of sound. This Mach's wave can be photographed with the help of the «Schlierenmethode»; some beautiful pictures are reproduced in «Handbuch der Physik» (Geiger und Scheel), vol. VII, p. 337—339. On one of these pictures one can see the reflection of the wave at the surface of a wall, along which the projectile was moving.

The very rapid increase in the resistance, encountered by the projectile when its velocity becomes greater than the velocity of sound, is due to the formation of this wave (wave-making resistance). The whole phenomenon is, however, much more complicated than the corresponding optical one (boundary conditions at the surface of the projectile, non-linearity of the equations of aerodynamics, etc.).

Of course, neither the optical nor the acoustical wave are actually discontinuous and their intensity is by no means infinite. Since the dispersion of sound can be neglected up to wave lengths, which are much smaller than the dimensions of the projectiles, the actual steepnes and the intensity of the acoustical wave are determined by the dimensions and the form of the projectile. On the contrary, the characteristics of the optical wave are determined by the dispersion of light in the medium, since the index of refraction n drops to 1 and even below 1 and the radiation is thereby cut off at a wave length far greater than the dimensions of the electron. The effect of dispersion will be considered in the next section, while this one shall be concluded with some further remarks on the hypothetical case $n = \text{const}$, $\beta n > 1$.

In this case the energy radiated by a point charge per unit path becomes infinite [cf. equation (3,6)]. One can, however, approximately account for the finite dimensions of the electron if one assumes, that the radiation of the electron is limited to the waves which are longer than its diameter d. Performing accordingly the integration in (3, 6) from $\omega = 0$ to $\omega = c/nd$ (i. e. from $\lambda = \infty$ to $\lambda/2\pi = d$) one obtains:

$$\frac{dW}{dl} = \frac{e^2}{2n^2d^2}\left(1 - \frac{1}{\beta^2 n^2}\right). \qquad (4,10)$$

This equation is almost identical with a result obtained by Sommerfeld (8) some 35 years ago. Sommerfeld calculated directly the vector sum $\mathbf{F}$ of electromagnetic forces of interaction of all the volume elements of a rigid spherical electron. In particular he investigated the case of an uniform motion of an electron in vacuo and found, that the resultant force $\mathbf{F}$ vanishes only so long as $v < c$, but is equal to

$$F = \frac{9e^2}{4\pi d^2}\left(1 - \frac{1}{\beta^2}\right) \qquad (4,11)$$

if $v > c$; this force is opposite to the direction of v. Hence a uniform motion of an electron with a velocity $v > c$ can be maintained only by an external force acting on the electron, equal to F and of an opposite direction. Sommerfeld concluded, that the work performed by this external force must be dissipated in radiation, so that in our notations $F = \frac{dW}{dl}$. In fact, our approximate expression (4,10) for $\frac{dW}{dl}$ in the case $n = 1$ considered by Sommerfeld differs from his expression (4, 11) for F only by a factor $\frac{2\pi}{9} \approx 0.7$.

It will be noted that Sommerfeld's paper was written prior to the establishment of the theory of relativity, when velocities greater than c were considered possible.

Some years later Sommerfeld and Klein (9) published a very enlightening discussion of the wave resistance encountered by projectiles at velocities v greater than the velocity of sound, and showed that the dependence of this resistance on v is very similar to the one expressed by the equation (4, 11).

§ 5. The dispersion of light substantially modifies the results obtained in § 4. The index of refraction n at very high frequencies never exceeds 1, so that the condition of radiation $\beta n > 1$ can be satisfied only within a finite range of frequencies. Let us assume, that for a given value of β this range extends from $\omega = \omega_1$ to $\omega = \omega_2$*.

In this section we will consider only the radiation of the electron, i. e. the field at large distances from the electron, and will neglect the part of the field which corresponds to frequencies lying outside of the range ω_1, ω_2. Inserting the asymptotic expression (2,18) of A_z in (2,7) one obtains:

$$\left.\begin{aligned} E_\rho &= \frac{e}{c\sqrt{2\pi\rho}} \int_{\omega_1}^{\omega_2} d\omega \cdot \frac{\sqrt{s}}{n} \cdot \frac{1}{\beta n} \cdot e^{i\omega\left(t - \frac{z}{v}\right) - is\rho + \frac{i\pi}{4}} + \text{C. C.}, \\ E_z &= -\frac{e}{c\sqrt{2\pi\rho}} \int_{\omega_1}^{\omega_2} d\omega \cdot \frac{\sqrt{s}}{n} \cdot \sqrt{1 - \frac{1}{\beta^2 n^2}} \cdot e^{i\omega\left(t - \frac{z}{v}\right) - is\rho + \frac{i\pi}{4}} + \text{C. C.}, \end{aligned}\right\} \qquad (5,1)$$

and a similar expression for H_φ.

These integrals can be approximately evaluated with the help of the well known approximate formula

$$\int_{\omega_1}^{\omega_2} d\omega \cdot \Phi(\omega) \cdot e^{if(\omega)} = \frac{\sqrt{2\pi}\,\Phi(\omega_0)}{\sqrt{|f''(\omega_0)|}}\, e^{if(\omega_0) \pm \frac{i\pi}{4}}, \qquad (5,2)$$

where ω_0 denotes that root of the equation

$$f'(\omega_0) = \left(\frac{df}{d\omega}\right)_{\omega_0} = 0, \qquad (5,3)$$

which lies within the range of integration ($\omega_1 < \omega_0 < \omega_2$), and where the upper or low-

* There may be also several such ranges separated by intervals in which $\beta n < 1$.

er sign is to be taken in the exponential according as $f''(\omega_0)$ is positive of negative.

If the equation (5,3) has no roots within the range of integration, the integral (5,2) vanishes in the first approximation. Equation (5,2) is valid under the condition that $e^{if(\omega)}$ goes through a large number of periods within the range of integration, whilst $\Phi(\omega)$ changes comparatively slowly; further the value of $f'''(\omega_0)\cdot\{f''(\omega_0)\}^{-3/2}$ must be small. We will assume, that in our case [equations (5,1)] these conditions are satisfied.

The equation (5,3) takes in our case the form

$$t-\frac{z}{v}-\rho\frac{ds}{d\omega}=0. \qquad (5,4)$$

Suppose that within the range of integration ω_1, ω_2 (i. e. within the range of frequencies, for which s is real) the values of the function g

$$g=v\frac{ds}{d\omega} \qquad (5,5)$$

are comprised between the limits $g_{\min}$ and $g_{\max}$. Then at any given moment t the radiation field will be different from zero only in the space between the cones

$$z+g_{\min}\rho=vt \text{ and } z+g_{\max}\rho=vt, \qquad (5,6)$$

since for values of ρ and z corresponding to points lying outside of this space the equation (5,4) will have no real roots. Hence, the field in any fixed point ρ, z will be different from zero only during a finite interval of time:

$$\Delta t=\frac{g_{\max}-g_{\min}}{v}\rho.$$

The duration Δt of the radiative impulse increases with the distance ρ from the z axis: on account of the dispersion the wave trains spread more and more the further they travel from the radiating electron.

The vertices of the cones (5,6) coincide with the momentary position of the electron. The cones lie in front of the electron if g is negative and behind the electron if g is positive. Now it follows from (5,5) and (2,17), that

$$g=\frac{1}{\sqrt{\beta^2n^2-1}}\left(\beta^2n^2-1+\beta^2n\omega\frac{dn}{d\omega}\right).$$

Further the phase velocity of light is $\frac{c}{n}=u$ and its group velocity w is defined by

$$\frac{1}{w}=\frac{d}{d\omega}\left(\frac{\omega}{u}\right)=\frac{1}{c}\left(n+\omega\frac{dn}{d\omega}\right).$$

Hence the expression for g may be written as follows:

$$g=\frac{v^2-uw}{w\sqrt{v^2-u^2}}. \qquad (5,7)$$

The group velocity w of light outside of the regions of anomalous dispersion is smaller than its phase velocity u and since v must be greater than u for the radiation to be possible, g is positive. Therefore, the cones (5,6), within which the radiation is confined, lie behind the electron.

If, however, the group velocity w in a medium is greater than the phase velocity u, then the radiation may outstrip the radiating body moving with a velocity $v>u$, since the velocity of propagation of the radiation is equal not to u, but to w. For instance, the group velocity of capillary ripples on the surface of water is greater than their phase velocity. Therefore, when a small obstacle, such as a fishing line, is moved forward through still water, the surface of water *in front* of the obstacle is covered with capillary ripples, while the longer gravity waves, for which $w<u$, are formed *behind* the obstacle. The ship waves are of the latter kind and lie behind the ship.

Let us return to the equations (5,1) and let us assume for definiteness that $s''(\omega)=\frac{d^2s}{d\omega^2}>0$. Using (5,2) one finds

$$E_\rho=\frac{2e}{\rho v n_0^2}\sqrt{\frac{s_0}{s''(\omega_0)}}\cos\left\{\omega_0\left(t-\frac{z}{v}\right)-s_0\rho\right\}, \qquad (5,8)$$

where the quantities ω_0, ρ and $t-z/v$ are connected by the relation [cf. (5,4) and (5,5)]:

$$z+g(\omega_0)\rho=vt. \qquad (5,9)$$

The expression for E_z differs from (5,8) by the factor $-\sqrt{\beta^2 n_0^2-1}$ and the expression for H_φ—by the factor βn_0^2, where $n_0\equiv n(\omega_0)$.

At a given moment t the intensity of the field will be at its maximum in those points of space, where the phase of the wave is a multiple of π:

$$\omega_0\left(t-\frac{z}{v}\right)-s_0\rho=m\pi, \qquad (5,10)$$

$$m=0,\ \pm1,\ \pm2,\ \pm\ldots$$

Eliminating the parameter ω_0 from the equations (5,9) and (5,10) one obtains an equation in ρ and $t-z/v$, which defines the position of the «crests» of the electromagnetic wave; the result depends on the law of dispersion. Applying this method to hydrodynamics one can obtain a very satisfactory representation of the pattern of the ship waves.

In conclusion we will show that the equation (5,9) has a simple physical meaning. Consider a partial cylindrical wave of frequency ω_0; its wave vector makes with the axis z an angle $\theta_0=\arccos 1/\beta n_0$ [cf. equation (2,18)]. Hence a wave packet formed by a superposition of waves of an infinitesimal range of frequencies $\omega_0, \omega_0+d\omega_0$ will be moving in the direction θ_0; its velocity will be equal to the group velocity w_0. Now the field of the electron is evidently stationary with respect to the electron; this must be true also for the part of the field due to the waves in the frequency range ω_0, $\omega_0+d\omega_0$. This property of being stationary is sufficient to determine the part of the field in question. In fact, the field formed by the superpositon of the waves ω_0, $\omega_0+d\omega_0$ can be stationary in respect to the electron only on a conical surface having the following property: the translation of any edge of the cone (i. e. of the line of intersection of the cone with any meridian plane) in the direction θ_0 with the velocity w_0 is equivalent to the translation of this edge in the direction z with the velocity v*. This condition can be expressed by the equation (Fig. 1):

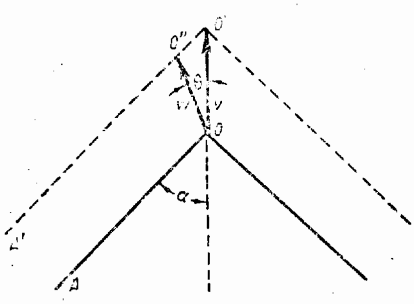

Fig. 1

$$\frac{w_0}{\sin\alpha}=\frac{v}{\sin(\pi-\alpha-\theta_0)},$$

where α denotes the angle between the edge of the cone and the axis z. Taking into account that $\cos\theta_0=\frac{1}{\beta n_0}=\frac{u_0}{v}$, one easily obtains:

$$\operatorname{ctg}\alpha=\frac{v-w_0\cos\theta_0}{w_0\sin\theta_0}=\frac{v^2-u_0w_0}{w_0\sqrt{v^2-u_0^2}}=g(\omega_0).$$

Since $\operatorname{ctg}\alpha=\frac{vt-z}{\rho}$, this is equivalent to the equation (5,9).

§ 6. It is of some interest to consider our problem also from the point of view of an observer moving with the electron.

Let us denote by $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{D}'$, $\mathbf{H}'$, the field vectors measured in the rest system $S'(x', y', z', t')$ of the electron, whereas

* Strictly speaking these translations are not exactly equivalent, since the former one brings the edge OA not in the position $O'A'$ but in the position $O''A'$. However, this difference, amounting to the field on the path $O'O''$ is covered by the radiation emitted by the electron on the path OO'.

E, B, D, H are the field vectors in the system $S(x, y, z, t)$ used up to now and connected with the ponderable medium. The components of **E** and **H** are given by the equations (2,21), the components of **D** differ from those of **E** by the factor $n^2(\omega)$ under the sign of the integral; finally $\mathbf{B}=\mathbf{H}$ (since we have assumed that $\mu=1$).

Evaluating **E′**, **B′**, etc. in terms of **E**, **B**, etc. with the help of the Lorentz transformation one obtains:

$$\left.\begin{aligned}
E'_\rho &= \frac{e\beta}{2c\sqrt{1-\beta^2}}\int\limits_{-\infty}^{+\infty} e^{i\omega\xi}\left(1-\frac{1}{\beta^2n^2}\right)\cdot\frac{\partial a}{\partial\rho}\cdot d\omega, \quad E'_z=\frac{ie}{2c^2}\int\limits_{-\infty}^{+\infty} e^{i\omega\xi}\left(\frac{1}{\beta^2n^2}-1\right)\omega\cdot a\cdot d\omega,\\
D'_\rho &= \frac{e\beta}{2c\sqrt{1-\beta^2}}\left(1-\frac{1}{\beta^2}\right)\int\limits_{-\infty}^{+\infty} e^{i\omega\xi}\frac{\partial a}{\partial\rho}\,d\omega, \quad D'_z=\frac{ie}{2c^2}\int\limits_{-\infty}^{+\infty} e^{i\omega\xi}\left(\frac{1}{\beta^2}-n^2\right)\omega\cdot a\cdot d\omega,\\
B'_\varphi &= \frac{e\beta}{2c\sqrt{1-\beta^2}}\int\limits_{-\infty}^{+\infty} e^{i\omega\xi}\left(\frac{1}{n^2}-1\right)\cdot\frac{\partial a}{\partial\rho}\cdot d\omega,
\end{aligned}\right\}\qquad (6,1)$$

where $\xi=-\frac{z'\sqrt{1-\beta^2}}{v}$.

All other components of the field vectors vanish in S'.

Thus the field in the rest system of the electron is stationary as it should be, and the magnetic intensity **H′** vanishes, but the magnetic induction **B′** is different from zero if $n\neq 1$. This magnetic field is due to the variations of the polarization of the medium as it moves throught the unhomogeneous electric field of the electron; magnetic fields of this kind were experimentally investigated by Röntgen and Eichenwald.

According to (6,1) the constant field of static charges (as well as of stationary currents) is influenced by the dispersive properties of the medium if this medium is moving. This is due to the peculiar connections between the vectors **D′**, **E′**, **B′**, **H′** in a moving dispersive medium, which can be deduced from the usual ones of the type of (2,3) by means of a Lorentz transformation. In the case of a stationary field this connection corresponds to some definite relations between the coefficients of the expansion of the vectors **D′**, **E′**, **B′**, **H′** in terms of periodic functions $e^{-2\pi i\frac{z'}{\lambda}}$ of the *space* coordinate z; these relations involve the value of value of $n(\omega)$ for $\omega=\frac{2\pi v}{\lambda\sqrt{1-\beta^2}}$.

Let us suppose that in some range of frequencies $\beta n(\omega)>1$. Investigating the problem in the system of reference S connected with the medium we have found that under this condition the electron continually radiates energy. The reaction of the radiation on the electron is equivalent to a force of resistance [cf. equation (3,6)]:

$$F=\frac{dW}{dl}=\frac{e^2}{c^2}\int\limits_{\beta n>1}\omega d\omega\left(1-\frac{1}{\beta^2n^2}\right),$$

which tends to slow the electron down to the velocity $\beta=\frac{1}{n_{\max}}$. Evidently a corresponding force F' must act on the electron also in its rest system S' and will tend to drag it along with the moving medium. With the help of the relativistic formulae for the transformation of the force one obtains (since F is antiparallel to the velocity of the electron):

$$F'=F=\frac{e^2}{c^2}\int\limits_{\beta n>1}\omega d\omega\left(1-\frac{1}{\beta^2n^2}\right).\qquad (6,2)$$

It is of some interest to calculate the force F' in the system S' without any recourse to the system S. Since **E′** and **H′** become infinite at the point where the electron is situated, one has to perform the calculation with the help of the stress-energy tensor.

The relativistic expression of the stress-energy tensor T^β_α was first given by Minkowsky and was generalized by Dällenbach* so as to include the cases where there is no linear connection with constant coefficients between the vectors **E**, **D**, **H**, and **B** (e. g. dispersive media and media showing hysteresis). According to Dällenbach

$$\left.\begin{aligned} 4\pi T_i^k &= E_i D_k + H_i B_k - \\ &\quad - \delta_{ik}\left\{\int \mathbf{D}d\mathbf{E} + \int \mathbf{B}d\mathbf{H}\right\}, \\ 4\pi T_4^k &= [\mathbf{E}, \mathbf{H}]_k, \\ 4\pi T_{k_4}^4 &= -[\mathbf{D}, \mathbf{B}]_k, \\ 4\pi T_4^4 &= \int \mathbf{E}d\mathbf{D} + \int \mathbf{H}d\mathbf{B}, \end{aligned}\right\} \quad (6,3)$$

where $i, k = 1, 2, 3$ and the integration in $\int \mathbf{D}d\mathbf{E}$, etc. has to be extended from the moment t_0 at which all the field vectors vanished to the moment t at which the value of T_α^β is to be determined.

It should be noted that Abraham raised objections against the Ansatz of Minkowsky (which are applied also to its generalization by Dällenbach) on account of the asymmetry of the stress-energy tensor ($T_{\alpha\beta} \neq T_{\beta\alpha}$) and proposed for this tensor another expression $T'_{\alpha\beta}$. This expression differs from (6,3) by certain additional terms $t_{\alpha\beta}$ which depend in a complicated way on the velocity of the medium and make $T'_{\alpha\beta}$ symmetrical:

$$T'_{\alpha\beta} = T_{\alpha\beta} + t_{\alpha\beta}; \quad T'_{\alpha\beta} = T'_{\beta\alpha}. \qquad (6,4)$$

We hope to show at another place that the objections of Abraham are irrelevant and that one has to accept for T_α^β the Minkowsky-Dällenbach expression (6,3). However, we need not enter here in this question since the forces acting on an *electric charge* placed in an *homogeneous* medium are certainly determined by the tensor (6,3) quite irrespective from the presence or absence of additional terms $t_{\alpha\beta}$ in the general expression for the stress-energy tensor. In fact, let us insert (6, 3) in the equations

$$f_i = \frac{\partial T_i^\alpha}{\partial x_\alpha}, \quad -A = c\frac{T_4^\alpha}{\partial x^\alpha}, \; \alpha = 1,2,3,4 \; (6,5)$$

which define the density **f** of the electromagnetic ponderomotive forces and the work A done by these forces per unit volume and unit time.

If the medium is homogeneous, so that the relations between the field vectors **E**, **D**, **B**, **H** do not explicitly involve space and time coordinates, one easily obtains with the help of the field equations:

$$\left.\begin{aligned} \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \; \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{div}\mathbf{D} = 4\pi\rho, \quad \operatorname{div}\mathbf{B} = 0, \end{aligned}\right\} (6,6)$$

the following result:

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}[\mathbf{j}, \mathbf{B}], \; A = \mathbf{jE}. \qquad (6,7)$$

Note that (6,7) follows from (6,3), (6,5) and (6,6) quite independently from the particular form of the relations between **E**, **H**, on one side, and **D**, **B**, on the other.

There is no doubt that (6,7) represents the forces acting on an electric charge. Any additional terms $t_{\alpha\beta}$ in the expression for $T_{\alpha\beta}$ correspond evidently to additional forces acting on the *medium* itself, and not on the electric charge *. Since we wish to calculate the forces acting on the electron and not on the medium we must thus use for $T_{\alpha\beta}$ the expression (6,3) quite irrespective of the controversy between Abraham and Minkowsky.

After this lengthy but necessary departure we can at last calculate the force F'

* References to the papers of Minkowsky, Dällenbach and Abraham can be found in Pauli, Relativitätstheorie, Leipzig, 1921, § 31. Pauli apparently agrees with the arguments of Abraham.

* If one uses the Ansatz (6,4) of Abraham, one obtains, for instance, in the case of an homogeneous medium at rest in an electric field $\mathbf{f} = \rho\mathbf{E} + \frac{1}{2}\operatorname{rot}[\mathbf{P}, \mathbf{E}]$, where $\mathbf{P} = \frac{\mathbf{D}-\mathbf{E}}{4\pi}$ is the polarization. Thus, in anisotropic media, where $[\mathbf{P}, \mathbf{E}]$ may be different from zero, one obtains in addition to the force $\rho\mathbf{E}$ acting on the electric charge also a force $\frac{1}{2}\operatorname{rot}[\mathbf{D}, \mathbf{E}]$ acting on the medium itself.

acting on the electron in an uniformly moving medium. Since the field is stationary $\frac{\partial T_i^4}{\partial t}=0$ and

$$F'_k = \oint T_{kn} \cdot dS,$$

where $\mathbf{n}$ is the normal to the surface element dS and where the integration can be extended over any closed surface S which encloses the electron. Let it be the surface of a cylinder of an infinitesimal radius ρ and of infinite length and let the axis of the cylinder coincide with the axis z. Evidently $F'_x = F'_y = 0$ and, since $\mathbf{H}'=0$, one obtains:

$$F'_z = \int_{-\infty}^{+\infty} T'_{z\rho} \cdot 2\pi\rho \cdot dz' = \frac{1}{2}\rho \int_{-\infty}^{+\infty} E'_z D'_\rho dz'.$$

Substituting the values of E'_z and D'_ρ from (6,1) and using

$$\int_{-\infty}^{+\infty} dz' e^{-\frac{iz'}{v}(\omega+\omega')\sqrt{1-\beta^2}} =$$

$$= 2\pi\delta\left(\frac{(\omega+\omega')}{v}\sqrt{1-\beta^2}\right) =$$

$$= \frac{2\pi v}{\sqrt{1-\beta^2}}\delta(\omega+\omega')$$

[cf. equation (3,2)], one obtains:

$$F'_z = -\frac{i\pi\rho e^2}{4c^2}\int_{-\infty}^{+\infty}\left(\frac{1}{\beta^2 n^2}-1\right)\omega\, a(\omega)\,\frac{\partial a(-\omega)}{\partial\rho}\,d\omega.$$

According to (3,3) and (3,4) this equation is equivalent to (6,2), where F' denotes the absolute value of F'_z.

It should be noted that $E'_z D'_\rho \neq E'_\rho D'_z$, so that a symmetrical «Ansatz» for $T_{\alpha\beta}$ would give quite a different result in disagreement with (6,2).

§ 7. Up to now we have assumed that the electron is moving in an infinite homogeneous medium with a constant velocity which never changed since $t=-\infty$. Evidently the field of an electron, the velocity of which was only approximately constant during a finite interval of time, will be under certain conditions in some respects similar to the field considered. We will be able to make a more precise statement if we investigate in some detail at least one particularly simple case, somewhat more similar to the actual motion of an fast electron in ponderable media than that considered up to now.

Suppose that an electron was at rest from $t=-\infty$ up to the moment $t=-t_0$, when it suddenly acquired a finite velocity v, and that v remained constant up to the moment $t=t_0$, when the electron was again suddenly brought to rest. Choosing the axis of coordinates in a suitable manner one finds, that the corresponding current density $\mathbf{j}$ is equal to

$$\left.\begin{array}{l} j_z = ev\delta(x)\delta(y)\delta(z-vt) \\ \qquad\qquad \text{if } -vt_0 < z < vt_0, \\ j_z = 0 \\ \qquad\qquad \text{if } |z| > vt_0. \end{array}\right\} \quad (7,1)$$

Expanding j_z in a Fourier series one obtains:

$$\left.\begin{array}{ll} j_z(\omega) = \frac{e}{2\pi}\delta(x)\delta(y)e^{-\frac{i\omega z}{v}}, & \text{if } |z| < vt_0, \\ j_z(\omega) = 0, & \text{if } |z| > vt_0. \end{array}\right\} (7,2)$$

The field of the electron can be determined by the method used in § 2, but in the present case it is somewhat simpler to make use of the well known solution of the equation (2,5):

$$A_\omega(x,y,z) =$$

$$= \frac{1}{c}\int \frac{j_\omega(x',y',z')}{R} e^{\frac{\pm i\omega nR}{c}} dx'dy'dz', \quad (7,3)$$

where

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2},$$

it being understood that A_ω and j_ω are the z-components of the corresponding vectors. The two signs in the exponential correspond to the retarded and to the advanced vector potentials of the field of the electron; rejecting the advanced potential we have to retain only the minus sign.

Substituting (7, 2) in (7,3) one finds:

$$A_\omega(\rho, z) = \frac{e}{2\pi c}\int_{-vt_0}^{+vt_0} \frac{dz'}{R} e^{-\frac{i\omega}{v}(z'+\beta nR)}, \quad (7,4)$$

where

$$R = \sqrt{\rho^2 + (z-z')^2}, \quad \rho^2 = x^2 + y^2.$$

At large distances from the electron ($R \gg vt_0$)

$$R \approx R_0 - \frac{zz'}{R_0} = R_0 - z' \cos\theta,$$

where $R_0 = \sqrt{\rho^2 + z^2}$ and $\cos\theta = \frac{z}{R_0}$. Neglecting the difference between R and R_0 in the denominator of the integrand one obtains:

$$A_\omega = \frac{e}{2\pi c R_0} e^{-\frac{i\omega n}{c}R_0} \int_{-vt_0}^{+vt_0} dz'\, e^{-\frac{i\omega z'}{v}(1-\beta n\cos\theta)} =$$

$$= \frac{e\beta q(\omega)}{\pi R_0 \omega} e^{-\frac{i\omega n R_0}{c}}, \quad (7,5)$$

where

$$q(\omega) = \frac{\sin\{\omega t_0 (1-\beta n \cos\theta)\}}{1-\beta n\cos\theta}. \quad (7,6)$$

Finally, using (2,4) and (2,7), one finds that in the wave zone of the electron, where $R_0 \gg \frac{c}{n\omega}$,

$$\left.\begin{aligned} H_\varphi &= -\frac{2e\beta}{\pi c R_0}\sin\theta \int_0^\infty n\, q(\omega)\sin\omega\left(t - \frac{nR_0}{c}\right)d\omega, \\ E_\rho &= -\frac{2e\beta}{\pi c R_0}\sin\theta \int_0^\infty q(\omega)\sin\omega\left(t - \frac{nR_0}{c}\right)d\omega, \\ E_z &= \frac{2e\beta}{\pi c R_0}\sin\theta\cos\theta \int_0^\infty q(\omega)\sin\omega\left(t - \frac{nR_0}{c}\right)d\omega; \end{aligned}\right\} \quad (7,7)$$

all other components of $\mathbf{H}$ and $\mathbf{E}$ vanish.

Thus at large distances from the electron $\left(R_0 \gg vt_0 \text{ and } R_0 \gg \frac{c}{n\omega}\right)$ the field consists of a superposition of spherical waves, the common origin of which coincides with the position of the electron. The polarization of these waves is the same as of the waves emitted by an dipole oscillating along the axis z, but their amplitude depends on the polar angle θ not only through the usual factors standing in (7,7) before the sign of the integral, but also through the factor $q(\omega)$ under the sign of the integral.

If $\omega t_0 \gg 1$, this factor has a sharp maximum for $\beta n\cos\theta = 1$ and thus enhances the waves belonging to the range of frequencies defined by $\beta n(\omega) > 1$ and emitted under an acute angle $\theta = \arccos\frac{1}{\beta n}$ to the direction of the motion of the electron. The radiation of frequencies outside of this range is mainly due to the acceleration experienced by the electron at the moments $t = \pm t_0$ and will show in general the same dependence on the polar angle θ as the usual dipole radiation.

It can be shown with the help of the formula

$$\int_{-\infty}^{+\infty} \sin(\omega t + \alpha)\sin(\omega' t + \beta)\,dt = \pi\delta(\omega - \omega'),$$

that the time integral of the Poynting vector is equal to

$$\frac{c}{4\pi}\int_{-\infty}^{+\infty} [\mathbf{E},\mathbf{H}]\,dt = \frac{e^2\beta^2}{c\pi^2 R_0^2}\sin^2\theta \frac{\mathbf{R}_0}{R_0}\int_0^\infty n q(\omega)\,d\omega.$$

Hence, the total amount of energy radiated by the electron is equal to

$$W = \frac{2e^2\beta^2}{\pi c}\int_0^\infty n\, J(\omega)\,d\omega, \quad (7,8)$$

where

$$J(\omega) = \int_0^\pi q^2(\omega) \sin^3 \theta \, d\theta. \qquad (7,9)$$

This integral can be evaluated in terms of elementary functions and of Si and Ci. For our purpose it will be sufficient to note the values of J for $\omega t_0 \gg 1$. Neglecting the rapidly oscillating terms of the type $\sin 2\omega t_0$ one obtains:

$$\left.\begin{aligned} J = J_1 &= \frac{1}{\beta^3 n^3}\left(\ln \frac{1+\beta n}{|1-\beta n|} - 2\beta n\right) \\ &\text{if } \beta n < 1; \\ J = J_1 &+ \frac{\pi \omega t_0}{\beta n}\left(1 - \frac{1}{\beta^2 n^2}\right) \\ &\text{if } \beta n > 1. \end{aligned}\right\} (7,10)$$

These expressions are valid under the condition that $\omega t_0 |1-\beta n| \gg 1$; J is always finite and has for $\beta n = 1$ the value

$$J = \ln(4\gamma\omega t_0) - 1,$$

where $\gamma = 1.781\ldots$

According to (7,9) the radiation of the electron can be divided in two parts. The one corresponds to $J = J_1$ and is due to the acceleration of the electron at the moments $t = \pm t_0$. Substituting for J in (7,8) the value of J_1 one obtains a divergent result; this was to be expected, since we have assumed the acceleration of the electron to be infinite.

However, this assumption hardly has any substantial influence on the second part of the radiation, which corresponds to the additional term in (7,9) for $\beta n > 1$. Substituting for J in (7,8) the difference $J - J_1$ one obtains:

$$\Delta W = \frac{2e^2 \beta t_0}{c} \int_{\beta n > 1} \omega d\omega \left(1 - \frac{1}{\beta^2 n^2}\right).$$

Dividing ΔW by the length of the path of the electron $2vt_0$ we arrive again to our previous equation (3,6).

The new derivation of this equation has, however, the advantage, that it makes clear the condition of its validity, namely the condition

$$\omega t_0 \gg 1. \qquad (7,10)$$

Somewhat extrapolating the results of this section we can define t_0 as the time, during which the velocity of the electron may be considered as constant. For our purpose the velocity v can be considered as constant during the time t_0, if the difference between vt_0 and the actual path travelled by the electron during t_0 is much less than the wave length of the radiation considered:

$$\frac{t_0^2}{2}\frac{dv}{dt} \ll \lambda = \frac{2\pi c}{n\omega},$$

or

$$\omega^2 t_0^2 \cdot \frac{1}{2\pi\omega}\frac{dv}{dt} \ll \frac{2c}{n}.$$

Both this condition and the condition (7,10) can be satisfied simultaneously by a suitable choice of t_0 only if

$$\frac{1}{2\pi\omega}\frac{dv}{dt} = T\frac{dv}{dt} \ll \frac{c}{n}, \qquad (7,11)$$

i. e. if the variation of βn during one period T of the wave considered is much less than 1. Only under this condition can the formula (3,6) be applied to the calculation of the rate of the radiation of frequency $\omega = \frac{2\pi}{T}$ emitted by the electron.

The rate, at which a fast electron in a ponderable medium loses its energy by ionization, is actually so small, that the condition (7,11) is satisfied for all frequencies corresponding to the visible radiation. Using the formula (3,6) one has, of course, to take in account the gradual slowing down of the electron and has to consider β as a function of the path travelled by the electron. The condition (7,11) is violated only at relatively large intervals of time, when the electron changes its velocity abruptly in a head—on collision with atomic nuclei or atomic electrons.

Actually the total intensity of the coherent radiation of electrons of any velocity in any ponderable medium is quite negligible in comparison with the ordinary «Bremsstrahlung». The experimental investigation of the Čerenkov radiation is made possible only by the difference in the spectral distribution of both radiations and by the fact, that in the visible region the intensity of the coherent radiation is much greater than that of the Bremsstrahlung.

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