Self-Stress, Covariance and the Classical Electron

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It is shown that the covariance problem of the classical electron cannot be divorced from the self-stress problem as claimed recently by Rohrlich. The basic invariance principle of a relativistic field theory (i.e., the formalism should be independent of the choice of the family of space-like hypersurfaces) is used to give another derivation of von Laue's theorem. This new derivation makes it quite clear that the energy-momentum expressions calculated from the Maxwell energy-stress tensor are surface dependent because of the nonvanishing of the self-stress. The fact that the early workers obtained the well-known factor of $\frac{4}{3}$ in the momentum expressions for one choice of surface, and Rohrlich and others find a factor of unity for another choice of surface, merely serves to confirm the violation of the basic invariance principle. Some additional remarks are made about the covariance problem of classical electrodynamics. The advantages and implications of the alternative proposal of Poincaré are elucidated.

1. INTRODUCTION

FOR many years it has been generally accepted that one of the fundamental objections to the Lorentz electron is the fact that the expressions for energy and momentum one calculates from the Maxwell energy-stress tensor of a charged particle do not have correct transformation properties, *and* that the reason for this is that the Lorentz electron has nonvanishing self-stress.[1] Recently, Rohrlich,[2] rediscovering an idea suggested independently by Kwal,[3] Mandel,[4] and Fermi,[5] has claimed that this lack of relativistic covariance of the classical electron is not due to the difficulty with the self-stress, but rather due to a lack of covariance in the definition of the expressions for energy and momentum. Thus, whereas the traditional objection to the classical electron starts from expressions of the form

$$P_\nu = \int M_\nu{}^0 d^3x, \quad (\nu = 0,1,2,3), \tag{1}$$

where $M_\nu{}^\mu$ are the components of the Maxwell energy-stress tensor of the classical electron in rectangular coordinates (we set $c=1$), the Kwal–Rohrlich suggestion is to replace (1) by

$$P_\nu = \int M_\nu{}^\mu n_\mu d\sigma, \tag{2}$$

where n_μ is a unit time-like vector tangent to the electron's trajectory, and $d\sigma$ is an element of a "flat" space-like hypersurface orthogonal to n_μ. As shown by Kwal and Rohrlich, the expressions (2) do have the correct transformation properties. It is therefore a conclusion of this work that one can divorce the classical self-stress problem from the covariance problem.

It is the purpose of this note to show that this suggestion actually violates the concept of "covariance" as it is understood in contemporary relativistic field theory (e.g., quantum electrodynamics) and that this violation follows from the nonvanishing of the self-stress. In other words, the usual objection to $M_\nu{}^\mu$ remains.

We shall also comment briefly on the significance of the violation of relativistic invariance in classical electron theory, and attempt to bring out clearly some of the important implications of the alternative proposal of Poincaré.

2. THE INVARIANCE PRINCIPLE

As pointed out by Weiss,[6] Tomonaga,[7] and Schwinger,[8] in a relativistic field theory, *the laws*

[1] A. Pais, *Developments in the Theory of the Electron* (Institute for Advanced Study and Princeton University, Princeton, New Jersey, 1948).
[2] F. Rohrlich, Am. J. Phys. **27**, 639 (1960); Phys. Today **15**, No. 3, 19 (1962). See also J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), p. 594; also W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), p. 390.
[3] B. Kwal, J. Phys. Radium **10**, 103 (1949).
[4] H. Mandel, Z. Physik **39**, 40 (1926). This discussion anticipates that of Kwal and Rohrlich, and until now has been overlooked.
[5] E. Fermi, Physik. Z. **23**, 340 (1922); Nuovo Cimento **25**, 159 (1923). Fermi's presentation differs from that of Mandel, Kwal, and Rohrlich in that he does not discuss the problem from the standpoint of the transformation properties of the energy-stress tensor.
[6] P. Weiss, Proc. Roy. Soc. (London) **169**, 102 (1938).
[7] S. Tomonaga, Prog. Theoret. Phys. (Kyoto) **1**, 1 (1946).
[8] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

of nature are to be formulated in a way that is independent of the choice of the family of space-like hypersurfaces; for brevity, we shall refer to this as the invariance principle. Using this principle, Schwinger derives in a well-known way, a necessary condition for an energy-stress tensor $T_\nu{}^\mu$ to yield covariantly conserved expressions for energy and momentum, i.e.,

$$T_\nu{}^\mu{}_{,\,\mu}=0, \tag{3}$$

where $T_\nu{}^\mu{}_{,\,\mu}\equiv\partial_\mu T_\nu{}^\mu$.

We say "necessary condition" because as we shall see below it is not sufficient, in particular $T_\nu{}^\mu=K\delta_\nu{}^\mu$, where K is a constant, satisfies (3), but such a tensor does *not* lead to expressions for energy and momentum that transform as a four-vector.[9] Condition (3) is of course *not* satisfied by the Lorentz electron. One has $M_\nu{}^\mu{}_{,\,\mu}=j^\mu F_{\nu\mu}$, where j^μ is proportional to a delta function for a point charge, and $F_{\nu\mu}$ is the Maxwell field tensor. This is the well-known self-force problem, which the Kwal–Rohrlich approach does not purport to solve.

To obtain a sufficiency condition on $T_\nu{}^\mu$, it is convenient to work with families of plane space-like hypersurfaces.[10] For the problem considered here, the above invariance principle reduces to the statement: The expressions for energy and momentum calculated from (2) are to be independent of the "tilt" of the space-like hypersurface. It follows immediately that (2) should reduce to (1), but (1) yields expressions that are not covariant, there is therefore a contradiction with covariance in the Kwal–Rohrlich suggestion.

To see this contradiction more directly, it is convenient to consider the electron in its rest frame, so that $M_\nu{}^\mu$ reduces to $(M_0{}^0,M_j{}^i)$; more generally, we may consider an arbitrary energy-stress tensor which, for a suitable choice of coordinates can be brought in the form $(T_0{}^0,T_j{}^i)$, where $T_0{}^0,T_j{}^i$ are functions only of the spatial coordinates. If we choose the normal as $n_\mu=(1,0,0,0)$, we find of course $P_0=\int T_0{}^0d^3x$, $P_j=0$. However, *without transforming out of the electron's rest frame,* let us merely tilt the space-like surface. We choose $n'_\mu=(1,\lambda_i)\gamma$, where $\gamma=(1-\lambda_i{}^2)^{-\frac{1}{2}}$, $\lambda_i{}^2<1$. In the rest frame with a tilted space-like hypersurface we have, since $T'_\nu{}^\mu=T_\nu{}^\mu$,

$$P'_\nu=\int T'_\nu{}^\mu n'_\mu d\sigma'=\gamma^{-1}\int T_\nu{}^\mu n'_\mu d^3x. \tag{4}$$

Hence, $P'_0=\int T_0{}^0d^3x=P_0$ as before, but we also find

$$P'_j=\int T_j{}^i\lambda_i d^3x. \tag{5}$$

Now P_j originally was zero, and by the invariance principle we should have $P_j'=P_j$. But this is true if, and only if

$$\partial P'_j/\partial\lambda_i=0, \tag{6}$$

and hence

$$\int T_j{}^i d^3x=0. \tag{7}$$

As is well known, if $T_j{}^i=M_j{}^i$, condition (7) is not fulfilled; this is the self-stress problem. Incidentally, we see that $T_j{}^i=K\delta_j{}^i$ also does not satisfy (7).

Another proof that condition (7) is necessary and sufficient to guarantee correct transformation properties follows from the well-known discussions of Einstein,[11] Planck,[12] and von Laue,[13] and is sometimes referred to as von Laue's theorem.[14] One considers the electron as seen in a moving frame, and demands that P_ν defined by $\int T_\nu{}^0d^3x$ transform correctly. However, we have arrived at the condition more directly by recognizing that the principle that permits P_ν to be calculated from $\int T_\nu{}^0d^3x$ in a moving frame also directly entails (7) in the particle's rest frame.

The Kwal–Rohrlich proposal, therefore, does not in any way remove the well-known difficulty with covariance. Its basic error is to demand that n_μ is tangent to the electron's trajectory, whereas in a covariant theory n_μ should be merely the normal to an *arbitrary* space-like hypersurface.

[9] It is therefore somewhat surprising to find that such an energy-stress tensor, or "cosmological term" with $K=\infty$, appears in quantum electrodynamics as the expectation value of the energy-stress tensor of the vacuum electron fluctuations as shown by J. Schwinger, Phys. Rev. **75**, 651 (1949).

[10] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 9.

[11] A. Einstein, Jb. Radioakt. **4**, 446 (1907).

[12] M. Planck, Ann. Physik **26**, 1 (1908).

[13] M. von Laue, Ann. Physik **35**, 524 (1911).

[14] G. Mie, Ann. Physik **40**, 7 (1913). See the historical account in Max Jammer, *Concepts of Mass* (Harvard University Press, Cambridge, Massachusetts, 1961), p. 197.

Actually, Rohrlich explicitly recognizes that this should be the case for the total energy-stress tensor of the system, but fails to draw the appropriate conclusions. Thus, if the total energy-stress tensor is of the form $T_\nu{}^\mu = M_\nu{}^\mu + N_\nu{}^\mu$, where $N_\nu{}^\mu$ describes all other contributions, then the total 4-momentum is given by

$$P_\nu = \int (M_\nu{}^\mu + N_\nu{}^\mu) n_\mu d\sigma, \tag{8}$$

where it is agreed that n_μ should be arbitrary. But then (8) reduces to

$$P_\nu = \int (M_\nu{}^0 + N_\nu{}^0) d^3x, \tag{9}$$

and the main point of the classical argument is that since $\int M_\nu{}^0 d^3x$ is not covariant, and also since P_ν must be covariant, the energy-stress tensor $N_\nu{}^\mu$ cannot vanish, and must be chosen such that in the rest frame of the electron $\int (M_j{}^i + N_j{}^i) d^3x = 0$, and also we must have from (3), $M_j{}^i{}_{,i} + N_j{}^i{}_{,i} = 0$, for stress-equilibrium. Thus, the conclusion of the classical argument is that a purely electromagnetic, relativistically-invariant description of the inertial properties of the electron is impossible: *an additional energy-stress tensor is needed.*

We note that the above discussion does not involve the divergent character of $\int M_0{}^0 d^3x$ for a point charge. As remarked by Pais,[1] the self-stress problem is more fundamental than the self-energy problem as far as relativistic transformation properties are concerned. However, as we shall see below it is merely the fact that $\int M_0{}^0 d^3x$ does not vanish and the structure of $M_\nu{}^\mu$ that gives rise to the self-stress problem. We note also that a relationship between the conditions (3) and (7) can be established by use of the identity:

$$\int (T_\nu{}^\mu x^\kappa)_{,\mu} d^4x = \int T_\nu{}^\kappa d^4x + \int T_\nu{}^\mu{}_{,\mu} x^\kappa d^4x. \tag{10}$$

Hence if (3) holds, $\int T_\nu{}^\mu{}_{,\mu} x^\kappa d^4x = 0$, and by the application of the divergence theorem

$$\int T_\nu{}^\mu x^\kappa n_\mu d\sigma = \int T_\nu{}^\kappa d^4x, \tag{11}$$

where n_μ is not restricted to a time-like direction. However if $\int T_\nu{}^\mu x^\kappa n_\mu d\sigma$ vanishes over remote time-like hypersurfaces bounding the four-dimensional volume element [call this condition (a)], then only values on the space-like hypersurfaces contribute. Now if there exists a static coordinate frame for which $T_\nu{}^\mu$ reduces to $(T_0{}^0, T_j{}^i)$ [call this condition (b)], we find that for a suitable choice of the time origin and orientation of the space-like hypersurfaces, (11) reduces to

$$\int T_0{}^0 x^0 d^3x = \int_0^{x^0} dt \int T_0{}^0 d^3x,$$
$$0 = \int_0^{x^0} dt \int T_j{}^i d^3x. \tag{12}$$

From which we see that it is the failure to satisfy (a) that leads to the difficulty with $K\delta_\nu{}^\mu$.

By means of (10), and conditions (a) and (b), we can readily relate the self-stress of the electron to the first moments of the self-force, one finds, upon elimination of a common time factor,

$$\int M_j{}^i d^3x = \int j^0 F_{0j} x^i d^3x. \tag{13}$$

Because of spherical symmetry, the off-diagonal terms do not contribute and we need only consider $\int M_i{}^i d^3x$. Also since $M \equiv M_\mu{}^\mu = 0$, we have finally

$$\int M_i{}^i d^3x = \int j^0 F_{0i} x^i d^3x = -\int M_0{}^0 d^3x. \tag{14}$$

Since $M_0{}^0$ is positive definite, $\int M_0{}^0 d^3x$ cannot vanish; and hence merely this, in conjunction with $M = 0$, entails the nonvanishing of the self-stress and the self-force, and consequently, the violation of the invariance principle.

3. CONCLUDING REMARKS ON COVARIANCE AND THE CLASSICAL ELECTRON

The above discussion brings out the fact that a classical electrodynamics based solely on the electromagnetic field can be covariant only in the *absence of charged particles*, since we have $M_\nu{}^\mu{}_{,\mu} = 0$, and the self-stress problem does not arise. In such a theory, one is therefore restricted to a class of singularity-free solutions (e.g., plane waves), for which no covariant description of the interaction with charged particles is possible, in particular, one cannot describe the emission and absorption of radiation. One cannot even confine the radiation to a box ("hohlraum") in a covariant manner, since the walls of the box must interact to reflect the radiation. Also in such a restricted theory there is no way to make "measurements". It is therefore not too surprising that Maxwell-field models of the electron were unable to meet the challenge of quantum theory; from a covariant standpoint alone, such models contain serious inconsistencies.

In another paper,[15] we shall give an alternative approach to the covariance problem of the classical electron based on the well-known suggestion of Poincaré.[16] This idea has the merit of recognizing that the Maxwell tensor does not provide us with a covariant description of a charged particle (in agreement with von Laue's theorem) and that at least one additional compensating energy-stress system (or "field"), such as the previously introduced $N_\nu{}^\mu$, is needed to obtain covariant inertial properties.

One may see in this recognition of the necessity for a compensating field (or fields) a dramatic foreshadowing of the fact that, later on, the demands of quantum theory for a more consistent description of the "mechanics" of the electron would lead people to introduce new fields. Indeed, from the standpoint of covariance alone, the simplest assumption about the compensating fields is to require that they form irreducible representations of the inhomogeneous Lorentz group: e.g., Klein–Gordon field, Dirac field, Proca field, etc.[17,18] Of course, this is a "simple assumption" only in retrospect and with the benefit of the work that has been done over half a century. But in any case, once we begin to recognize the new fields as (what might have been) compensating fields, the wave theory of the electron becomes somewhat clearer from the standpoint of the classical models: It is a theory in which the "compensating field" plays the dominant role in the mechanics of the electron, and what was formerly the dominant field—the Maxwell field—enters in only as a perturbation.

In conclusion, it would appear that a resolution of the self-stress problem of the classical electron that is in strict conformity with the (restricted) principle of relativity leads us, via a broad interpretation of Poincaré's idea; into the domain of the quantum theory of the electron. It would be very interesting to know whether there are any other relativistically-invariant possibilities, in view of the generally recognized incompleteness in the contemporary theory of the electron such as the arbitrariness of e/m and the fine-structure constant, and the mathematical difficulties inherent in the manipulation of infinite quantities as though they were finite and "small."

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[15] F. R. Tangherlini, Nuovo Cimento **26**, 497 (1962). Although our treatment is within the broader framework of general relativity, the essential elements of a special relativistic treatment are preserved in a form-invariant way. For a recent special relativistic treatment see P.A.M. Dirac, Proc. Roy. Soc. (London) **A268**, 57 (1962).

[16] H. Poincaré, Rend. Circ. Mat. Palermo **21**, 124 (1906). Actually an earlier recognition that Lorentz's theory of the electron required compensating stresses is due to M. Abraham, Physik. Z. **5**, 674 (1904). However, Abraham was attempting to construct a purely electromagnetic theory of inertia and hence regarded the need for these stresses in the Lorentz model as unsatisfactory.

[17] The inhomogeneous Lorentz group has been designated the Poincaré group by Wigner, Bargmann, and Wightman, see, e.g., A. S. Wightman, *Dispersion Relations and Elementary Particles*, edited by C. DeWitt and R. Omnes (John Wiley & Sons, Inc., New York, 1960), p. 164.

[18] Compensation of the classical electron by means of a scalar field and a vector field have been considered, respectively, by E. G. G. Stückelberg, Helv. Phys. Acta **14**, 51 (1941); and F. Bopp, Ann. Physik **38**, 345 (1940). One might also feel that an alternative deduction would be to assume that the compensation is of gravitational origin. However, this way seems barred because in generally covariant notation $M_\nu{}^\mu$ satisfies $M_\nu{}^\mu{}_{;\mu}=j^\mu F_\nu{}^\mu$, whereas an energy-stress tensor must satisfy $T_\nu{}^\mu{}_{;\mu}=0$ [i.e., the analog of (3)] to be acceptable by itself as a source for Einstein's field equations. Poincaré's conjecture (Section 8 of the above cited paper) was that there is some relation between the cause that produces gravitation and that which produces the compensation.