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Radiation Reaction as a Retarded Self-Interaction*

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It is shown that the exact Lorentz-Dirac equation of motion for a strictly point charge is nothing more than the usual Lorentz force law when the retarded self-field of the particle is properly taken into account, as required for the very consistency of the idea of energy-momentum localization in the field.

In two previous papers[1,2] an approach to the classical theory of radiation reaction that uses no advanced fields has been presented. It was shown that the decomposition of the retarded Liénard-Wiechert field of a classical point charge into a velocity field $F_{\mathrm{I}}^{\mu\nu}$ and an acceleration field $F_{\mathrm{II}}^{\mu\nu}$ (both retarded) induces a splitting of the energy-momentum tensor of such a field into a bound part $T_{\mathrm{I}}^{\mu\nu}$ and an emitted part $T_{\mathrm{II}}^{\mu\nu}$, the divergences of which are given by

$$\partial_\nu T_{\mathrm{I}}^{\mu\nu}(x) = -\int_{-\infty}^{+\infty} d\tau\, \delta^{(4)}(x-z(\tau)) \times [m_{\mathrm{Coul}}\, a^\mu(\tau) - \tfrac{2}{3}e^2 \dot{a}^\mu(\tau)] \tag{1a}$$

(here, m_{Coul} is the linearly divergent electromagnetic mass of the particle), and

$$\partial_\nu T_{\mathrm{II}}^{\mu\nu}(x) = -\int_{-\infty}^{+\infty} d\tau\, \delta^{(4)}(x-z(\tau))\, \tfrac{2}{3}e^2 a^2(\tau) v^\mu(\tau). \tag{1b}$$

These results were used to obtain the Lorentz-Dirac equation of motion,

$$m a^\mu = \tfrac{2}{3}e^2(\dot{a}^\mu - a^2 v^\mu) + F^\mu, \tag{2}$$

for the point charge under the action of an external force F^μ. This was accomplished by writing down the energy-momentum balance relation

$$\partial_\nu T_{\mathrm{bare}}^{\mu\nu} + \partial_\nu T_{\mathrm{I}}^{\mu\nu} = -\partial_\nu T_{\mathrm{II}}^{\mu\nu} - f^\mu,$$

where $T_{\mathrm{bare}}^{\mu\nu}$ is the mechanical energy tensor for a point particle of mass m_{bare} such that $m_{\mathrm{bare}} + m_{\mathrm{Coul}} = m$ (m is the observed mass of the particle), and f^μ is the force density corresponding to the external force F^μ.

The procedure sketched above to obtain the equation of motion presents a feature common to most derivations of this kind,[3] namely the main emphasis lies in the action of the particle *on* the field. In fact what one does is to compute the variation of energy-momentum in the field (which is due to changes in the state of motion of the source), and then to postulate that the field has to react back on the source in such a way that the sum of the change in the energy-momentum of the particle per unit of time and the corresponding change in the field is equal to the external force.

Our purpose in this note is to examine in more detail this reaction of the field on the particle. We shall show that the equation of motion (2) is nothing more than the usual Lorentz force law (with an external force F^μ included),

$$m a^\mu = e F^{\mu\nu} v_\nu + F^\mu, \tag{3}$$

when the field $F^{\mu\nu}$ is taken to be the average *retarded* field produced by the particle itself on the the world line.[4] Before going into the proof we want to emphasize that the need of taking into account the self-field is by no means an *ad hoc* assumption but a necessary consistency requirement. In fact, if one wants to maintain the idea that there are energy and momentum localized in the field, and that these quantities are described in the usual way by means of an energy-momentum tensor $T^{\mu\nu}$, then because of the relation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} j_\nu \tag{4}$$

he is forced to conclude that $F^{\mu\nu} j_\nu$ is the electromagnetic force density acting on the charge distribution.[5] The point now is that as long as one stays within the framework of continuous charge distributions, there is no distinction between the field at a given point produced by all charges except the one at the point in consideration and the field obtained when the contribution of the point itself is added, because the amount of charge at one point is always zero for a continuous distribution. However, if point charges are introduced into the theory the situation changes drastically, and the inclusion of the self-field becomes unavoidable, because Eq. (4) is valid only if the field is a solution of Maxwell's equations with the current j^μ, and if the self-field is subtracted out, what is left is a solution with $j^\mu = 0$.

Having made clear that the self-field has to be included, the next step is to include it. The problem arises immediately that the retarded field diverges on the world line. Moreover, it does not only diverge, but the "limit" depends on the way in which the singularity is approached. However, this last feature suggests a way out of both problems: Since there is no preferred direction of approach, we define the field at the singularity as the average value over all possible directions. More precisely: At a given time, in the instantaneous rest system of the particle, we take the integral of each component of the field $F^{\mu\nu}_{\rm ret}$ over the surface of a sphere of radius ϵ centered at the position of the particle, divide it by $4\pi\epsilon^2$, and let ϵ go to zero at the end. The value of the limit in an arbitrary Lorentz frame is defined by a Lorentz transformation from the rest frame.

In geometrical terms, in Minkowski space this translates into an integration over the two-dimensional intersection $\tilde{\Sigma}(\epsilon)$ of a tube of radius ϵ around the world line with a hyperplane orthogonal to the four-velocity $v^\mu(\tau)$ that intercepts the world line at $z(\tau)$, as shown in Fig. 1. With the notation explained in Fig. 1, the value $F^{\mu\nu}_{\rm ret}(z)$ of the retarded field at the point z on the world line is therefore defined as[6]

$$F^{\mu\nu}_{\rm ret}(z)=\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon^2}\int_{\tilde{\Sigma}(\epsilon)} d^2\sigma\, F^{\mu\nu}_{\rm ret}(z+\epsilon u). \qquad (5)$$

The above integral is readily evaluated with the help of the following expansion found by Dirac[7]:

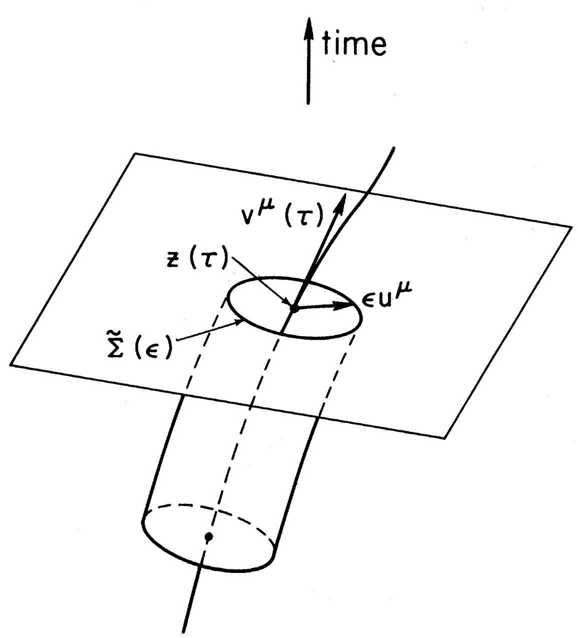

FIG. 1. Minkowski geometry used in averaging the field. $\tilde{\Sigma}(\epsilon)$ is the intersection of a tube of radius ϵ that surrounds the world line and a spacelike hyperplane orthogonal to $v^\mu(\tau)$ that intercepts the world line at $z(\tau)$ (three-space proper to the particle at the time τ). The vector from $z(\tau)$ to a generic point on $\tilde{\Sigma}(\epsilon)$ is written as ϵu^μ, with $u_\alpha u^\alpha=+1$.

$$F^{\mu\nu}_{\rm ret}(z+\epsilon u)=2e\bigg[\frac{1}{\epsilon^2}v^{[\mu}u^{\nu]}-\frac{1}{2\epsilon}(v^{[\mu}a^{\nu]}+a_u v^{[\mu}u^{\nu]}) + \tfrac{3}{4}a_u v^{[\mu}a^{\nu]}+\tfrac{1}{8}a^2 v^{[\mu}u^{\nu]} - \tfrac{1}{2}\dot{a}^{[\mu}u^{\nu]}-\tfrac{2}{3}\dot{a}^{[\mu}v^{\nu]}+O(\epsilon)\bigg]$$

(here $a_u=a_\alpha u^\alpha$), and the relations

$$\int_{\tilde{\Sigma}(\epsilon)} d^2\sigma\, u^\alpha=\int_{\tilde{\Sigma}(\epsilon)} d^2\sigma\, u^\alpha u^\beta u^\gamma=0,$$

$$\frac{1}{4\pi\epsilon^2}\int_{\tilde{\Sigma}(\epsilon)} d^2\sigma\, u^\alpha u^\beta=\tfrac{1}{3}(\eta^{\alpha\beta}+v^\alpha v^\beta).$$

The result is

$$F^{\mu\nu}_{\rm ret}(z)=2e\left[-\left(\lim_{\epsilon\to 0}\frac{2}{3\epsilon}\right)v^{[\mu}a^{\nu]}-\tfrac{2}{3}\dot{a}^{[\mu}v^{\nu]}\right], \qquad (6)$$

and consequently we get for the Lorentz force

$$eF^{\mu\nu}_{\rm ret}v_\nu=-\left(\lim_{\epsilon\to 0}\frac{2e^2}{3\epsilon}\right)a^\mu+\tfrac{2}{3}e^2(\dot{a}^\mu-a^2v^\mu). \qquad (7)$$

The first term of the right-hand side diverges, and it represents the infinite Coulomb mass of the point charge which is to be absorbed in the usual way into the observed finite mass m of the particle. We see then that when the result (7) is not introduced in (3), the equation of motion (2) is obtained, as we stated at the beginning.[8]

As a last point, it is interesting to remark that in spite of the fact that the energy-momentum of the acceleration field is totally radiated away, it would be erroneous to expect that the contribution of $F^{\mu\nu}_{\rm II}$ in (7) would give the negative of the emission rate, $-\frac{2}{3}e^2a^2v^\mu$, and one can indeed verify without any detailed calculation that this is not the case by noticing that $F^{\mu\nu}_{\rm II}v_\mu v_\nu$ equals zero, whereas $-\frac{2}{3}e^2a^2v^\mu v_\mu=\frac{2}{3}e^2a^2$ does not vanish. The reason is that $F^{\mu\nu}_{\rm II}$ is not a solution of the Maxwell equation $\partial_\nu F^{\nu\mu}=-4\pi j^\mu$, and therefore a relation analogous to (4), with $T^{\mu\nu}$ replaced by $T^{\mu\nu}_{\rm II}$ and $F^{\mu\nu}$ replaced by $F^{\mu\nu}_{\rm II}$, does not hold. This means that even though $eF^{\mu\nu}_{\rm II}v_\nu$ is the force that the acceleration field exerts on its own source, its negative is *not* the rate of change of the energy-momentum of that field. Similar arguments hold for the velocity field, and one sees that in order to find out how the total energy-momentum lost by the particle splits into an emitted and a bound part, one has to make a detailed analysis of the properties of the field, as has been done in Ref. 1. On the other hand, if our approach is going to be consistent, Eq. (4) should hold for the *full* retarded field, and this is what actually happens. In fact, adding (1a) and (1b) and performing the time integration, we get

$$\partial_\nu T^{\mu\nu}(x) = -\delta^{(3)}(\vec{x} - \vec{z}(x^0)) \times [m_{\text{Coul}}\, a^\mu - \tfrac{2}{3}e^2(\dot{a}^\mu - a^2 v^\mu)]/v^0.$$

Next, recalling that

$$j^\mu(x) = e\delta^{(3)}(\vec{x} - \vec{z}(x^0))v^\mu/v^0,$$

and using Eq. (7), we find

$$F^{\mu\nu}_{\text{ret}}(x) j_\nu(x) = -\delta^{(3)}(\vec{x} - \vec{z}(x^0)) \times [m_{\text{Coul}}\, a^\mu - \tfrac{2}{3}e^2(\dot{a}^\mu - a^2 v^\mu)]/v^0.$$

This shows that with our definition of $F^{\mu\nu}$ on the world line, the relation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} j_\nu$$

holds everywhere.

Note added in proof. It is natural at this point to ask whether $F^{\mu\nu}$ given by (6) is the *only* possible solution (at $x = z$) to (4) regarded as an equation for the unknown $F^{\mu\nu}(z)$. This is a meaningful question because, on account of Gauss's integral theorem, the divergence $\partial_\nu T^{\mu\nu}(z)$ can be evaluated with a knowledge of the field off the world line where it is well defined and not singular. The answer is in the affirmative[9] provided that a quite natural assumption is made. In short, the assumption is that the field on the world line should have as much in common as possible with the retarded field that exists off the world line ("maximal matching"). In other words, the field existing off the world line and Maxwell's equations determine completely the field on the world line. The equation of motion for the particle then follows uniquely.

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[1]C. Teitelboim, Phys. Rev. D1, 1572 (1970); 2, 1763 (E) (1970).

[2]C. Teitelboim, Phys. Rev. D3, 297 (1971).

[3]This short communication is not the place to give an exhaustive bibliography on the subject. A particularly complete list of references with interesting comments on them can be found in the book by H. Arzeliès, *Rayonnement et Dynamique du Corpuscle Chargé Fortement Accéléré* (Gauthier Villars, Paris, 1966). Some references, besides the ones quoted in Ref. 1, that contain treatments of the radiation-reaction problem without advanced fields are M. Mathisson, Z. Physik 67, 826 (1931); Proc. Cambridge Phil. Soc. 38, 40 (1942); L. Infeld and P. R. Wallace, Phys. Rev. 57, 797 (1940); L. Infeld and J. Plebanski, Bull. Acad. Polon. Sci. 4, 347 (1956); N. E. Fremberg, Proc. Roy. Soc. (London) A188, 18 (1946); Lund Univ. Mat. Sem. 7, 1946 (unpublished). In the last two papers conclusions similar to the ones reached in this note are obtained by defining the value of the self-field on the world line using a procedure of analytical continuation. Another interesting paper is the one by M. A. Mathisson, Proc. Cambridge Phil. Soc. 36, 331 (1940), where it is shown how to get finite equations in "the passage to mechanics" from the general energy-momentum balance relation $\partial_\nu T^{\mu\nu} = f^\mu$.

[4]It is worthwhile to emphasize here that the particle is regarded strictly as a point charge. It is therefore meaningless to speak of a charge distribution inside the particle or of the action of one piece of the charge on another. This viewpoint is different from the one of H. A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat* (Tuebner, Leipzig, 1909), who thought of the particle as having a finite extension, inside of which some continuous charge distribution existed, and derived the radiation-damping formula by looking at the retarded action of each piece of the charge onto the others. His derivation cannot really be thought of as a self-interaction approach, because no part of the charge interacts with itself [see discussion below Eq. (4)]. The procedure used in this paper to define the field produced by the particle on its world line involves an average over a region where the charge *is not* in contrast with the integrations of Lorentz over the charge distribution in the electron.

[5]See for example F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), pp. 92–93.

[6]It is evident that for a point where the field is not singular, such an averaging procedure gives exactly the value of the field at that point, for any choice of the slicing spacelike plane. Therefore the integral (5) can be taken as a redefinition of the field at every space-time event.

[7]P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938). See also Ref. 5, p. 143; we use the same notation and metric conventions that are used there.

[8]It might seem strange to see a self-energy $2e^2/3\epsilon$ instead of the $e^2/2\epsilon$ usually arising in the calculations of electromagnetic mass. However, there is actually no difference between the two expressions, since the limit $\epsilon \to 0$ is to be taken. The ϵ here does not have any physical significance (the particle is strictly a point charge; see Ref. 4), and the quantity $2e^2/3\epsilon$ is *not* meant to be the electromagnetic-field energy outside a sphere of radius ϵ in the case of a Coulomb field.

[9]C. Teitelboim and C. A. López (unpublished).