Magnetic-Dipole Models

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A recent book rejects the amperian-current model of a magnetic dipole and accepts the magnetic-charge model, because the first model would lead to inconsistencies in the interpretation of Poynting's theorem. It is shown that the inconsistencies are only apparent. The choice between the two models should be based on measurement of the force on a magnetic dipole, which, in nonstationary cases, is different for the two models.

A RECENT book[1] rejects the amperian-current model of a magnetic dipole and accepts the magnetic-charge model. On p. 25 it says that the use of the first model "makes it impossible to develop a macroscopic theory of electromagnetism that is both self-consistent and in agreement with experimental evidence." I disagree with this and with the supporting reasons, given in Sec. 7.10, "Poynting's Theorem for the Amperian-Current Model."

Section 7.10 gives *a self-consistent interpretation of Poynting's theorem for the amperian-current model, which the authors consider not to agree with experimental evidence.*

If the amperian-current model is used, a magnetization $\mathbf{M}$ is equivalent to a current density $\mathbf{J}_m = \text{curl}\mathbf{M}$. As a result, Poynting's theorem contains a term $\mathbf{E}\cdot\text{curl}\mathbf{M}$, which consistency requires to be interpreted as the density of the power absorbed by magnetized matter. On p. 312 it says: "This interpretation is very disturbing for three reasons. In the first place, different parts of a permanent magnet would appear to absorb or deliver power in the presence of a static electric field. Although it can be readily shown that the net power absorbed by the magnet would always be equal to zero, the continuous transfer of energy from one part of the magnet to another is unacceptable unless we can point to some other physical mechanism capable of transferring the same energy back. In the second place, $\mathbf{J}_m = \text{curl}\mathbf{M}$ implies that a surface amperian current must be present on any surface at which the tangential component of $\mathbf{M}$ is discontinuous. This implies, in turn, that energy may be absorbed or delivered by a magnetized body right at its surface, with an infinite volume density. This second conclusion is also unacceptable from a physical standpoint. Finally, when energy is dissipated because of hysteresis in the constituent relation of the magnetic material, the space distribution of the energy dissipated as given by $\mathbf{E}\cdot\text{curl}\mathbf{M}$ does not agree with the space distribution of the heat generated, as determined by measurements."

To study these reasons, let us consider a part of a body composed of amperian currents. This can be done in two ways. First, we can imagine a part with a boundary that does not intersect amperian currents. The magnetization of such a part is equivalent to a volume current of density $\text{curl}\mathbf{M}$ and a surface current of density $\mathbf{M}\times\mathbf{n}$, where $\mathbf{n}$ is a unit vector directed as the outward normal. Since the surface currents of adjacent parts are equal but opposed, they can be disregarded in many cases. The part is then considered as carrying only a volume current. This corresponds to a boundary that is allowed to intersect amperian currents. This conception of a part is used in the book.

To study the power absorbed by a magnetized body, we first consider parts whose boundaries do not intersect amperian currents. The power absorbed by an amperian current I of moment $\mathbf{m}$ is

$$I\oint d\mathbf{s}\cdot\mathbf{E} = I\int da\mathbf{n}\cdot\text{curl}\mathbf{E} = \mathbf{m}\cdot\text{curl}\mathbf{E}.$$

The density of absorbed power is thus

$$p_1 = \mathbf{M}\cdot\text{curl}\mathbf{E} = -\mathbf{M}\cdot\partial\mathbf{B}/\partial t. \tag{1}$$

The corresponding form of Poynting's theorem

[1] R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces* (John Wiley & Sons, Inc., New York, 1960).

can be written as

$$\mathrm{div}(\mathbf{E}\times\mathbf{H})+\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{\mathbf{B}}{\mu_0}\cdot\frac{\partial\mathbf{B}}{\partial t}=-\mathbf{E}\cdot\left(\mathbf{J}_f+\frac{\partial\mathbf{P}}{\partial t}\right)+\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial t}. \quad (2)$$

Combination of the two magnetic terms leads to a power-conversion density $\mathbf{H}\cdot\partial\mathbf{B}/\partial t$, in agreement with experimental evidence.

Next, we follow the authors by considering the power absorbed by the volume current. The density of absorbed power is

$$p_2=\mathbf{E}\cdot\mathrm{curl}\mathbf{M}=\mathbf{M}\cdot\mathrm{curl}\mathbf{E}-\mathrm{div}(\mathbf{E}\times\mathbf{M})=-\mathbf{M}\cdot\partial\mathbf{B}/\partial t-\mathrm{div}(\mathbf{E}\times\mathbf{M}). \quad (3)$$

The corresponding form of Poynting's theorem can be written as

$$\mathrm{div}\left(\mathbf{E}\times\frac{\mathbf{B}}{\mu_0}\right)+\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{\mathbf{B}}{\mu_0}\cdot\frac{\partial\mathbf{B}}{\partial t}=-\mathbf{E}\cdot\left(\mathbf{J}_f+\frac{\partial\mathbf{P}}{\partial t}\right)+\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathrm{div}(\mathbf{E}\times\mathbf{M}). \quad (4)$$

It is the last term of (3) that has worried the authors. The term is the source density of a power flow of density $-\mathbf{E}\times\mathbf{M}$ in the amperian currents: A part of the circuit of an amperian current absorbs power from the field, another delivers power to the field (see the Appendix). That is why the term does not appear in (1), related as it is to parts whose boundaries do not intersect amperian currents.

Let us now consider the three reasons quoted above. As a comparison of (2) and (4) shows, the energy transferred in the amperian currents is transferred back by an electromagnetic power flow of density $\mathbf{E}\times\mathbf{B}/\mu_0-\mathbf{E}\times\mathbf{H}=\mathbf{E}\times\mathbf{M}$. The power flow in the amperian currents carries off power absorbed with volume density $-\mathrm{div}\times(\mathbf{E}\times\mathbf{M})$, which may be infinite at the surface. The space distribution of the energy dissipated because of hysteresis is given by $-\mathbf{M}\cdot\partial\mathbf{B}/\partial t$. The reasons against the interpretation of $\mathbf{E}$ curl$\mathbf{M}$ as the density of absorbed power are therefore invalid.

Section 7.10 also gives *an interpretation of Poynting's theorem in agreement with experimental evidence, which the authors consider to be inconsistent with the amperian-current model.*

On pp. 312–313 it says about a theory based on the amperian-current model: "While $\mathbf{B}/\mu_0$ is taken as the fundamental magnetic vector, $\mathbf{E}\times\mathbf{H}$ rather than $\mathbf{E}\times\mathbf{B}/\mu_0$ is interpreted as representing the density of the electromagnetic power flow. This interpretation seems to us inconsistent for the following reason. When the amperian-current model is adopted, $\mathbf{H}$ becomes a "mixed" vector, involving the state of magnetization of matter as well as the magnetic part of the electromagnetic field. Thus, $\mathbf{E}\times\mathbf{H}$ cannot represent an inherent property of the electromagnetic field, which, by its very definition, must be independent of the state of matter."

The following considerations may dispel this feeling of inconsistency. One need not consider $\mathbf{H}$ and $\mathbf{B}$ such that one of them measures the magnetic field, while the other is a mixed vector. When the field is integrated along a line that does not intersect dipoles, its measure is the component of $\mathbf{H}$ parallel to the line. When the field is integrated over a surface that does not intersect dipoles, its measure is the component of $\mathbf{B}$ normal to the surface. This does not depend on the dipole model used. These are precisely the integrations that appear in the integral form of Maxwell's equations. Similar remarks hold for $\mathbf{E}$ and $\mathbf{D}$. Therefore, I prefer to consider $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$ to be equally "fundamental." From this point of view the concept of "inherent property of the electromagnetic field" as a property "independent of the state of matter" loses its sense. Hence, I feel no inconsistency between the use of the amperian-current model and the interpretation of $\mathbf{E}\times\mathbf{H}$ as the density of electromagnetic power flow.

A choice between the two magnetic-dipole models must be based on experimentally determinable differences in properties. A dipole is characterized by the field it produces and by the force and couple exerted on it in free space. The fields and couples are the same for the models, but the forces are different. The force on an amperian current I of moment $\mathbf{m}$ is given by

$$\mathbf{F}/\mu_0 = I\oint d\mathbf{s}\times\mathbf{H}$$

$$= I\int da(\mathbf{n}\times\nabla)\times\mathbf{H} = (\mathbf{m}\times\nabla)\times\mathbf{H}$$

$$= (\mathbf{m}\cdot\nabla)\mathbf{H} + \mathbf{m}\times(\nabla\times\mathbf{H}) - \mathbf{m}(\nabla\cdot\mathbf{H}),$$

$$\mathbf{F} = (\mu_0\mathbf{m}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{m}\times(\partial\epsilon_0\mathbf{E}/\partial t). \qquad (5)$$

The force on a magnetic-charge dipole of moment $\mathbf{m}$ is given by

$$\mathbf{F} = (\mu_0\mathbf{m}\cdot\nabla)\mathbf{H} - (\partial\mu_0\mathbf{m}/\partial t)\times\epsilon_0\mathbf{E}. \qquad (6)$$

The last term is the force on a magnetic current [Eq. (7.11)]. The difference between the forces is $\epsilon_0\mu_0\partial(\mathbf{E}\times\mathbf{m})/\partial t$, the time rate of the difference in electromagnetic momentum of the models [Eq. (5.36)]. It would be interesting if (5) and (6) could be tested experimentally.

As long as there is no experimental evidence against the amperian-current model and in favor of the magnetic-charge model, it is unjustified to reject the first and accept the second. On the contrary, atomic theory leads to the amperian-current model. An experimental result that could be explained only by using the magnetic-charge model would be rather revolutionary. Hence, we should adhere to the amperian-current model unless experimental evidence compels us to abandon it.

APPENDIX. POWER FLOW IN AMPERIAN CURRENTS IN AN ELECTRIC FIELD

We divide the circuit of an amperian current I in striplike subcircuits by straight lines perpendicular to the electric field. In each subcircuit we imagine a current I of such a direction that the current in the original circuit remains the same. We thus arrive at a combination of striplike dipoles that is equivalent to the original dipole. A striplike dipole absorbs or delivers power only at the ends, line elements ds_1 and ds_2 of the original circuit. A power $Id\mathbf{s}_1\cdot\mathbf{E}$ is supplied by the field to ds_1 and an equal power is supplied by ds_2 to the field. There is a power flow in the dipole.

Let us consider a rectangular cylinder of height h having cross sections of area σ and normal $\mathbf{n}$. A striplike dipole in the cylinder produces a power flow through the cross sections that intersect the strip. If $\mathbf{d}$ is the vector from ds_1 to ds_2, the proportion of the cross sections through which the power $Id\mathbf{s}_1\cdot\mathbf{E}$ flows in the direction of $\mathbf{n}$ is $\mathbf{d}\cdot\mathbf{n}/h$. The mean power flow in the direction of $\mathbf{n}$ is $(Id\mathbf{s}_1\cdot\mathbf{E})(\mathbf{d}\cdot\mathbf{n}/h)$. Since

$$(d\mathbf{s}_1\cdot\mathbf{E})\mathbf{d} = (\mathbf{E}\cdot\mathbf{d})d\mathbf{s}_1 - \mathbf{E}\times(d\mathbf{s}_1\times\mathbf{d}),$$

while $\mathbf{E}\cdot\mathbf{d}=0$ and $Id\mathbf{s}_1\times\mathbf{d}$ is the dipole moment $\mathbf{m}$, this equals $-(\mathbf{E}\times\mathbf{m})\cdot\mathbf{n}/h$. To find the total mean power flow, we add the contributions of all the strips of all the amperian currents in the cylinder. Since $\Sigma\mathbf{m}=\mathbf{M}h\sigma$, we obtain $-(\mathbf{E}\times\mathbf{M})\cdot\mathbf{n}\sigma$. This is the component in the direction of $\mathbf{n}$ of a vector $-(\mathbf{E}\times\mathbf{M})\sigma$. The mean power-flow density is $-\mathbf{E}\times\mathbf{M}$.