Exact solution to the field equations in the case of an ideal, infinite solenoid

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The calculation of the induced electromotive force in a circular coil placed around the outside of an infinitely long, ideal solenoid is found in many textbooks. This calculation has led to some confusion since we appear to have a changing magnetic field producing an electric field in a region where there is no magnetic field. An exact solution of Maxwell's equations is given when the source is a steady alternating current. The results of the two methods are compared. In addition, this solution makes it possible to give an exact calculation of the radiation reaction and compare it with the radiated power. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The calculation of the induced electromotive force in a circular coil placed around the outside of an infinitely long, ideal solenoid is one of the most common applications of Faraday's law. The calculation uses the quasistatic approximation and begins by utilizing magnetostatics to show that the magnetic field produced by a current flowing through an ideal solenoid is equal to $\mu_0 nI$ inside the solenoid and zero outside. I is the current through the windings of the solenoid and n is the number of turns per unit length. The induced emf is obtained by assuming that the magnetic flux can be calculated from the static value by simply allowing the current to be a function of time. Thus the flux through a circular path perpendicular to the axis of the solenoid and lying outside the solenoid is $\mu_0 nI(t)\pi a^2$, where a is the radius of the solenoid. For $r>a$, Faraday's law gives for the electric field, E_ϕ, the value,

$$2\pi rE_\phi=-\mu_0 n\pi a^2\frac{dI}{dt} \quad \text{or} \quad E_\phi=-\frac{\mu_0 na^2}{2r}\frac{dI}{dt}. \tag{1}$$

Since the magnetic field is uniform inside the solenoid, we get for

$$r<a, \quad 2\pi rE_\phi=-\mu_0 n\pi r^2\frac{dI}{dt} \quad \text{or} \quad E_\phi=-\frac{\mu_0 nr}{2}\frac{dI}{dt}. \tag{2}$$

The disturbing feature of this calculation is that we appear to have, for $r>a$, a changing magnetic field producing an electric field in a region where there is no magnetic field. This result appears to be incompatible with the whole idea of local fields.

How do textbooks deal with this apparent paradox? In a very unsystematic review of lower division and junior level texts, I have found a wide variety of responses. A majority take the easy way out and ignore the matter. In several texts it is stated that the magnetic field is not zero, but negligible outside the solenoid. This appears to be a reference to the static case and the fact that an ideal solenoid is only an approximation to a real solenoid. In one text, the author emphasizes that the Faraday electric field is not directly related to the value of B at points on the path. In a once very popular upper division text, the authors explain that while the magnetic induction is zero outside the solenoid, the vector potential is not zero and thus there is no paradox.

Interestingly enough, the best explanation of this paradox I have found is in *Selected Solutions for Physics*[1] by Halliday, Resnick, and Derringh. In the textbook,[2] the student is asked, as part of a problem, to explain the paradox. The suggested response in the solutions' manual is that if the magnetic field inside the solenoid increases, additional lines of flux must snake into the solenoid, cutting the coil as they do so.

A bit of reflection on this matter leads one to the conclusion that the assumption that there is no magnetic field outside a long, ideal solenoid is simply not valid. We are, after all, dealing with time-dependent phenomena and thus we should use Maxwell's equations in all their glory. In other words, the cause of this paradox is our use of magnetostatics to calculate a time-varying magnetic field. This situation is described by Griffiths[3] as follows:

> I must warn you, now, of a small fraud that tarnishes many applications of Faraday's law: Faraday's law, of course, pertains to *changing* magnetic fields, and yet we would like to use the apparatus of magneto*statics* (Ampère's law, the Biot–Savart law, and the rest) to *calculate* those fields. Technically, any result derived in this way is only approximately correct. But in practice the error is usually negligible unless the field fluctuates *extremely* rapidly.

One of the purposes of this paper is to show that it is possible to find an exact solution to Maxwell's equations when the source of the fields is an oscillating alternating current in a long, ideal solenoid. This calculation is a simple application of the standard techniques of mathematical physics. I have not found it in any of the standard references, although a closely related calculation of the fields produced by an alternating current flowing longitudinally in a cylindrical conductor can be found in Ref. 4. As we shall see, the exact solution possesses some interesting features.

II. THE EXACT SOLUTION FOR OUTGOING, SINUSOIDAL WAVES

We now are going to calculate the electric and magnetic fields produced by a surface current flowing uniformly around the circumference of an infinitely long cylinder of radius a. Although the surface current is uniform, we will describe the surface current density as being equal to $I(t)n$, where $I(t)$ is the current through a single turn and n is the number of turns per unit length, in order to make contact with the usual treatments of the solenoid.

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The symmetry of the system implies that we should use cylindrical coordinates and that the fields depend only on r and t. We shall see that we can satisfy both the field equations and the boundary conditions if the electric field has a component only in the ϕ direction, $E_\phi(r,t)$, and the magnetic induction field only a component in the z direction, $B_z(r,t)$. We assume the current varies with time as $I(t)=I_0 \cos \omega t=\mathrm{Re}\{I_0 e^{-i\omega t}\}$. The symbol Re indicates the we must take the real part of the quantity which follows.

Our assumptions concerning the form of $\mathbf{E}$ and $\mathbf{B}$ imply that $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$. Assuming that $E_\phi(r,t) =\mathrm{Re}\{E(r)e^{-i\omega t}\}$ and $B_z(r,t)=\mathrm{Re}\{B(r)e^{-i\omega t}\}$, we then have for the two remaining Maxwell's equations:

$$\nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

becomes

$$-\frac{dB}{dr}=-\frac{i\omega}{c^2}E \quad\text{or}\quad E=-\frac{ic^2}{\omega}\frac{dB}{dr} \tag{3}$$

and

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

becomes

$$\frac{1}{r}\frac{d(rE)}{dr}=i\omega B. \tag{4}$$

Equation (3) is only valid if $r\neq a$.

We can eliminate $E(r)$ between these two equations and we get,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB}{dr}\right)=\frac{i\omega}{c^2}\frac{1}{r}\frac{d(rE)}{dr}=-\frac{\omega^2}{c^2}B=-k^2B, \tag{5}$$

where k is the wave number, $k=2\pi/\lambda$. Thus $kc=\omega$.

Rearranging, we get the following equation for $B(r)$:

$$\frac{d^2B}{dr^2}+\frac{1}{r}\frac{dB}{dr}+k^2B= \quad\text{or}\quad r^2\frac{d^2B}{dr^2}+r\frac{dB}{dr}+k^2r^2B=0. \tag{6}$$

This is Bessel's equation of order 0.[5,6] It has two independent solutions, $J_0(kr)$ and $N_0(kr)$. We require a solution which is finite at $r=0$. Since $N_0(kr)$ has a singularity at the origin, our solution for $r<a$ must have the form,

$$B(r)=CJ_0(kr). \tag{7}$$

C is a constant to be determined by the boundary conditions.

For $r>a$, we require a solution which gives only outgoing waves. With the time dependence of $e^{-i\omega t}$, this implies that $B(r)$ must behave like e^{ikr} for large r, since $\mathrm{Re}\{e^{ikr-i\omega\tau+\alpha}\}=\cos(kr-\omega t+\alpha)$ is the correct form for traveling waves in the $+r$ direction. This requires that our solution to Bessel's equation be the correct linear combination or $J_0(kr)$ and $N_0(kr)$. This linear combination is called the Hankel function, $H_0^{(1)}(kr)$, where

$$H_0^{(1)}(kr)=J_0(kr)+iN_0(kr). \tag{8}$$

This is the correct linear combination since for large r, $H_0^{(1)}(kr)$ has the form

$$H_0^{(1)}(kr)\rightarrow\sqrt{\frac{2}{\pi kr}}\exp\{i(kr-\pi/4)\}. \tag{9}$$

Thus for $r>a$,

$$B(r)=FH_0^{(1)}(kr). \tag{10}$$

F, like C, is a constant to be determined by the boundary conditions.

Once $B(r)$ is known, $E(r)$ can be calculated from Eq. (3). We can simplify our results by using the fact that

$$\frac{dJ_0(kr)}{dr}=-kJ_1(kr) \quad\text{and}\quad \frac{dH_0^{(1)}(kr)}{dr}=-kH_1^{(1)}(kr). \tag{11}$$

$J_1(kr)$ and $H_1^{(1)}(kr)$ are solutions to Bessel's equation of order 1.

We then get for the electric field from Eq. (3),

$$E(r)=\frac{ic^2k}{\omega}CJ_1(kr)=icCJ_1(kr) \quad (r<a) \tag{12}$$

and

$$E(r)=\frac{ic^2k}{\omega}FH_1^{(1)}(kr)=icFH_1^{(1)}(kr) \quad (r>a). \tag{13}$$

The constants C and F must be chosen to fit the boundary conditions[7] that (1) E is continuous at $r=a$ and (2) $B(r\rightarrow a-)-B(r\rightarrow a+)=\mu_0 I_0 n\equiv K$. Thus $icCJ_1(ka) =icFH_1^{(1)}(ka)$ or $CJ_1(ka)=FH_1^{(1)}(ka)$, and $CJ_0(ka)-FH_0^{(1)}(ka)=K$.

Solving the last two equations for C and F,

$$C=\frac{KH_1^{(1)}(ka)}{H_1^{(1)}(ka)J_0(ka)-J_1(ka)H_0^{(1)}(ka)}$$

and

$$F=\frac{KJ_1(ka)}{H_1^{(1)}(ka)J_0(ka)-J_1(ka)H_0^{(1)}(ka)}.$$

We can simplify the denominator by combining $H_1^{(1)}(ka)=J_1(ka)+iN_1(ka)$ with the Wronskian,

$$J_1(ka)N_0(ka)-J_0(ka)N_1(ka)=\frac{2}{\pi ka}, \tag{14}$$

to obtain

$$H_1^{(1)}J_0-J_1H_0^{(1)}=(J_1+iN_1)J_0-J_1(J_0+iN_0)$$
$$=-i\{J_1N_0-J_0N_1\}=-\frac{2i}{\pi ka}.$$

Thus we have $C=(i\pi kaK/2)H_1^{(1)}(ka)$ and $F=(i\pi kaK/2)J_1(ka)$, where $K=\mu_0 I_0 n$.

Substituting these values for C and F into Eqs. (7) and (10), we then get for the magnetic induction field,

$$B_z(r,t)=\begin{cases}\mathrm{Re}\left\{\dfrac{i\pi kaK}{2}H_1^{(1)}(ka)J_0(kr)e^{-i\omega t}\right\} & (r<a)\\ \mathrm{Re}\left\{\dfrac{i\pi kaK}{2}J_1(ka)H_0^{(1)}(kr)e^{-i\omega t}\right\} & (r>a)\end{cases}. \tag{15}$$

$H_0^{(1)}(kr)$ and $H_1^{(1)}(ka)$ are complex quantities, but $J_n(kr)$ and $N_n(kr)$ are real. Thus if we use $H_0^{(1)}(kr)=J_0(kr)+iN_0(kr)$ and $H_1^{(1)}(ka)=J_1(ka)+iN_1(ka)$, we can obtain from Eq. (15)

$$B_z(r,t)=\begin{cases}\dfrac{\pi kaK}{2}J_1(ka)J_0(kr)\sin\omega t-\dfrac{\pi kaK}{2}N_1(ka)J_0(kr)\cos\omega t & (r<a)\\ \dfrac{\pi kaK}{2}J_1(ka)J_0(kr)\sin\omega t-\dfrac{\pi kaK}{2}J_1(ka)N_0(kr)\cos\omega t & (r>a)\end{cases}. \tag{16}$$

We now calculate the electric field, $E_\phi(r,t)$, in the same way and we get from Eqs. (12) and (13),

$$E_\phi(r,t)=\begin{cases}-\dfrac{c\pi kaK}{2}J_1(kr)N_1(ka)\sin\omega t-\dfrac{c\pi kaK}{2}J_1(kr)J_1(ka)\cos\omega t & (r<a)\\ -\dfrac{c\pi kaK}{2}N_1(kr)J_1(ka)\sin\omega t-\dfrac{c\pi kaK}{2}J_1(ka)J_1(kr)\cos\omega t & (r>a)\end{cases}. \tag{17}$$

Equations (16) and (17) are the exact solution to the field equations if the current varies with time as $I(t)=I_0\cos\omega t$.

III. ANALYSIS AND DISCUSSION

Before discussing the preceding calculation, it must be pointed out that the mathematical model used has limited physical reality. In a sense this is the price we pay for having an exactly solvable model. This lack of physical reality occurs for two reasons. First of all, there is the usual unphysical character of an infinitely long source in a three-dimensional world.

The second reason for the lack of physical reality is that we require that all points on the infinitely long, cylindrical source have, at any instant, the same surface current. Now propagation times are limited by the velocity of light, so that for an infinitely long solenoid we must have a infinite set of synchronized radio frequency generators driving each segment of the solenoid. However, we still have the problem of propagation around the circumference of the solenoid. This requires that the propagation time around the solenoid $2\pi a/c$ be much smaller than the period of the oscillations: $2\pi a/c\ll 2\pi/\omega$ or $\omega a/c\ll 1$ or $ka\ll 1$. Thus we require that the wave length be much larger than the radius of the solenoid. But it should be emphasized that the solution given in Sec. II is not an approximation but is an exact solution for the idealized model described in that section.

In order to compare the exact solution with the usual quasistatic model, we are now going to consider the limiting case of both ka and kr being much smaller than 1. In order to do this, we need the limiting form for small arguments of the various Bessel functions which occur in Eqs. (16) and (17). These are

$$J_0(x)\approx 1 \quad N_0(x)\approx(2/\pi)\ln(x)$$

$$J_1(x)\approx \tfrac{1}{2}x \quad N_1(x)\approx -2/\pi x.$$

Substituting these limiting forms into Eqs. (16) and (17) and using $kc=\omega$, we obtain

$$B_z(r,t)=\begin{cases}\dfrac{\pi(ka)^2K}{4}\sin\omega t+K\cos\omega t & (r<a)\\ \dfrac{\pi(ka)^2K}{4}\sin\omega t-\dfrac{(ka)^2K\ln(kr)}{2}\cos\omega t & (r>a)\end{cases} \tag{18}$$

and

$$E_\phi(r,t)=\begin{cases}\dfrac{Kr}{2}\omega\sin\omega t-\dfrac{\pi(ka)^2Kr}{8}\omega\cos\omega t & (r<a)\\ \dfrac{Ka^2}{2r}\omega\sin\omega t-\dfrac{\pi(ka)^2Kr}{8}\omega\cos\omega t & (r>a)\end{cases}. \tag{19}$$

Now if we neglect terms of order $(ka)^2$, we obtain

$$B_z(r,t)=\begin{cases}K\cos\omega t & (r<a)\\ 0 & (r>a)\end{cases} \tag{20}$$

and

$$E_\phi(r,t)=\begin{cases}\dfrac{Kr}{2}\,\omega \sin \omega t & (r<a)\\ \dfrac{Ka^2}{2r}\,\omega \sin \omega t & (r>a)\end{cases}. \tag{21}$$

Recalling that $K=\mu_0 I_0 n$, $I(t)=I_0 \cos \omega t$ and $dI/dt=-I_0\omega \sin \omega t$, we see that Eq. (21) is identical to Eqs. (1) and (2). We also see that Eq. (20) validates the use of the magnetostatic calculation as a basis for obtaining the time-varying magnetic field. That the usual method of calculating the induced electromotive force is very accurate in the long wavelength limit is to be expected. What is at first unexpected is that the second of Eq. (21) arises from the presence of $N_1(kr)$ in Eq. (17). The $N_1(kr)$ is there due to the fact that we need a solution which represents outgoing traveling waves. It is at first surprising that the radiation condition is involved since the usual calculation leading to Eq. (1) does not involve radiation. However, we must remember the limiting case $ka\to 0$ and $kr\to 0$ has the effect of changing the character of the underlying set of differential equations from Helmholtz to Laplace. The presence of the logarithm in the second of Eq. (18) is simply a product of the unphysical two-dimensional model we have used.

Schelkunoff[8] has suggested that it should be possible to calculate the electromagnetic field iteratively near a source by a method of successive approximation, using alternatively Eqs. (3) and (4). He also points out that this has to be done judiciously, and that often the method fails after the first few iterations. We can see why that could be so if we look at the transition from Eqs. (16) and (17) to Eqs. (20) and (21).

We would like to close this section by developing a by-product of our exact solution which may be of pedagogical interest. We can easily calculate the power radiated per unit length of solenoid and compare it with the power needed to move current around the solenoid.

Now the form of $H_m^{(1)}(x)$ for large arguments is

$$H_m^{(1)}(x)\approx \sqrt{\frac{2}{\pi x}}\exp\{i(x-m\pi/2-\pi/4)\}.$$

Thus from the second of Eq. (15)

$$B_z(r,t)\approx \mathrm{Re}\left\{\frac{i\pi Kka}{2}J_1(ka)\sqrt{\frac{2}{\pi x}}\exp i(kr-\pi/4-\omega t)\right\}$$
$$\approx -\frac{\pi Kka}{2}J_1(ka)\sqrt{\frac{2}{\pi kr}}\sin(kr-\pi/4-\omega t).$$

We also have in a similar way

$$E_z(r,t)\approx -\frac{c\pi Kka}{2}J_1(ka)\sqrt{\frac{2}{\pi kr}}\times\sin(kr-\pi/4-\omega t).$$

The Poynting vector points outward and has a magnitude of

$$S=\frac{1}{\mu_0}EB=\frac{1}{\mu_0}c\left(\frac{\pi Kka}{2}\right)^2[J_1(ka)]^2\frac{2}{\pi kr}[\sin(kr-\pi/4-\omega t)]^2.$$

If we average this over one period we have

$$S_{\mathrm{av}}=\frac{1}{\mu_0}c\left(\frac{\pi Kka}{2}\right)^2[J_1(ka)]^2\frac{2}{\pi kr}\frac{1}{2}.$$

The average power radiated through a cylinder of radius r and unit length is

$$P_{\mathrm{av}}=S_{\mathrm{av}}2\pi r=\frac{1}{\mu_0}c\left(\frac{\pi Kka}{2}\right)^2[J_1(ka)]^2\frac{2}{k}$$
$$=\frac{\mu_0 I_0^2 n^2 a^2 \pi^2 \omega}{2}[J_1(ka)]^2. \tag{22}$$

We can compare this with the power required to move the current $I(t)$ through one "turn" of the coil in the presence of the electric field, $E_\phi(r,t)$.

The power required per "turn" is $-2\pi a I(t)E_\phi(a,t)$. Multiply this by n to get the power per unit length and we get

$$P=-2\pi a n I(t)E_\phi(a,t)=-2\pi a n I_0\cos(\omega t)E_\phi(a,t).$$

Using Eq. (17) and averaging over one period, we get,

$$P_{\mathrm{av}}=2\pi a I_0 n\frac{c\pi kaK}{2}[J_1(ka)]^2\frac{1}{2}$$
$$=\frac{\mu_0 I_0^2 n^2 a^2 \pi^2 \omega}{2}[J_1(ka)]^2,$$

which is identical to Eq. (13). This is one of those rare problems where we can calculate explicitly the radiation reaction.

In a recent issue of this journal, French[9] asks, in essence, does there exist a simple calculation of the induced emf in a coil of wire encircling a long solenoid which does not make use of the quasistatic approximation? The present calculation, although straightforward, probably does not constitute an affirmative answer to the question, since it requires a fair amount of properties of special functions and the like.

IV. CONCLUSIONS

The use of magnetostatics to calculate the time-dependent flux is a very common technique to obtain the induced emf from Faraday's law. It is an approximation, which, as we have shown in the case of the ideal solenoid, is very accurate in the quasistatic limit. The fact that this approximation is so accurate often means that no mention is made of the fact that we are dealing with an approximation. This can be confusing to students and, on occasion, to authors of textbooks.

The solenoid problem plays a key role in illustrating this because the approximation can lead, in this case, to the conclusion that a changing magnetic field in one region can produce an electric field in a region where there is no magnetic field. But the point must be made that the use of this approximation is not unique to the solenoid problem, but what is unusual about the solenoid is that it demonstrates through the appearance of a physically impossible situation that we are dealing with an approximation. It would appear that the best way to deal with all this would be to make explicit mention of the fact the we are using a very accurate, but approximate, method to calculate induced electric fields.

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Evaluation of the eigenvalues of multiple quantum-well potentials

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We present a method for determining the bound and quasibound state eigenvalues of an arbitrary multiple quantum-well potential based on the poles of the reflection coefficient at the first boundary of the potential. The method has the advantage of computational simplicity and can be used to approximate the eigenvalues of a coordinate-dependent potential. As examples, the method is used to determine the bound and tunneling state eigenvalues of a three-boundary potential and the bound state eigenvalues of a five quantum-well potential. The method is compared with other methods of analysis of quantum-well potentials. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The common use of quantum-well superlattices in optoelectric devices has stimulated interest in the energy eigenstates of particles confined in quantum wells.[1] The derivation of the eigenvalues of a particle trapped in a single quantum well (or "potential box") is given in almost every text on quantum mechanics.[2,3] In contrast, analyses of the states of particles in multiple quantum-well potentials are generally limited to the case of the (infinite) periodic potential. The analysis of the periodic (Kronig–Penney) potential exploits the perfect translational symmetry of the potential well,[2,3] while the relative simplicity of the single well case allows for a simple derivation of the eigenvalue equations. Since the latter equations assume the form of transcendental relations even in these relatively simple cases, however, the lack of interest in analyses of more complex potentials wells can be explained by the complexity of the resulting eigenvalue equations. In the case of any discrete step potential these equations derive strictly from the continuity conditions on the solution of Schrödinger's equation and its derivative at each potential step. But the form of the equations becomes more complex as the number of boundaries increases, and analytic formulas for the eigenvalues can be derived only in the simplest limiting cases. More generally, therefore, the eigenvalues must be extracted from the eigenvalue equations by either graphical or purely numerical methods.

In a relatively recent paper in this Journal[4] Kalotas and Lee present a method for determining the eigenvalues of the bound states of a multiple step potential based on the zeros of a particular element of a 2×2 matrix derived from a matrix formulation of the continuity conditions on the Ψ function and its derivative at each of the boundaries of the potential. Here we present an even more straightforward method for determining the "eigenvalues" of both the bound and the quasibound states of the same (general) potential, based on the poles of the reflection coefficient determined by the potential. The method has the advantage of directly determining the bound state eigenvalues of the given potential without the need for a detailed application of the continuity conditions at all boundaries. In addition, the method provides a suscinct formula for the bound state eigenvalues of a multiple quantum-well potential which can easily be adapted to the case where the effective mass of the confined particle is different in the separate regions of the potential.

In Sec. II we review the equations which describe a particle in a quantum-well potential and define the bound and quasibound states of the particle. We then connect these states to a condition on the reflection coefficient at the first boundary of the potential, which is expressed in a general form in Eq. (23). The condition is made use of in Sec. II C and in Appendix A to derive explicit equations for the eigenvalues of a particle in two-, three-, and four-boundary potentials, respectively. The computational efficiency of the "resonance method" developed in Sec. II is then illustrated in Sec. III by a determination of both the bound and the tunneling states of a three-boundary potential well and the bound states of a five quantum-well potential. An extension of the method to the case of coordinate-dependent potentials is suggested in Sec. IV. In Appendix B, we connect the reflection coefficient to certain other quantities used in the description of the scattering produced by a general potential.

II. THEORY

We are interested in a solution of the Schrödinger equation in the case of a discrete quantum-well potential in one dimension, which we represent in the general form