CHAPTER XV

THE PROPERTIES OF MOVING ELECTRIC CHARGES

275. As the properties of moving electric charges are of great importance in the explanation of many physical phenomena, we shall consider briefly some of the simpler properties of a moving charge and other closely allied questions.

Magnetic Force due to a Moving Charged Sphere.

The first problem we shall discuss is that of a uniformly charged sphere moving with uniform velocity along a straight line. Let e be the charge on the sphere, a its radius, and v its velocity; let us suppose that it is moving along the axis of z, then when things have settled down into a steady state the sphere will carry its Faraday tubes along with it. If we neglect the forces due to electromagnetic induction, the Faraday tubes will be uniformly distributed round the sphere and the number passing normally through unit area at a point P will be $e/4\pi OP^2$, O being the centre of the charged sphere. These tubes are radial and are moving with a velocity v parallel to the axis of z, hence the component of the velocity at right angles to their direction is $v \sin\theta$, where θ is the angle OP makes with the axis of z; by Art. 265 these moving tubes will produce a magnetic force at P equal to

$$4\pi\,(e/4\pi\,.\,OP^2)\,v\sin\theta = ev\sin\theta/OP^2.$$

The direction of this force is at right angles to the tubes, i.e. at right angles to OP; at right angles also to their direction of motion, i.e. at right angles to the axis of z; thus the lines of magnetic force will be circles whose planes are at right angles to the axis of z and whose centres lie along this axis. Thus we see that the magnetic field outside the charged sphere is the same as that given by Ampère's rule for an element of current ids, parallel to the axis of z, placed at the centre of the sphere, provided $ev = ids$.

276. As the sphere moves, the magnetic force at P changes, so that in addition to the electrostatic forces there will be forces due to electromagnetic induction, these will be proportional to the intensity of the magnetic induction multiplied by the velocity of the lines of magnetic induction, i.e. the force due to electromagnetic induction at a point P will be proportional to $\mu\,(ev \sin\theta/OP^2) \times v$, where μ is the magnetic permeability of the medium; while the electrostatic force will be $e/K \,.\, OP^2$, where K is the specific inductive capacity of the medium. The ratio of the force due to electromagnetic induction to the electrostatic force is $\mu K v^2 \sin\theta$ or $\sin\theta v^2/V^2$, where V is the velocity of light through the medium surrounding the sphere; hence in neglecting the electromagnetic induction we are neglecting quantities of the order v^2/V^2. The direction of the force due to electromagnetic induction at P is along NP, if PN is the normal drawn from P to the axis of z; this force tends to make the Faraday tubes congregate in the plane through the centre of the sphere at right angles to its direction of motion; when the sphere is moving with the velocity of light it can be shown that all the Faraday tubes are driven into this plane.

Increase of Mass due to the Charge on the Sphere.

277. Returning to the case when the sphere is moving so slowly that we may neglect v^2/V^2; we see that since H, the magnetic force at P, is $ev \sin\theta/OP^2$, and at P there is kinetic energy equal to $\mu H^2/8\pi$ per unit volume (see Art. 163), the kinetic energy per unit volume at P is

$$\mu e^2 v^2 \sin^2\theta/8\pi \,.\, OP^4.$$

Integrating this for the volume outside the sphere, we find that the kinetic energy outside the sphere is $\frac{\mu e^2 v^2}{3a}$, where a is the radius of the sphere. Thus if m be the mass of the uncharged sphere the kinetic energy when it has a charge e is equal to

$$\frac{1}{2}\left(m + \frac{2}{3}\frac{\mu e^2}{a}\right) v^2.$$

Thus the effect of the charge is to increase the mass of the sphere by $2\mu e^2/3a$. It is instructive to compare this case with another, in which there is a similar increase in the effective mass of a body; the case we refer to is that of a body moving through a liquid.

Thus when a sphere moves through a liquid it behaves as if its mass were $m + \frac{1}{2}m'$, where m is the mass of the sphere, and m' the mass of liquid displaced by it. Again when a cylinder moves at right angles to its axis through a liquid its apparent mass is $m + m'$, where m' is the mass of the liquid displaced by the cylinder. In the case of an elongated body like a cylinder, the increase in mass is much greater when it moves sideways than when it moves point foremost, indeed in the case of an infinite cylinder the increase in the latter case vanishes in comparison with that in the former; the increase in mass being $m' \sin^2\theta$, where θ is the angle the direction of motion of the cylinder makes with its axis. In the case of bodies moving through liquids the increase in mass is due to the motion of the body setting in motion the liquid around it, the site of the increased mass is not the body itself but the space around it where the liquid is moving. In the electrical problem we may regard the increased mass as due to mass bound by the Faraday tubes and carried along with them as they move about. We shall for brevity speak of the source of this mass as the ether, not postulating however for this ether any property other than that of supplying mass for the Faraday tubes. From the expression for the energy per unit volume we see that the increase in mass is the same as if a mass $4\pi\mu N^2$ per unit volume were bound by the tubes, and had a velocity given to it equal to the velocity of the tubes at right angles to themselves, the motion of the tubes along their length not setting this mass in motion. Thus on this view the increased mass due to the charge is the mass of ether set in motion by the tubes. If we regard atoms as made up of charges of positive and negative electricity, it is possible to regard all mass as electrical in its origin, and as arising from the ether set in motion by the Faraday tubes connecting the electrical charges of which the atoms are supposed to be made up. For a development of this view the reader is referred to the author's *Conduction of Electricity through Gases; Electricity and Matter;* and "Mass, Energy and Radiation," *Phil. Mag.*, June 1920.

Momentum in the Electric Field.

278. The view indicated above, that the Faraday tubes set the ether moving at right angles to the direction of these tubes, suggests that at each point in the field there is momentum whose direction is at right angles to the tubes, and by symmetry in the plane through the tube and the line along which the centre of the charged sphere moves. As the mass of the ether moved per unit volume at P is $4\pi\mu N^2$ where N is the density of the Faraday tubes at P, the momentum per unit volume would, on this view, be $4\pi\mu N^2 v \sin\theta$. This is equal to BN where B is the magnetic induction and N the density of the Faraday tubes at P, the direction of the momentum being at right angles to B and N. We shall now prove that this expression for the momentum is general and is not limited to the case when the field is produced by a moving charged sphere.

279. Since the magnetic force due to moving Faraday tubes is (Art. 265) equal to 4π times the density of the tubes multiplied by the components of the velocity of the tubes at right angles to their direction, and is at right angles both to the direction of the tubes and to their velocity; we see if α, β, γ are the components of the magnetic force parallel to axes of x, y, z at a place where the densities of the Faraday tubes parallel to x, y, z are f, g, h, and where u, v, w are the components of the velocity of the tubes, α, β, γ are given by the equations

$$\alpha = 4\pi (hv - gw), \quad \beta = 4\pi (fw - hu), \quad \gamma = 4\pi (gu - fv).$$

If all the tubes are not moving with the same velocity we shall have

$$\alpha = 4\pi (h_1v_1 - g_1w_1 + h_2v_2 - g_2w_2 + h_3v_3 - g_3w_3 + \dots)$$

with similar expressions for β, γ. Here u_1, v_1, w_1 are the components of the velocity of the tubes f_1, g_1, h_1; u_2, v_2, w_2 those of the tubes f_2, g_2, h_2 and so on.

Now T the kinetic energy per unit volume at P is equal to

$$\frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2) = \frac{\mu}{8\pi} \times 16\pi^2 . (\{\Sigma (hv - gw)\}^2 + \{\Sigma (fw - hu)\}^2 + \{\Sigma (gu - fv)\}^2)$$
$$= 2\pi\mu . \{(\Sigma (hv - gw))^2 + (\Sigma (fw - hu))^2 + (\Sigma (gu - fv))^2\};$$

the momentum per unit volume parallel to x due to the tubes f_1, g_1, h_1 is equal to $\frac{dT}{du_1}$, i.e. to

$$- 4\pi\mu \{h_1\Sigma (fw - hu) - g_1\Sigma (gu - fv)\}$$
$$= \mu (g_1\gamma - h_1\beta).$$

Similarly that due to the tubes f_2, g_2, h_2 is equal to

$$\mu (g_2\gamma - h_2\beta),$$

and so on, thus P the total momentum parallel to x per unit volume is given by the equation

$$P = \mu (\gamma\Sigma g - \beta\Sigma h)$$
$$= \mu (\gamma g - \beta h),$$

where f, g, h are the densities parallel to x, y, z of the whole assemblage of Faraday tubes. Similarly Q, R, the components of the momentum parallel to y and z, are given respectively by the equations

$$Q = \mu (\alpha h - \gamma f),$$
$$R = \mu (\beta f - \alpha g).$$

Thus we see that the vector P, Q, R is perpendicular to the vectors $\alpha, \beta, \gamma, f, g, h$, and its magnitude is $BN \sin\theta$ where B is the magnetic induction at the point, N the density of the Faraday tubes and θ the angle between B and N; hence we see that each portion of the field possesses an amount of momentum equal to the vector product of the magnetic induction and the dielectric polarization.

280. Before considering the consequences of this result, it will be of interest to consider the connection between the momentum and the stresses which we have supposed to exist in the field. We have seen (Arts. 45, 46) that the electric and magnetic forces in the field could be explained by the existence of the following stresses:

$$\alpha \begin{cases} (1) \text{ a tension } \dfrac{KR^2}{8\pi} \text{ along the lines of electric force;} \\ (2) \text{ a pressure } \dfrac{KR^2}{8\pi} \text{ at right angles to these lines;} \end{cases}$$

here K is the specific inductive capacity, and R the electric force;

$$\beta \begin{cases} (1) \text{ a tension } \dfrac{\mu H^2}{8\pi} \text{ along the lines of magnetic force;} \\ (2) \text{ a pressure } \dfrac{\mu H^2}{8\pi} \text{ at right angles to these lines;} \end{cases}$$

here μ is the magnetic permeability of the medium and H the magnetic force.

Let us consider the effect of these tensions on an element of volume bounded by plane faces perpendicular to the axes of x, y, z. The stresses α are equivalent to a hydrostatic pressure $KR^2/8\pi$ and a tension $KR^2/4\pi$ along the lines of force. The effect of the hydrostatic pressure on the element of volume is equivalent to forces

$$-\frac{d}{dx}\left(\frac{KR^2}{8\pi}\right)\Delta x\Delta y\Delta z, \quad -\frac{d}{dy}\left(\frac{KR^2}{8\pi}\right)\Delta x\Delta y\Delta z,$$
$$-\frac{d}{dz}\left(\frac{KR^2}{8\pi}\right)\Delta x\Delta y\Delta z,$$

parallel to the axes of x, y, z respectively, Δx, Δy, Δz being the sides of the element of volume.

Let us now consider the tension $KR^2/4\pi$. We know that a stress N in a direction whose direction cosines are l, m, n is equivalent to the following stresses:

Nl^2	acting on the face	$\Delta y\Delta z$	parallel to	x,
Nlm	,,	,,	,,	,,	y,
Nln	,,	,,	,,	,,	z,
Nlm	,,	,,	$\Delta x\Delta z$	,,	x,
Nm^2	,,	,,	,,	,,	y,
Nmn	,,	,,	,,	,,	z,
Nln	,,	,,	$\Delta x\Delta y$	,,	x,
Nmn	,,	,,	,,	,,	y,
Nn^2	,,	,,	,,	,,	z.

Thus the effect of these stresses on the element of volume is equivalent to a force parallel to x equal to

$$\left\{\frac{d}{dx}(Nl^2)+\frac{d}{dy}(Nlm)+\frac{d}{dz}(Nln)\right\}\Delta x\Delta y\Delta z;$$

the forces parallel to y and z are given by symmetrical expressions.

In our case the tension is along the lines of force, hence $l=\frac{X}{R}$, $m=\frac{Y}{R}$, $n=\frac{Z}{R}$, where X, Y, Z are the components of the electric force, hence substituting these values for l, m, n and putting $N=\frac{KR^2}{4\pi}$, we see that the tension produces a force parallel to x equal to

$$\left(\frac{d}{dx}\frac{KX^2}{4\pi}+\frac{d}{dy}\frac{KXY}{4\pi}+\frac{d}{dz}\frac{KXZ}{4\pi}\right)\Delta x\Delta y\Delta z.$$

The force parallel to x due to the hydrostatic pressure and this tension is equal to

$$\left(-\frac{d}{dx}\frac{K(X^2+Y^2+Z^2)}{8\pi}+\frac{d}{dx}\frac{KX^2}{4\pi}+\frac{d}{dy}\frac{KXY}{4\pi}+\frac{d}{dz}\frac{KXZ}{4\pi}\right)\Delta x\Delta y\Delta z;$$

when the medium is uniform, this may be written

$$\frac{K}{4\pi}\left\{Y\left(\frac{dX}{dy}-\frac{dY}{dx}\right)-Z\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+X\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)\right\}\Delta x\Delta y\Delta z.$$

Now $\qquad KX,\ KY,\ KZ = 4\pi f,\ 4\pi g,\ 4\pi h,$

and by equation (4) Art. 234,

$$\frac{dX}{dy}-\frac{dY}{dx}=\frac{dc}{dt},\quad \frac{dZ}{dx}-\frac{dX}{dz}=\frac{db}{dt},\quad \frac{dY}{dz}-\frac{dZ}{dy}=\frac{da}{dt},$$

while $\qquad K\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=4\pi\rho;$

thus the force parallel to x due to the electric stresses may be written

$$\left(g\frac{dc}{dt}-h\frac{db}{dt}+X\rho\right)\Delta x\Delta y\Delta z.$$

In the same way the magnetic stresses may be shown to give a force parallel to x equal to

$$\frac{\mu}{4\pi}\left\{\beta\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)-\gamma\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\alpha\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)\right\}\Delta x\Delta y\Delta z;$$

since by Art. 234

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi\frac{df}{dt},\quad \frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi\frac{dg}{dt},\quad \frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi\frac{dh}{dt},$$

and $\qquad \mu\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=4\pi\sigma,$

where σ is the density of the magnetism, the magnetic stresses give rise to a force parallel to x equal to

$$\left(c\frac{dg}{dt}-b\frac{dh}{dt}+\alpha\sigma\right)\Delta x\Delta y\Delta z;$$

hence the system of electric and magnetic stresses together gives rise to a force parallel to x equal to

$$\left(\frac{d}{dt}(cg - bh) + X\rho + \alpha\sigma\right)\Delta x \Delta y \Delta z.$$

The terms $X\rho$ and $\alpha\sigma$ represent the forces acting on the charged bodies and the magnets in the element of volume, and are equal to the rate of increase of momentum parallel to x of these bodies, the remaining term

$$\frac{d}{dt}(cg - bh)\,\Delta x \Delta y \Delta z$$

equals the rate of increase of the x momentum in the ether in the element of volume. This agrees with our previous investigation; for we have seen (p. 391) that the momentum parallel to x per unit volume is equal to $gc - hb$.

281. A system of charged bodies, magnets, circuits carrying electric currents &c. *and the ether* forms a self-contained system subject to the laws of dynamics; in such a system, since action and reaction are equal and opposite, the whole momentum of the system must be constant in magnitude and direction, if any one part of the system gains momentum some other part or parts must lose an equal amount. If we take the incomplete system got by leaving out the ether, this is not true. Thus take the case of a charged body struck by an electric wave, the electric force in the wave acts on the body and imparts momentum to it, no other material body loses momentum, so that if we leave out of account the ether we have something in contradiction to the third law of motion. If we take into account the momentum in the ether there is no such contradiction, as the momentum in the electric waves after passing the charged body is diminished as much as the momentum of that body is increased.

282. Another interesting example of the transference of momentum from the ether to ordinary matter is afforded by the pressure exerted by electric waves, including light waves, when they fall on a slab of a substance by which they are absorbed. Take the case when the waves are advancing normally to the slab. In each unit of volume of the waves there is a momentum equal to the

product of the magnetic induction B and the dielectric polarization N; B and N are at right angles to each other, and are both in the wave front; the momentum which is at right angles to both B and N is therefore in the direction of propagation of the wave. In the wave $B = 4\pi\mu NV$, so that $BN = \frac{1}{4\pi}\frac{B^2}{\mu V}$, V being the velocity of light; B is a periodic function, and may be represented by an expression of the form $B_0 \cos(pt - nx)$, x being the direction of propagation of the wave; the mean value of B^2 is therefore $\frac{1}{2}B_0^2$. Thus the average value of the momentum per unit volume of the wave is $\frac{1}{8\pi}\frac{B_0^2}{\mu V}$, the amount of momentum that crosses unit area of the face of the absorbing substance per unit time is therefore $\frac{1}{8\pi\mu}\frac{B_0^2}{V} \times V$, or $\frac{1}{8\pi\mu}B_0^2$. As the wave is supposed to be absorbed by the slab no momentum leaves the slab through the ether, so that in each unit of time $\frac{B_0^2}{8\pi\mu}$ units of momentum are communicated to the slab for each unit area of its face exposed to the light: the effect on the slab is the same therefore as if the face were acted upon by a pressure $B_0^2/8\pi\mu$. It should be noticed that μ is the magnetic permeability of the dielectric through which the waves are advancing, and not of the absorbing medium.

If the slab instead of absorbing the light were to reflect it, then if the reflection were perfect each unit area of the face would in unit time be receiving $B_0^2/8\pi\mu$ units of momentum in one direction, and giving out an equal amount of momentum in the opposite direction; the effect then on the reflecting surface would be as if a pressure $2 \times B_0^2/8\pi\mu$ or $B_0^2/4\pi\mu$ were to act on the surface. This pressure of radiation as it is called was predicted on other grounds by Maxwell; it has recently been detected and measured by Lebedew and by Nichols and Hull by some very beautiful experiments.

283. If the incidence is oblique and not direct, then if the reflection is not perfect there will be a tangential force as well as a normal pressure acting on the surface. For suppose i is the angle of incidence, B_0 the maximum magnetic induction in the incident light, B_0' that in the reflected light, then across each unit of wave front in the incident light $B_0^2/8\pi\mu$ units of momentum in the direction of the incident light pass per unit time, therefore each unit of surface receives per unit time $\cos iB_0^2/8\pi\mu$ units of momentum in the direction of the incident light, or $\cos i \sin iB_0^2/8\pi\mu$ units of momentum parallel to the reflecting surface. In consequence of reflection

$$\cos i \sin iB_0'^2/8\pi\mu$$

units of momentum in this direction leave unit area of the surface in unit time, thus in unit time

$$\cos i \sin i\,(B_0^2 - B_0'^2)/8\pi\mu$$

units of momentum parallel to the surface are communicated to the reflecting slab per unit time, so that the slab will be acted on by a tangential force of this amount. Professor Poynting succeeded in detecting this tangential force.

Since the direction of the stream of momentum is changed when light is refracted, there will be forces acting on a refracting surface, also when in consequence of varying refractivity the path of a ray of light is not straight the refracting medium will be acted upon by forces at right angles to the paths of the ray; the determination of these forces, which can easily be accomplished by the principle of the Conservation of Momentum, we shall leave as an exercise for the student.

284. We shall now proceed to illustrate the distribution of momentum in some simple cases.

Case of a Single Magnetic Pole and an Electrified Point.

Let A be the magnetic pole, B the charged point, m the strength of the pole, e the charge on the point, then at a point P the magnetic induction is m/AP^2 and is directed along AP, the dielectric polarization is $e/4\pi BP^2$ and is along BP, hence the momentum at P is

$$\frac{me \sin APB}{4\pi \,.\, AP^2 \,.\, BP^2}$$

and its direction is the line through P at right angles to the plane APB. The lines of momentum are therefore circles with their centres along AB and their planes at right angles to it, the resultant momentum in any direction evidently vanishes. There will however be a finite *moment* of momentum about AB: this we can easily show by

integration to be equal to em. Thus in this case the distribution of momentum is equivalent to a moment of momentum em about AB. The distribution of momentum is similar in some respects to that in a top spinning about AB as axis. Since the moment of momentum of the ether does not depend upon the distance between A and B it will not change either in magnitude or direction when A or B moves in the direction of the line joining them. If however the motion of A or B is not along this line, the direction of the line AB and therefore the direction of the axis of the moment of momentum of the ether, changes. But the moment of momentum of the system consisting of the ether, the charge point, and the pole must remain constant; hence when the momentum in the ether changes, the momentum of the system consisting of the pole and the charge must change so as to compensate for the change in the momentum of the ether. Thus suppose the charged point moves from B to B' in the time δt, then in that time the moment of momentum in the ether changes from em along AB to em along AB'; this change in the moment of momentum of the ether is equivalent to a moment of momentum whose magnitude is $em\delta\theta$, where $\delta\theta = \angle BAB'$, and whose axis is at right angles to AB in the plane BAB'. The change in the moment of momentum of the pole and point must be equal and opposite to this. Since the resultant momentum of the ether vanishes in any direction, the change in the momentum of the pole must be equal and opposite to the change in momentum of the point, and these two changes must have a moment of momentum equal to $em\delta\theta$: we see that this will be the case if δI the change in momentum of the point is at right angles to the plane BAB' and equal to $\frac{em\delta\theta}{AB}$, while the change in momentum in the pole is equal and opposite to this. This change in momentum $\frac{em\delta\theta}{AB}$ occurring in the time δt may be regarded as produced by a force F acting on the point at right angles to the plane BAB' and given by the equation

$$F = \frac{em}{AB} \cdot \frac{\delta\theta}{\delta t}.$$

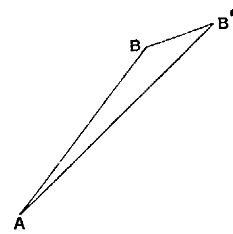

Fig. 135

Now
$$\delta\theta = \frac{BB' \sin ABB'}{AB},$$
or if v be the velocity of the point,
$$\delta\theta = \frac{v\delta t \sin ABB'}{AB},$$
or
$$\frac{\delta\theta}{dt} = \frac{v \sin ABB'}{AB};$$
thus
$$F = \frac{emv}{AB^2} \sin ABB'$$
$$= evH \sin \phi,$$
where H is the magnetic force at the point and ϕ the angle between H and the direction in which the point is moving; from this we see that a moving charged point in a magnetic field is acted on by a force at right angles to the velocity of the point, at right angles also to the magnetic force at the point, and equal to the product of the charge, the magnetic force and the velocity of the point at right angles to the magnetic force. Thus we see that we can deduce the expression for the force acting on a charged point moving across the lines of magnetic force directly from the principle of the Conservation of Momentum. We should have got an exactly similar expression if we had supposed the charge at rest and the pole in motion; in this case we must take v to be the velocity of the pole and ϕ the angle between v and AB.

285. From the expression given on page 391 for the momentum in the field we can prove that the momentum in the ether due to a charged point at P and the magnetic force produced by a current flowing round a small closed circuit, is equivalent to a momentum passing through P whose components F, G, H parallel to the axes of x, y, z respectively are given by the equations
$$F = \mu i a \left(m \frac{d}{dz}\frac{1}{r} - n \frac{d}{dy}\frac{1}{r} \right),$$
$$G = \mu i a \left(n \frac{d}{dx}\frac{1}{r} - l \frac{d}{dz}\frac{1}{r} \right),$$
$$H = \mu i a \left(l \frac{d}{dy}\frac{1}{r} - m \frac{d}{dx}\frac{1}{r} \right),$$
where i is the current flowing round the circuit, a the area of the circuit and l, m, n the direction cosines of the normal to its plane,

x, y, z are the coordinates of P and r the distance of P from the centre of the circuit, the charge at P is supposed to be the unit charge. We see that
$$\frac{dF}{dy} - \frac{dG}{dx} = \mu i a \left\{ m \frac{d^2}{dy\,dz}\frac{1}{r} - n \left(\frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dx^2}\frac{1}{r} \right) + l \frac{d^2}{dx\,dz}\frac{1}{r} \right\},$$
or since
$$\frac{d^2}{dx^2}\frac{1}{r} + \frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dz^2}\frac{1}{r} = 0,$$
$$\frac{dF}{dy} - \frac{dG}{dx} = \mu i a \left(l \frac{d^2}{dx\,dz}\frac{1}{r} + m \frac{d^2}{dy\,dz}\frac{1}{r} + n \frac{d^2}{dz^2}\frac{1}{r} \right)$$
$$= \mu i a \frac{d}{dz} \left(l \frac{d}{dx}\frac{1}{r} + m \frac{d}{dy}\frac{1}{r} + n \frac{d}{dz}\frac{1}{r} \right)$$
$$= c,$$
c being the z component of the magnetic induction at P due to the small circuit. We have similarly if a and b are the x and y components respectively of this induction
$$\frac{dH}{dx} - \frac{dF}{dz} = b,$$
$$\frac{dG}{dz} - \frac{dH}{dy} = a.$$

The usual expression for the electromotive force due to induction follows at once from the principle of the Conservation of Momentum. For the momentum in the ether is equivalent to a momentum through P whose components are F, G, H. Suppose that in consequence of the motion of the circuit or the alteration of the current through it, F, G, H become $F + \delta F$, $G + \delta G$, $H + \delta H$, then the momentum in the ether still passes through P but has now components $F + \delta F$, $G + \delta G$, $H + \delta H$ instead of F, G, H; but the momentum of the whole system, point circuit and ether must remain constant; thus to counterbalance the changes in momentum δF, δG, δH at P due to the ether, we must have changes in momentum of the unit charge at P equal to $-\delta F$, $-\delta G$, $-\delta H$. Suppose that the time taken by the changes δF, δG, δH is δt, then in the time δt the x momentum of the unit charge at P must change by $-\delta F$, i.e. the unit charge must be acted on by force $-\frac{dF}{dt}$. Thus there is at P an electric force whose component parallel to x is $-\frac{dF}{dt}$, similarly the components parallel to y and z are $-\frac{dG}{dt}$, $-\frac{dH}{dt}$. The electric force whose components we have just found is the force due to electromagnetic induction, and its magnitude is that given by Faraday's law. To prove this we notice that the line integral of the electric force round a fixed circuit of which ds is an element is equal to

$$-\int\left(\frac{dF}{dt}\frac{dx}{ds}+\frac{dG}{dt}\frac{dy}{ds}+\frac{dH}{dt}\frac{dz}{ds}\right)ds$$

$$=-\frac{d}{dt}\int\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds$$

$$=-\frac{d}{dt}\int\left\{l\left(\frac{dG}{dz}-\frac{dH}{dy}\right)+m\left(\frac{dH}{dx}-\frac{dF}{dz}\right)+n\left(\frac{dF}{dy}-\frac{dG}{dx}\right)\right\}dS$$

by Stokes' theorem; here l, m, n are the direction cosines of the normal to a surface filling up the closed curve, dS is an element of this surface. Substituting the values already given for $\frac{dG}{dz}-\frac{dH}{dy}$, &c. the preceding expression becomes

$$-\frac{d}{dt}\int(la+mb+nc)\,dS;$$

the integral in this expression is the number of lines of magnetic induction passing through the closed circuit, hence we see that the line integral of the electric force due to induction round a closed circuit equals the rate of diminution in the number of lines of magnetic induction passing through the circuit; this however is exactly Faraday's law of induction (see Art. 229).

286. When a charged particle is moving so rapidly that v^2/V^2 cannot be neglected, the distribution of the Faraday tubes round the particle is no longer uniform and the expression $2\mu e^2 v/3a$ given in Art. 277 for the momentum of the charged sphere has to be modified.

When v approaches V, the value of momentum v, the apparent mass, increases rapidly with v; thus if an appreciable amount of the mass of a body is due to electric charge, the mass of the body will increase with the velocity, it is only however when the velocity of the body approaches that of light that this increase becomes appreciable, in the limiting case where the velocity is that of light the apparent mass would be infinite. The influence of velocity on the apparent mass of particles travelling with great velocities has

been detected by Kaufmann by some very interesting experiments, a short account of which will be found in the author's *Conduction of Electricity through Gases*, page 533. Kaufmann found that a particle moving with a velocity about five per cent. less than the velocity of light, had a mass about three times that with small velocities.

The increase in the mass of a slowly moving charged sphere is $2\mu e^2/3a$, i.e. 4 (potential energy of the sphere)/$3V^2$, thus if this mass were to move with the velocity of light its kinetic energy would be two-thirds of the electrical potential energy. The proportion between the increase in the mass due to electrification and the electrical potential energy can be shown to hold for any system of electrified bodies as well as for the simple case of the charged sphere.

287. Effects due to changes in the velocity of the moving charged body. We shall take first the case of a charged sphere moving so slowly that the lines of force are symmetrically distributed around it, and consider what will happen when the sphere is suddenly stopped. The Faraday tubes associated with the sphere have inertia and are in a state of tension, thus any disturbance communicated to one end of a tube will travel along the tube with a finite and constant velocity—the velocity of light. Let us suppose that the stoppage of the particle takes a finite small time τ. We can find the configuration of the tubes, after a time t has elapsed since the sphere began to be stopped, in the following way. Describe with the centre of the charged sphere as centre two spheres, one having the radius Vt, the other the radius $V(t-\tau)$. Then since no disturbance can have reached the portions of the Faraday tubes situated outside the surface of the outer sphere these tubes will be in the positions they would have occupied if the sphere had not been stopped, while since the disturbance has passed over the tubes within the inner sphere, these tubes will be in their final position. Thus consider a tube which when the particle was stopped was along the line OPQ, O being the centre of the charged sphere, this will be the final position of the tube; hence at the time t the portion of this tube inside the inner sphere will be in the position OP, the portion $P'Q'$ outside the outer sphere will be in the position it would have occupied if the sphere had not been stopped, i.e. if O' is the position to which O would have come if the sphere had not been stopped,

$P'Q'$ will be a straight line passing through O'. Thus to preserve its continuity the tube must bend round in the shell between the surfaces of the two spheres, and take the position $OPP'Q'$. Thus the tube which before the sphere was stopped was radial, has now, in the shell, a tangential component, and this implies a tangential electric force; this tangential force is, as the following calculation shows, much greater than the radial force at P before the sphere was brought to rest.

Let us suppose that δ, the thickness of the shell, is so small that the portion of the Faraday tube inside it may be regarded as straight,

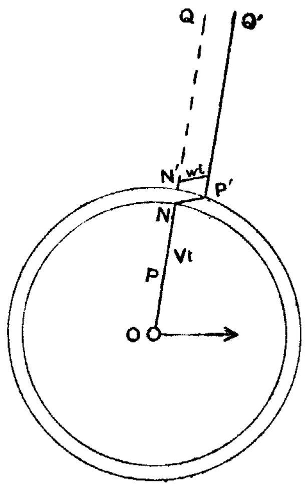

Fig. 136

then, if T is the tangential force inside the pulse, R the radial force, we have

$$\frac{T}{R}=\frac{P'N'}{PN'}=\frac{OO'\sin\theta}{\delta}=\frac{wt\sin\theta}{\delta} \quad\text{..............(1)},$$

where w is the velocity with which the sphere was moving before it was stopped, and θ the angle OP makes with the direction of motion of the sphere; t is the time since the sphere was stopped. Since $OP=Vt$ and $R=e/K\,.\,OP^2$, K being the specific inductive capacity of the medium, we have, writing r for OP,

$$T=\frac{ew\sin\theta}{KV\,.\,r\delta}.$$

Thus the tangential force varies inversely as the distance and not as the square of the distance.

The tangential Faraday tubes move radially outwards with the velocity V, they will therefore produce a magnetic force at right angles to the plane of the pulse and in the opposite direction to the magnetic force at P before the sphere was stopped; this force is equal to

$$V\times 4\pi\,.\,\frac{KT}{4\pi}=\frac{ew\sin\theta}{r\delta};$$

the magnetic force before the sphere was stopped was $ew\sin\theta/r^2$, thus the magnetic force in the pulse, which however only lasts for a very short time, exceeds that in the steady field in the proportion of r to δ.

Thus the pulse produced by the stoppage of the sphere is the seat of very intense electric and magnetic forces; the pulses formed by the stoppage of the negatively electrified particles of the cathode rays form, in my opinion, the well-known Röntgen rays.

288. Energy in the Pulse. The energy due to the magnetic force in the field is per unit volume

$$\frac{\mu}{8\pi}\frac{e^2w^2\sin^2\theta}{\delta^2r^2};$$

integrating this through the pulse we find that the energy due to the magnetic force in the pulse is

$$\frac{\mu e^2w^2}{3\delta^2}.$$

The energy due to the tangential electric force in the pulse is per unit volume

$$\frac{KT^2}{8\pi}=\frac{e^2w^2\sin^2\theta}{8\pi\,.\,KV^2\delta^2r^2};$$

integrating this through the pulse we find that this energy is equal to $\frac{\mu e^2w^2}{3\delta}$, since $\mu K=\frac{1}{V^2}$.

Thus the total energy in the pulse is $\frac{2}{3}\frac{\mu e^3w^2}{\delta}$; and this energy radiates away into space. The energy in the field before the sphere was stopped was $\frac{1}{3}\mu e^2w^2/a$, where a is the radius of the sphere (see Art. 277). Thus if δ is not much greater than the diameter of the sphere a very considerable fraction of the kinetic energy is radiated away when the particle is stopped.

289. Distribution of Momentum in the Field. There is no momentum inside the surface of the sphere whose radius is $b(t-T)$, there is a certain amount of momentum in the pulse, and momentum in the opposite direction in the region outside the pulse; we shall leave it as an exercise for the student to show that the momentum in the pulse is equal and opposite to that outside it, so that as soon as the sphere is reduced to rest the whole momentum in the field is zero.

290. Case of an Accelerated Charged Body. The preceding method can be applied to the case when the charged body has its velocity altered in any way, not necessarily reduced to zero. Thus if the velocity instead of being reduced to zero is diminished by δw, we can show in just the same way as before that the magnetic force H in the pulse is given by the equation

$$H = \frac{e\Delta w \,.\, \sin\theta}{r\delta},$$

and the tangential electric force T by

$$T = \frac{e\Delta w \sin\theta}{KVr\delta}.$$

Now $\delta = V\delta t$ if δt is the time required to change the velocity by Δw, hence we have

$$H = \frac{e}{V}\frac{\Delta w}{\delta t}\frac{\sin\theta}{r}, \qquad T = \frac{e}{KV^2}\frac{\Delta w}{\delta t}\frac{\sin\theta}{r};$$

but $\Delta w/\delta t = -f$, where f is the acceleration of the particle, hence

$$H = -\frac{e}{V}\frac{f\sin\theta}{r}, \qquad T = -\frac{e}{KV^2}\frac{f\sin\theta}{r}.$$

It must be remembered that f is not the acceleration of the sphere at the time when H and T are estimated but at the time r/V before this. We see that when the velocity of the sphere is not uniform, part of the magnetic and electric force will vary inversely as the distance from the centre of the sphere, while the other part will vary inversely as the square of this distance; at great distances from the sphere the former part will be the most important.

The energy in the pulse emitted whilst the velocity is changing is equal to

$$\frac{2}{3}\frac{e^2 f^2}{V^2}d,$$

where d is the thickness of the pulse; since $d = V\delta t$, where δt is the time the acceleration lasts, the energy emitted in the time δt is

$$\frac{2}{3}\frac{e^2 f^2}{V}\delta t,$$

thus the rate of emission of energy is $2e^2f^2/3V$.

291. Magnetic and Electric Forces due to a charged particle vibrating harmonically through a small distance. The magnetic force proportional to the acceleration which we have just investigated arises from the motion of the tangential part of the Faraday tubes—the portion $P'N'$ of Fig. 134; the radial tubes are however also in motion, their velocity at right angles to their length being $w\sin\theta$, where w is the velocity of the particle when its acceleration is f, i.e. at a time r/V before the force is estimated. This motion of the radial tubes produces a magnetic force $ew\sin\theta/r^2$ in the same direction as that due to the acceleration. Thus H the magnetic force at P is equal to

$$\frac{ew\sin\theta}{r^2} + \frac{ef\sin\theta}{Vr},$$

and is at right angles to OP and to the axis of z along which the particle is supposed to be moving. Let the velocity of the particle along this line be $\omega\sin pt$ and its acceleration therefore $\omega p\cos pt$. The magnetic force at P at the time t will depend upon the velocity and acceleration of the particle at the time $t - \frac{r}{V}$, these are respectively $\omega\sin p\left(t-\frac{r}{V}\right)$ and $\omega p\cos p\left(t-\frac{r}{V}\right)$, thus H the magnetic force at P is given by the equation

$$H = \frac{e\omega\sin\theta\sin p\left(t-\frac{r}{V}\right)}{r^2} + \frac{e\omega\sin\theta p\cos p\left(t-\frac{r}{V}\right)}{Vr}.$$

If α, β, γ are the components of this force parallel to the axes of x, y, z, then

$$\alpha = -\frac{y}{r\sin\theta}H, \qquad \beta = \frac{x}{r\sin\theta}H, \qquad \gamma = 0.$$

Hence

$$\alpha = \frac{d}{dy}\frac{e\omega \sin p\left(t - \frac{r}{V}\right)}{r}, \quad \beta = -\frac{d}{dx}\frac{e\omega \sin p\left(t - \frac{r}{V}\right)}{r}.$$

If X, Y, Z are the components of the electric force, we have by equation (1), page 363,

$$K\frac{dX}{dt} = \frac{d\gamma}{dy} - \frac{d\beta}{dz} = \frac{d^2}{dx\,dz}\frac{e\omega \sin p\left(t - \frac{r}{V}\right)}{r},$$

$$K\frac{dY}{dt} = \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = \frac{d^2}{dy\,dz}\frac{e\omega \sin p\left(t - \frac{r}{V}\right)}{r},$$

$$K\frac{dZ}{dt} = \frac{d\beta}{dx} - \frac{d\alpha}{dy} = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\frac{e\omega \sin p\left(t - \frac{r}{V}\right)}{r}.$$

Hence the periodic parts of X, Y, Z are given by the equations

$$KX = -\frac{1}{p}\frac{d^2}{dx\,dz}\frac{e\omega \cos p\left(t - \frac{r}{V}\right)}{r},$$

$$KY = -\frac{1}{p}\frac{d^2}{dy\,dz}\frac{e\omega \cos p\left(t - \frac{r}{V}\right)}{r},$$

$$KZ = \frac{1}{p}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\frac{e\omega \cos p\left(t - \frac{r}{V}\right)}{r}.$$

In addition to these there are the components

$$-\frac{e}{K}\frac{d}{dx}\frac{1}{r}, \quad -\frac{e}{K}\frac{d}{dy}\frac{1}{r}, \quad -\frac{e}{K}\frac{d}{dz}\frac{1}{r},$$

of the electrostatic force due to the charge at O. In this investigation ω is supposed to be so small compared with V that ω^2/V^2 may be neglected.