### *Motion of a Charged Sphere*

We shall apply these results to a very simple and important case—the steady motion of a charged sphere. If the velocity of the sphere is small compared with that of light then the Faraday tubes will, as when the sphere is at rest, be uniformly distributed and radial in direction. They will be carried along with the sphere. If $e$ is the charge on the sphere, $O$ its centre, the density of the Faraday tubes at $P$ is $\frac{e}{4\pi\, OP^2}$; so that if $v$ is the velocity of the sphere, $\theta$ the

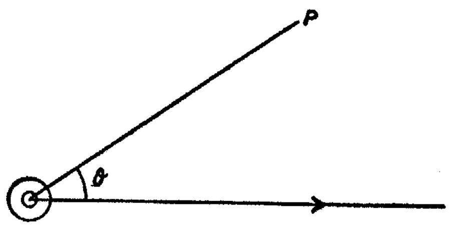


FIG. 6.

angle between $OP$ and the direction of motion of the sphere, then, according to the above rule, the magnetic force at $P$ will be $\frac{ev\sin\theta}{r^2}$, the direction of the force will be at right angles to $OP$, and at right angles to the direction of motion of the sphere; the lines of magnetic force will thus



be circles, having their centres on the path of the centre of the sphere and their planes at right angles to this path. Thus, a moving charge of electricity will be accompanied by a magnetic field. The existence of a magnetic field implies energy; we know that in a unit volume of the field at a place where the magnetic force is $H$ there are $\frac{\mu H^2}{8\pi}$ units of energy, where $\mu$ is the magnetic permeability of the medium. In the case of the moving sphere the energy per unit volume at $P$ is $\frac{\mu e^2 v^2 \sin^2\theta}{8\pi\, OP^4}$. Taking the sum of this energy for all parts of the field outside the sphere, we find that it amounts to $\frac{\mu e^2 v^2}{3a}$, where $a$ is the radius of the sphere. If $m$ is the mass of the sphere, the kinetic energy in the sphere is $\frac{1}{2} m v^2$; in addition to that we have the energy outside the sphere, which as we have seen is $\frac{\mu e^2 v^2}{3a}$; so that the whole kinetic energy of the system is $\frac{1}{2}\left(m + \frac{2\mu}{3}\frac{e^2}{a}\right) v^2$, or the energy is the same as if the mass of the sphere were $m + \frac{2\mu}{3}\frac{e^2}{a}$ instead of $m$. Thus, in consequence of the electric charge, the mass of the sphere is measured by $\frac{2\mu e^2}{3a}$. This is a very important result, since it shows that part of the mass of a charged sphere is due to its charge. I shall later on have to bring before you considerations which show that it is not impossible that the whole mass of a body may arise in the way.

Before passing on to this point, however, I should like to illustrate the increase which takes place in the mass of the sphere by some analogies drawn from other branches of physics. The first of these is the case of a sphere moving through a frictionless liquid. When the sphere moves it sets the fluid around it moving with a velocity proportioned to its own, so that to move the sphere we we have not merely to move the substance of the sphere itself, but also the liquid around it; the consequence of this is, that the sphere behaves as if its mass were increased by that of a certain volume of the liquid. This volume, as was shown by Green in 1833, is half the volume of the sphere. In the case of a cylinder moving at right angles to its length, its mass is increased by the mass of an equal volume of the liquid. In the case of an elongated body like a cylinder, the amount by



which the mass is increased depends upon the direction in which the body is moving, being much smaller when the body moves point foremost than when moving sideways. The mass of such a body depends on the direction in which it is moving.

Let us, however, return to the moving electrified sphere. We have seen that in consequence of its charge its mass is increased by $\frac{2\mu e^2}{3a}$; thus, if it is moving with the velocity $v$, the momentum is not $mv$, but $\left(m + \frac{2\mu e^2}{3a}\right)v$. The additional momentum $\frac{2\mu e^2}{3a}v$ is not in the sphere, but in the space surrounding the sphere. There is in this space *ordinary mechanical momentum*, whose resultant is $\frac{2\mu e^2}{3a}v$ and whose direction is parallel to the direction of motion of the sphere. It is important to bear in mind that this momentum is not in any way different from ordinary mechanical momentum and can be given up to or taken from the momentum of moving bodies. I want to bring the existence of this momentum before you as vividly and forcibly as I can, because the recognition of it makes the behavior of the electric field entirely analogous to that of a mechanical system. To take an example, according to Newton's Third Law of Motion, Action and Reaction are equal and opposite, so that the momentum in any direction of any self-contained system is invariable. Now, in the case of many electrical systems there are apparant violations of this principle; thus, take the case of a charged body at rest struck by an electric pulse, the charged body when exposed to the electric force in the pulse acquires velocity and momentum, so that when the pulse has passed over it, its momentum is not what it was originally. Thus, if we confine our attention to the momentum in the charged body, *i.e.*, if we suppose that momentum is necessarily confined to what we consider ordinary matter, there has been a violation of the Third Law of Motion, for the only momentum recognized on this restricted view has been changed. The phenomenon is, however, brought into accordance with this law if we recognize the existence of the momentum in the electric field; for, on this view, before the pulse reached the charged body there was momentum in the pulse, but none in the body; after the pulse passed over the body there was some momentum in the body and a smaller amount in the pulse,



the loss of momentum in the pulse being equal to the gain of momentum by the body.

We now proceed to consider this momentum more in detail. I have in my "Recent Researches on Electricity and Magnetism" calculated the amount of momentum at any point in the electric field, and have shown that if $N$ is the number of Faraday tubes passing through a unit area drawn at right angles to their direction, $B$ the magnetic induction, $\theta$ the angle between the induction and the Faraday tubes, then the momentum per unit volume is equal to $NB \sin \theta$, the direction of the momentum being at right angles to the magnetic induction and also to the Faraday tubes. Many of you will notice that the momentum is parallel to what is known as Poynting's vector—the vector whose direction gives the direction in which energy is flowing through the field.

### *Moment of Momentum Due to an Electrified Point and a Magnetic Pole*

To familiarize ourselves with this distribution of momentum let us consider some simple cases in detail. Let us begin with the simplest, that of an electrified point and a magnetic pole; let $A$, Fig. 7, be the point, $B$ the pole. Then, since the momentum at any point $P$ is at right angles to $A\ P$, the direction of the Faraday tubes and also to $B\ P$, the magnetic induction, we see that the momentum will be perpendicular to the plane $A\ B\ P$; thus, if we draw a series of lines such that their direction at any point coincides with the direction of the momentum at that point, these lines will form a series of circles whose planes are perpendicular to the line $A\ B$, and whose centres lie along that line. This distribution of momentum, as far as direction goes, is that possessed by a top spinning around $A\ B$. Let us now find what this distribution of momentum throughout the field is equivalent to.

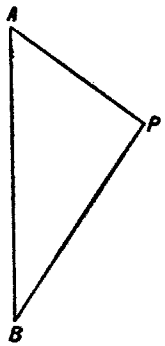

FIG. 7.

It is evident that the resultant momentum in any direction is zero, but since the system is spinning round $A\ B$, the direction of rotation being everywhere the same, there will be a finite moment of momentum round $A\ B$. Calculating the value of this from the expression for the momentum given above, we obtain the very simple expression $e\,m$ as the value of the moment of momentum about $A\ B$, $e$ being the charge on the point and $m$ the strength of the pole. By means of this



expression we can at once find the moment of momentum of any distribution of electrified points and magnetic poles.

To return to the system of the point and pole, this conception of the momentum of the system leads directly to the evaluation of the force acting on a moving electric charge or a moving magnetic pole. For suppose that in the time $\delta t$ the electrified point were to move from $A$ to $A'$, the moment of momentum is still $e\,m$, but its axis is along $A'\ B$ instead of $A\ B$. The moment of momentum of the field has thus changed, but the whole moment of momentum of the system comprising point, pole, and field must be constant, so that the change in the moment of momentum of the field must be accompanied by an equal and opposite change in the moment of momentum of the pole and point. The momentum gained by the point must be equal and opposite to that gained by the pole, since the whole momentum is zero. If $\theta$ is the angle $A\ B\ A'$, the change in the moment of momentum is $e\,m \sin\theta$, with an axis at right angles to $A\ B$ in the plane of the paper. Let $\delta I$ be the change in the momentum of $A$, —

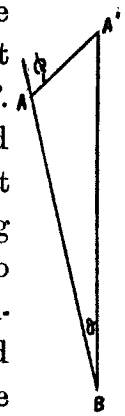

FIG. 8.

$\delta I$ that of B, then $\delta I$ and $-\delta I$ must be equivalent to a couple whose axis is at right angles to $A B$ in the plane of the paper, and whose moment is $e m \sin \theta$. Thus $\delta I$ must be at right angles to the plane of the paper and

$$\delta I . A B = e m \sin \theta = \frac{e m A A' \sin \phi}{A B}$$

Where $\phi$ is the angle $B A A'$. If $v$ is the velocity of $A, A A' = v \delta t$ and we get

$$\delta I = \frac{e m v \sin \phi \, \delta t}{A B^2}$$

This change in the momentum may be supposed due to the action of a force $F$ perpendicular to the plane of the paper, $F$ being the rate of increase of the momentum, or $\frac{\delta I}{\delta t}$. We thus get $F = \frac{e m v \sin \phi}{A B^2}$; or the point is acted on by a force equal to $e$ multiplied by the component of the magnetic force at right angles to the direction of motion. The direction of the force acting on the point is at right angles to its velocity and also to the magnetic force. There is an equal and opposite force acting on the magnetic pole.

The value we have found for $F$ is the ordinary expression for the mechanical force acting on a moving charged particle in a magnetic field; it



may be written as $e v H \sin \phi$, where $H$'s is the strength of the magnetic field. The force acting on unit charge is therefore $v H \sin \phi$. This mechanical force may be thus regarded as arising from an electric force $v H \sin \phi$, and we may express the result by saying that when a charged body is moving in a magnetic field an electric force $v H \sin \phi$ is produced. This force is the well-known electromotive force of induction due to motion in a magnetic field.

The forces called into play are due to the *relative* motion of the pole and point; if these are moving with the same velocity, the line joining them will not alter in direction, the moment of momentum of the system will remain unchanged and there will not be any forces acting either on the pole or the point.

The distribution of momentum in the system of pole and point is similar in some respects to that in a top spinning about the line $A B$. We can illustrate the forces acting on a moving electrified body by the behavior of such a top. Thus, let Fig. 9 represent a balanced gyroscope spinning about the axis $A B$, let the ball at $A$ represent the electrified point, that at $B$ the magnetic pole. Suppose the instrument is spinning with $A B$

horizontal, then if with a vertical rod I push against $A\,B$ horizontally, the point $A$ will not merely move horizontally forward in the direction in which it is pushed, but will also move vertically upward or downward, just as a charged

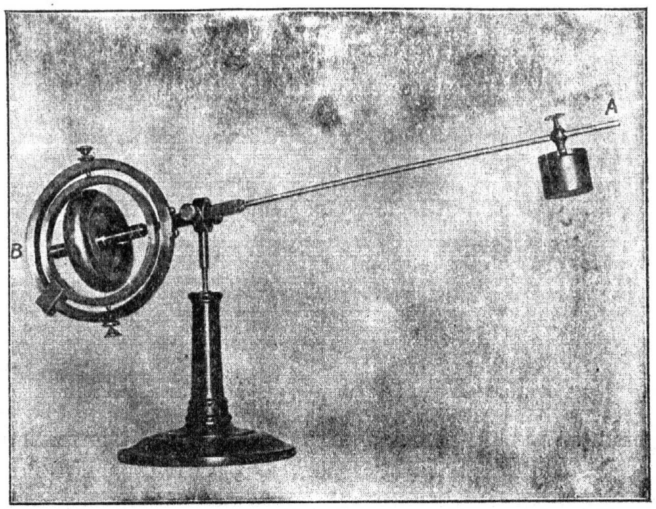


FIG. 9.

point would do if pushed forward in the same way, and if it were acted upon by a magnetic pole at $B$.

### *Maxwell's Vector Potential*

There is a very close connection between the momentum arising from an electrified point and a



magnetic system, and the Vector Potential of that system, a quantity which plays a very large part in Maxwell's Theory of Electricity. From the expression we have given for the moment of momentum due to a charged point and a magnetic pole, we can at once find that due to a charge $e$ of electricity at a point $P$, and a little magnet $A\,B$; let the negative pole of this magnet be at $A$, the positive at $B$, and let $m$ be the strength of either pole. A simple calculation shows that in this case the axis of the resultant moment of momentum is in the plane $P\,A\,B$ at right angles to $P\,O$, $O$ being the middle point of $A\,B$, and that the magnitude of the moment of momentum is equal to $e.m.\;A\,B\,\frac{\sin\phi}{OP^2}$, where $\phi$ is the angle $A\,B$ makes with $O\,P$. This moment of momentum is equivalent in direction and magnitude to that due to a momentum $e.m.\;A\,B\,\frac{\sin\phi}{OP^2}$ at $P$ directed at right angles to the plane $P\,A\,B$, and another momentum equal in magnitude and opposite in direction at $O$. The vector $m\;A\,B\,\frac{\sin\phi}{OP^2}$ at $P$ at right angles to the plane $P\,A\,B$ is the vector called by Maxwell the Vector Potential at $P$ due to the Magnet.

Calling this Vector Potential $I$, we see that the momentum due to the charge and the magnet is equivalent to a momentum $e\,I$ at $P$ and a momentum $-e\,I$ at the magnet.

We may evidently extend this to any complex system of magnets, so that if $I$ is the Vector Potential at $P$ of this system, the momentum in the field is equivalent to a momentum $e\,I$ at $P$ together with momenta at each of the magnets equal to

$-e$ (Vector Potential at $P$ due to that magnet).

If the magnetic field arises entirely from electric currents instead of from permanent magnets, the momentum of a system consisting of an electrified point and the currents will differ in some of its features from the momentum when the magnetic field is due to permanent magnets. In the latter case, as we have seen, there is a moment of momentum, but no resultant momentum. When, however, the magnetic field is entirely due to electric currents, it is easy to show that there is a resultant momentum, but that the moment of momentum about any line passing through the electrified particle vanishes. A simple calculation shows that the whole momentum in the field is equivalent to a momentum $e\,I$ at the electrified



point $I$ being the Vector Potential at $P$ due to the currents.

Thus, whether the magnetic field is due to permanent magnets or to electric currents or partly to one and partly to the other, the momentum when an electrified point is placed in the field at $P$ is equivalent to a momentum $e\,I$ at $P$ where $I$ is the Vector Potential at $P$. If the magnetic field is entirely due to currents this is a complete representation of the momentum in the field; if the magnetic field is partly due to magnets we have in addition to this momentum at $P$ other momenta at these magnets; the magnitude of the momentum at any particular magnet is $-e$ times the Vector Potential at $P$ due to that magnet.

The well-known expressions for the electromotive forces due to Electro-magnetic Induction follow at once from this result. For, from the Third Law of Motion, the momentum of any self-contained system must be constant. Now the momentum consists of (1) the momentum in the field; (2) the momentum of the electrified point, and (3) the momenta of the magnets or circuits carrying the currents. Since (1) is equivalent to a momentum $e\,I$ at the electrified particle, we see that changes in the momentum of the field must be accompanied by changes in the momentum of the particle. Let $M$ be the mass of the electrified particle, $u$, $v$, $w$ the components parallel to the axes of $x$, $y$, $z$ of its velocity, $F$, $G$, $H$, the components parallel to these axes of the Vector Potential at $P$, then the momentum of the field is equivalent to momenta $eF$, $eG$, $eH$ at $P$ parallel to the axes of $x$, $y$, $z$; and the momentum of the charged point at $P$ has for components $Mu$, $Mo$, $Mw$. As the momentum remains constant, $Mu + eF$ is constant, hence if $\delta u$ and $\delta F$ are simultaneous changes in $u$ and $F$,

$$M\delta u + e\delta F = 0;$$

$$\text{or } m\frac{du}{dt} = -e\frac{dF}{dt}.$$

From this equation we see that the point with the charge behaves as if it were acted upon by a mechanical force parallel to the axis of $x$ and equal to $-e\frac{dF}{dt}$, *i.e.*, by an electric force equal to $-\frac{dF}{dt}$. In a similar way we see that there are electric forces $-\frac{dG}{dt}$, $-\frac{dH}{dt}$, parallel to $y$ and $z$ respectively. These are the well-known expressions of the forces due to electro-magnetic induction, and we see that they are a direct consequence of the



principle that action and reaction are equal and opposite.

Readers of Faraday's Experimental Researches will remember that he is constantly referring to what he called the "Electrotonic State"; thus he regarded a wire traversed by an electric current as being in the Electrotonic State when in a magnetic field. No effects due to this state can be detected as long as the field remains constant; it is when it is changing that it is operative. This Electrotonic State of Faraday is just the *momentum* existing in the field.