Radiation Reaction fields for an accelerated dipole for scalar and electromagnetic radiation

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Abstract

The radiation reaction fields are calculated for an accelerated changing dipole in scalar and electromagnetic radiation fields. The acceleration reaction is shown to alter the damping of a time varying dipole in the EM case, but not the scalar case. In the EM case, the dipole radiation reactionfield can exert a force on an accelerated monopole charge associated with the accelerated dipole. The radiation reaction of an accelerated charge does not exert a torque on an accelerated magnetic dipole, but an accelerated dipole does exert a force on the charge. The technique used is that originally developed by Penrose for non-singular fields and extended by the author for an accelerated monopole charge.

It is well known that an accelerated charge radiates and that the emitted field from the accelerated charge exerts a force (the radiation reaction force) back on the charge itself. This radiation reaction force is usually derived either by an appeal to a balance of the energy and momentum emitted by the charge or by a detailed examination of the energy-momentum tensor just near the charge. These techniques invariably have difficulties with the fact that the fields from a point charge diverge, and necessitate re-normalisation of various quantities (eg the mass) in order to extract reasonable results. However, it was discovered by Penrose and Unruh that the radiation reaction field for a charge could be extracted from the radiation field by means of an integral over the future null cone of the the particle. This integral, which in the absence of any sources, exactly gives the field strength at a point, also gives a finite result whan applied to the field emanating from a point charge, and that finite result is exactly the radiation reaction fields at the particle. In this paper I will show that this approach can be generalised to the case of an accelerating dipole and leads to finite radiation reaction fields at the location of the point dipole. These fields lead to the damping forces on that accelerating dipole. These results will ultimately be applied in another paper on the question of the equilibrium polarisation of an accelerating particle with spin, but seem to be of sufficient general interest that they are here separated out for detailed study.

In this paper, I will derive the radiation reaction fields at the location of point source for a massless scalar field, and the reaction fields at the location of a point dipole source for the electromagnetic field, using the Penrose integral to do so. Thus I will begin by giving a very brief review of the Newmann-Penrose (NP) spinor formalism, and the natural null tetrad metric for an accelerated path in Minkowski spacetime. That accelerated path is assumed for the purposes of this paper to consist of an acceleration restricted to a plane (linear or circular acceleration, but with non-constant acceleration).

I. NULL METRIC, TETRAD AND NP SPIN FORMALISM

This section will be a very quick review of the NP spinor formalism, presented as much to specify the notation I will use as for any other purpose. For a more complete introduction to spinors, see Penrose and Rindler [2]

Given a path in spacetime, define a $ur\theta\phi$ coordinate system as follows:

Parameterize the curve by its length parameter u. (ie, $\eta_{\mu\nu}\frac{dx^\mu}{du}\frac{dX^\nu}{du} = 1$). At each point along the curve, define the future directed null cone centered at that point along the curve, and label that null cone by the parameter u.

On each of the null cones, choose the direction of the acceleration vector, and call it the $\theta = 0$ direction. Along each of the null generators of the cone (null lines which originate at the point along the curve at the vertex of the cone) define a radial affine parameter r, such that the tangent vector n^μ to the null curve parameterized by r (which is a null vector) has a dot product with the tangent vector to the original curve of unity– Ie, $n^\mu\frac{dx_\mu}{du} = 1$. Now, the two spheres defined by u and r constant are metric two spheres on which we will define angular coordinates θ and ϕ. They are chosen so that the metric on these two spheres is the usual two sphere metrics.. Because of the way that r has been defined, the circumference of these two spheres will be r so the metric on these two spheres will be $r^2(d\theta^2 + d\phi^2)$. The direction $\theta = 0$ is at the null generator pointing in the direction of the acceleration vector, and θ and ϕ will label the null generators. This procedure will define a new coordinate system centered on the path of the particle. The metric of flat spacetime in these coordinates is

$$\begin{aligned} ds^2 = &\left(1 + 2f(u)r\cos(\theta) - r^2(g(u)\cos(\phi) + f(u)\sin(\theta))^2 - r^2g(u)^2\sin(\phi)^2\cos(\theta)^2\right)du^2 \\ &+2dudr + 2(f(u)\sin(\theta) + g(u)\cos(\phi))r^2dud\theta - 2g(u)\sin(\theta)\cos(\theta)\sin(\phi)r^2dud\phi \\ &-r^2d\theta^2 - r^2\sin(\theta)^2d\phi^2 \end{aligned} \tag{1}$$

where $f(u)$ and $g(u)$ will be related to the acceleration. $f(u)$ is the acceleration, and $g(u)$ is the rate of change of the direction of the acceleration. The acceleration is assumed to be confined to the place $\phi = 0$ or π.

We now define a null complex tetrad for this metric. This is a set of four vectors l^μ, n^μ, m^μ and $\bar{m}^\mu$ such that each of these vectors is null (has zero inner product with itself) and such that

$$l^\mu n_\mu = m^\mu\bar{m}_\mu = 1 \tag{2}$$

and all other inner products are zero. The vector n^μ has already been defined as the vector tangent to the null generators of the null cones. The vector m^μ is defined as a complex vector lying tangent to the two surface of the u and r constant spheres. $\bar{m}^\mu$ is the complex conjugate to μ. Thus the null vector l^μ will be orthogonal to the these two spheres (as is n^μ). l^μ is assumed to be such that $l^\mu\frac{dX_\mu}{du}$ is greater than zero (ie, is a future pointing null vector.) For definiteness, I will choose $m^\mu = e^\mu_\theta + ie^\mu_\phi$ where e_θ and e_ϕ are the unit vectors which lie along the $u,\ r,\ \phi$ constant line and $u,\ r,\ \theta$ constant lines respectively.

In the above $ur\theta\phi$ coordinate system, these vectors therefor are

$$n^\mu = [0,1,0,0] \tag{3}$$
$$m^\mu = \frac{1}{\sqrt{2}}\left[0,0,\frac{1}{r},\frac{i}{r\sin(\theta)}\right] \tag{4}$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}}\left[0,0,\frac{1}{r},\frac{-i}{r\sin(\theta)}\right] \tag{5}$$
$$l^\mu = [1, -\frac{1}{2} - f(u) r\cos(\theta),\ f(u)\sin(\theta) + g(u)\cos(\phi),\ \ - g(u)\cot(\theta)\sin(\phi)] \tag{6}$$

In addition to the null vectors, the formalism defines a set of two dimensional complex spinors. In particular they assumed that there are two separate two dimensional spinor spaces which are anti-unitarily related to each other (these are just the two unitarily inequivalent spin 1/2 representations of the Lorentz group). Tensors over these two spaces are designated by indexed symbols whose indices are capital Roman letters. Tensors over the one representation have plain indices and tensors over the other are designated with primes on their indices. Because of their anti-unitary relationship, there exists a mapping from one type of tensor to the other, which we will denote by complex conjugation. Thus $(w^A)^* = \bar{w}^{A'}$. This is defined so that the inner product of complex conjugate vectors is just the ordinary complex conjugation. $(w^A v_A)^* = \bar{w}^{A'}\bar{v}_{A'}$. A complete set of basis vectors on these spinor spaces are designated by ι^A, o^A in the one case, and $\iota^{A'}$, $o^{A'}$ in the other case.They are chosen so that $(\iota^A)^* = \iota^{A'}$, etc. These spinor spaces are related to the spacetime vectors by means of the spin matrices $\sigma^\mu_{AA'}$, matrices which for a given vector v_μ represent a mapping from the one spinor space to the other via $V_\mu \sigma^\mu_{AA'}$. If we choose the basis vectors appropriately in the spinor space, then these matrices are just the four Pauli spin matrices,$\mathbf{1}, \sigma_x, \sigma_y, \sigma_z$. In particular I will assume that they are chosen so that

$$l_\mu \sigma^\mu = (\mathbf{1} + \sigma_z)/2 \tag{7}$$
$$n_\mu \sigma^\mu = (\mathbf{1} - \sigma_z)/2 \tag{8}$$
$$m_\mu \sigma^\mu = (\sigma_x + i * \sigma_y)/2 \tag{9}$$
$$\bar{m}_\mu \sigma^\mu = \sigma_x - i * \sigma_y)/2 \tag{10}$$

with ι^A and $\iota_{A'}$ both represented by $(0,1)$ and o_A and $o_{A'}$ both represented by $(1,0)$. Thus, we have

$$l_\mu \sigma^\mu_{AA'} = o_A o_{A'} \tag{11}$$
$$n_\mu \sigma_{AA'} = \iota_A \iota_{A'} \tag{12}$$
$$m_\mu \sigma^\mu_{AA'} = o_A \iota_{A'} \tag{13}$$
$$\bar{m}_\mu \sigma^\mu_{AA'} = \iota_A o_{A'} \tag{14}$$

From now on I will freely alternate between the spinor representation and the vector representation. Thus a tensor $S_{\mu\nu}$ can also be written as $S_{AA'BB'} = S_{\mu\nu}\sigma^\mu_{AA'}\sigma^\nu_{BB'}$, etc.

These spinor spaces also have a metric defined on them, a metric which must be compatible with the Lorentzian metric. The metric, designated by either ϵ^{AB} for the spinor, or $\epsilon^{A'B'}$ for the anti-spinor space is an antisymmetric metric, such that $\epsilon^{AB} = -\epsilon^{BA}$. The inverse metric is ϵ_{AB} which is also antisymmetric and is chosen so that $\epsilon_{AB}\epsilon^{CB} = \delta_A^C$. Indices are raised and lowered by means of the metric, but because of the antisymmetry of the metric, the order of the indices is of crucial importance. My convention, following Penrose and Rindler [2], is

$$\omega^A = \epsilon^{AB}\omega_B = -\omega_B\epsilon^{BA} \tag{15}$$

$$\omega_A = \omega^B\epsilon_{BA}. \tag{16}$$

and similarly for the prime space. The basis vectors ι^A and o^A are chosen to obey $o_A\iota^A = 1$, which is compatible with the relation of these spinor bases to the spacetime null tetrad, and the normalisation of those tetrad vectors.

This antisymmetry of the metric and the attendant risk of confusion in index manipulations (together with the presence of the two separate types of spinors) is probably the greatest impediment to the adoption of spinors as a standard approach in special and general relativity. Given these disadvantages, there must be some advantages to the spinor formalism which would persuade any but masochists to adopt the formalism. The key advantages are firstly that spinors allows a unified treatment of the various fields of whatever spin type in one simple notational system. Ie, scalar, spin 1/2, vector, spin 3/2, ... fields can all be treated very similarly, in an extremely compact and transparent way. The second advantage is that spinor space is a two dimensional space. This means that there is only antisymmetric tensor of rank two, and all other antisymmetric tensors of rank two must be proportional to this tensor. Furthermore, since the metric is antisymmetric, it can be chosen as the fiducial antisymmetric tensor. Any two indices (of the same type) of a spinor tensor can always be written as a combination of symmetric and antisymmetric pairs. Thus, we can write any tensor

$$S_{...}^{...A..B..} = S_{...}^{...(A|..|B)..} + \frac{1}{2}\epsilon_{CD}S_{...}^{....C..D..}\epsilon^{AB} \tag{17}$$

Any tensor can thus be written as the sum of tensors which are totally symmetric on their indices times products of the metric tensors. This ability to represent any tensor as either entirely symmetric tensors or multiples of the metric is the key power of the spinor notation, and achieves its greatest power in the representation of massless fields. A massless field of spin s is represented by a tensor with s indices, all of which are completely symmetric, say $\Psi_{AB..S}$ where the tensor is symmetric under interchange of any two indices. Furthermore, the equations of motion of a spin s massless field are simply written as $\nabla^{AA'}\Psi_{AB...S} = 0$, where $\nabla_{AA'}$ is the covariant derivative, defined on spinors such that $\sigma^\mu_{AA'}$ and the metric ϵ_{AB} are covariantly constant. It is the compactness of the spinor notation, and the transparency of the symmetries of the fundamental tensors which give the spinor notation its power.

In the following I will be primarily interested in the electromagnetic field, $F^{\mu\nu}$. Writing this in spinor form, we have $F_{AA'BB'}$ with the antisymmetry ensuring that $F_{AA'BB'} = -F_{BB'AA'}$. But using the above reduction, we note that this can be written as

$$F_{AB} = \frac{1}{2}(\epsilon^{C'D'}F_{AC'BD'}\epsilon_{A'B'} + \epsilon^{CD}F_{CA'DB'}\epsilon_{AB}) \tag{18}$$

I will use the notation that

$$F_{AB} = \frac{1}{2}\epsilon^{C'D'}F_{AC'BD'} \tag{19}$$

so that

$$F_{AA'BB'} = F_{AB}\epsilon_{A'B'} + F_{A'B'}\epsilon_{AB} \tag{20}$$

Maxwell's equations become

$$\nabla^{AA'}F_{AB} = 4\pi J_B{}^{A'} \tag{21}$$
$$\nabla^{AA'}F_{A'B'} = 4\pi J^A{}_{B'} \tag{22}$$

where J^μ is the current source for the Maxwell field. The reality of $F_{\mu\nu}$ and J_μ ensure that $(F_{AB})^* = F_{A'B'}$ and $(J_{AB'})^* = J_{BA'}$.

Finally, if $G_{\mu\nu}$ is another antisymmetric tensor, then

$$F^{\mu\nu}G_{\mu\nu} = 2(F^{AB}G_{AB} + F^{A'B'}G_{A'B'}) \tag{23}$$

and if F and G are both real tensors, then

$$F^{AB}G_{AB} = (F^{A'B'}G_{A'B'})^* \tag{24}$$

Having established the notation, I will now state the theorem without proof [1,2]. Given a spin s massless field without source, then the following integral gives the value of the field $\Psi(0)_{AB...S}$ at the apex (r=0) of the null cone emanating from a point in spacetime.

$$T(0)^{AB..S}\Psi(0)_{AB..S} = \frac{(-1)^{2s+1}}{2\pi}\int_{r,u\ \mathrm{const}} r\sin(\theta)T_{AB..S}\iota^A\iota^B...\iota^S o^D o^E...o^T \Big(o^X o^{Y'}\nabla_{XY'}\Psi_{DE..T} \\ -(2s+1)\Psi_{DE..T}o^W\iota^X o^{Y'}\nabla_{XY'}o_W\Big)\,d\theta d\phi \tag{25}$$

Here $T_{AB..S}$ is any covariantly constant spinor field, $\nabla_{XX'}T_{AB..S} = 0$. (In fact it need only be covariantly constant along the null cone of interest). This theorem states that the value of any massless field in flat spacetime can be determined by the integral over a sphere on the null cone emanating from that point. This expression is a generalisation to a dynamic massless field of arbitrary spin of the Kirkoff type integrals for static fields in terms of the integral over some surface surrounding the point in question of the normal derivatives of the field and the Green's function for that field.

Although I have stated the theorem in terms of integrals over metric spheres on the null cone (r =const.), it can also be generalised to the integral over arbitrary two surfaces on the null cone. However, I will not use that generalisation here.

While the use of this integral for the value of the field at points in the spacetime where the field is regular is interesting but unexceptional, a surprising result [1] is that this integral also gives finite values if the point of interest is the location of a point charge (with its divergent Coulomb field). In fact, this integral (or rather the average of this integral over the future null cone emanating from the location of that charge at some time, and the past directed null cone emanating from that same point) gives exactly the radiation reaction field for an accelerating charge.

$$F^{\mu\nu}_{RR} = -\frac{4}{3}\frac{dx(u)^{[\mu}}{du}\frac{D^3x(u)^{\nu]}}{Du^3} \tag{26}$$

where the square brackets around the indices indicates anti-symmetrization, ($S^{[\mu\nu]} = \frac{1}{2}(S^{\mu\nu} - S^{\nu\mu})$. Ie, this Penrose integral automatically averages out the divergent field of the point particle to give just the finite radiation reaction contribution. Thus this Penrose integral approach differs substantially from techniques like the Dirac [3] or Abraham–Lorentz [4] which suffer from divergences and the necessity for re-normalisations.

The purpose of this paper will be to apply the above formula to the calculation of the radiation reaction field of an accelerating and time varying point dipole source for the electromagnetic field. Surprisingly, considering the fact the "Coulomb" portion of the field now diverges as $1/r^3$ rather than as $1/r^2$ for the case of a point charge, the Penrose integral is still finite, and gives a field which agrees with the radiation reaction field for a time varying unaccelerated dipole calculated by other methods [6]. I will therefor assume that this finite field is also the correct radiation reaction field for an accelerated dipole. I have so far been unable to prove that this is consistent with the radiation damping one would calculate by more traditional techniques (eg, from the energy momentum tensor), but I can see no reason why it would not.

II. SCALAR FIELD

In order to gain practice, let me first calculate the radiation reaction field produced by a source which radiates scalar radiation. Consider a scalar field coupled to a point source travelling along the line $x^\mu(u)$ in flat spacetime, with a time varying source for the scalar field of intensity $m(u)$, The Lagrangian for the massless scalar field is assumed to be

$$S = \int \frac{1}{2}\sqrt{-g}\phi_{,\mu}\phi_\nu d^3x + 4\pi \int m(u)\phi(\mathbf{x}(u))du \tag{27}$$

which gives the equation for ϕ of

$$\nabla^\mu\nabla_\mu\phi = 4\pi \int m(u)\delta^4(x^\mu - x(u)^\mu(u))du \tag{28}$$

In the $ur\theta\phi$ coordinates, the retarded Green's function for the scalar field in $ur\theta\phi$ coordinates is particularly simple, it is just $1/r$. Thus the retarded solution for the scalar field is just

$$\phi(u, r, \theta\phi) = \frac{m(u)}{r} \tag{29}$$

We can now substitute this expression into the equation for the radiation reaction field. In order to write this in a slightly more transparent form, recall that $l^{AA'} = o^A o^{A'}$. Thus

$$o^X o^{Y'}\nabla\phi = l^\mu\phi_{,\mu} \tag{30}$$

$$\begin{aligned} o_A o^X o^{Y'}\nabla_{XY'}o^A &= o_A o_{A'} o^X o^{Y'}\nabla_{XY'}(o^A \iota^{A'}) \\ &= l_\mu m^\alpha \nabla_\alpha m^\mu \end{aligned} \tag{31}$$

since $o_A o^A = 0$. This gives us,

$$\begin{aligned}\phi(0) &= -\frac{1}{2\pi}\int r\sin(\theta)\left(\frac{\dot m(u)}{r} - (\frac{1}{2} + rf(u)\cos(\theta))(-\frac{m(u)}{r^2}) - \frac{1}{2}\frac{m(u)}{r^2}\right)d\theta d\phi \\ &= -2\dot m(u) \qquad (32)\end{aligned}$$

An interesting application is where the source for the scalar field is an internal oscillator with configuration variable q, such that $m = \epsilon q$. Ie, the Lagrangian is

$$\int\sqrt{(-g)}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}d^4x + \int\left(\frac{1}{2}(q_{,u}^2 - \Omega^2 q^2) + \epsilon q\phi(x(u))\right)du \qquad (33)$$

The equation of motion for the oscillator, including the effect of the radiation reaction field, is now

$$-q_{,u,u} - \Omega^2 q + \epsilon(\phi_0(x(u)) - 2\epsilon q_{,u}) = 0 \qquad (34)$$

where ϕ_0 is the value of the background field at the location of the particle. Ie, the radiation reaction field acts as a simple damping term to the internal harmonic oscillator, with damping coefficient $2\epsilon^2$. In another paper I will use this to investigate the emission of radiation from an accelerating detector in interaction with the scalar field.

III. ELECTROMAGNETIC RADIATION REACTION FIELDS

To find the radiation reaction fields for the electromagnetic field, we must first solve the equations for the electromagnetic fields from a point dipole. First define the vector tangent to the path of the particle parallel transported over the null cone.

$$T^\mu = l^\mu + \frac{1}{2}n^\mu \qquad (35)$$

In addition, define the vectors

$$Z^\mu = (-l^\mu + \frac{1}{2}n^\mu)\cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta)(m^\mu + \bar m^\mu) \qquad (36)$$

$$X^\mu = \left((-l^\mu + \frac{1}{2}n^\mu)sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)(m^\mu + \bar m^\mu)\right)\cos(\phi) + \frac{i}{\sqrt{2}}(m^\mu - \bar m^\mu)\sin(\phi) \qquad (37)$$

$$Y^\mu = \left((-l^\mu + \frac{1}{2}n^\mu)\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)(m^\mu + \bar m^\mu)\right)\sin(\phi) - \frac{i}{\sqrt{2}}(m^\mu - \bar m^\mu)\cos(\phi) \qquad (38)$$

which are all vectors which are parallel over the whole of the surface of the cone u constant (but are not parallel off that cone.) For future needs, let me define the basis vectors $e^{(i)\mu}$ such that

$$e^{(0)\mu} = T^\mu \qquad (39)$$

$$e^{(1)\mu} = X^\mu \qquad (40)$$

$$e^{(2)\mu} = Y^\mu \qquad (41)$$

$$e^{(3)\mu} = Z^\mu \qquad (42)$$

Define the dipole moment

$$\mathcal{D}^\mu(u) = d_x(u)X^\mu + d_y(u)Y^\mu + d_z(u)Z^\mu = d_{(i)}e^{(i)\mu} \tag{43}$$

as the dipole moment vector, with $d_{(0)}$ zero. Furthermore, define

$$S^{\mu\nu} = \frac{1}{r}(T^\mu\mathcal{D}^\nu - T^\nu\mathcal{D}^\mu) \tag{44}$$

Then the vector potential for the electric dipole moment $\mathcal{D}$ is

$$A^\mu(u, r, \theta, \phi) = \nabla_\nu S^{\mu\nu} \tag{45}$$

with electromagnetic field

$$F_{\mu\nu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu \tag{46}$$

Also define the tensor

$$\mathcal{A}^{\mu\nu} = a_{(ij)}e^{(i)\mu}e^{(j)\nu} \tag{47}$$

where $a_{(ij)}$ is antisymmetric in ij. Then the Penrose integral equation becomes

$$\begin{aligned}\mathcal{A}^{\mu\nu}F_{RR\mu\nu} = &-\frac{1}{2}\frac{1}{2\pi}\left[\int r\sin(\theta)\mathcal{A}_{AB}\iota^A\iota^B o_C o_D \left(o^X o^{Y'}\nabla_{XY'}F_{CD}\right.\right.\\ &\left.\left.-3F_{CD}o^W\iota^X o^{X'}\nabla_{XX'}o_W\right)d\theta d\phi\right] + ComplexConjugate\\ = &-\frac{1}{4\pi}\int r\sin(\theta)\mathcal{A}_{\mu\nu}\bar{m}^\mu n^\nu l^\rho m^\sigma(l^\tau\nabla_\tau F_{\rho\sigma} - 3F_{\rho\sigma}m^\alpha\bar{m}^\beta\nabla_\beta l_\alpha)d\theta d\phi + CC\end{aligned} \tag{48}$$

The first factor of half arises from the averaging over the future and past null cones (the contribution from the past null cone being zero).

This integral, though very messy, can be evaluated, and gives the radiation reaction field. After extensive calculation, aided in an essential way with the GRTensorII computer algebra system [7], the result for the E and B fields are

$$E^{(i)}_{RR} = -\frac{2}{3}\frac{D^3_{FW}\mathcal{D}^{(i)}}{Du^3} - \frac{D_{FW}\mathcal{D}^{(i)}}{Du}f^2(u) + \mathcal{D}^{(i)}f(u)\dot{f}(u) \tag{49}$$

$$B^{(i)}_{RR} = \frac{1}{3}\left(\frac{D^2_{FW}a_{(j)}}{Du^2}\mathcal{D}_{(k)} - 2\frac{D_{FW}a_{(j)}}{Du}\frac{D_{FW}\mathcal{D}_{(k)}}{Du}\right)\epsilon^{(i)(j)(k)} \tag{50}$$

where $\frac{D_{FW}}{Du}$ is the Fermi Walker derivative of the quantity along the path of the particle

$$\frac{D_{FW}S^\mu}{Du} = T^\nu\nabla_\nu S^\alpha(\delta^\mu_\alpha - T^\mu T_\alpha). \tag{51}$$

and

$$a^{(i)} = e^{(i)}_\mu T^\nu\nabla_\nu T^\mu \tag{52}$$

are the components of the acceleration in the XYZ frame.

We can find the radiation reaction fields for the case for a magnetic dipole moment, $\mathcal{M}^\mu$ by simply taking the dual of all of the above equations, for which we get

$$B^{(i)}_{RR} = -\frac{2}{3}\frac{D^3_{FW}\mathcal{M}^{(i)}}{Du^3} - \frac{D_{FW}\mathcal{M}^{(i)}}{Du} f^2(u) + \mathcal{M}^{(i)} f(u)\dot{f}(u) \tag{53}$$

$$E^{(i)}_{RR} = -\frac{1}{3}\left(\frac{D^2_{FW}a_{(j)}}{Du^2}\mathcal{M}_{(k)} - 2\frac{D_{FW}a_{(j)}}{Du}\frac{D_{FW}\mathcal{M}_{(k)}}{Du}\right)\epsilon^{(i)(j)(k)} \tag{54}$$

We note that if the acceleration is not equal to zero, and the dipole moment is periodic in time, the acceleration increases the radiation reaction field, since $D^3_{FW}\frac{M^{(i)}}{Du^3} \approx -Omega^2\frac{D_{FW}M^{(i)}}{Du}$, unlike for the scalar field where the radiation reaction field was independent of the acceleration.

One of the most interesting features of the radiation reaction field is that an accelerated magnetic dipole will have an electric component to its radiation reaction field. Thus, if the point dipole also has a non-zero charge, then the accelerated dipole will exert a force on that accelerated charge in addition to the normal radiation reaction force from the charge itself. On the other hand, since the radiation reaction field for an accelerated charge has no magnetic component in the rest frame of that charge, the accelerated charge will not alter the motion of the magnetic dipole moment.

If the magnetic dipole moment is proportional to the angular momentum, via the magnetic moment, μ of the point charge distribution, then the equation of motion for the dipole moment under constant magnitude of acceleration is

$$\frac{D_{FW}\mathbf{M}}{Du} = \mu\mathbf{M}\times\mathbf{B} = \mu\mathbf{M}\times(\mathbf{B_0}+\mathbf{B_{RR}}) \tag{55}$$

This solution is subject to runaway behaviour just as is the accelerated charge. In this case the runaway behaviour is in the direction in which the angular momentum points, the angular momentum itself of course being conserved by the above equation. Choosing coordinates $\theta\phi$ such that the B_0 points in the direction of $\theta = 0$, and the magnetic moment vector points in the direction $\Theta(u)$, $\Phi(u)$, and choosing $g(u) = 0$, $F(u)$ constant, the equation of motion for Θ, Φ are

$$\Theta_{,u} = -\frac{2\mu^2}{3}\left(\Phi_{,uuu}\sin(\Theta(u)) + 3\left(\Phi_{,u}(\sin(\Theta(u)))_{,u}\right)_{,u} - \sin(\Theta)(\Phi^3_{,u} + f^2\Phi_{,u})\right) \tag{56}$$

$$\Phi(u)_{,u} = -\mu B_0 - \frac{2\mu^2}{3\sin(\Theta(u))}\left(\Theta_{,uuu} - 3\cos(\Theta(u))(\sin(\Theta(u))\Phi_{,u})_{,u} - (\Theta_{,u})^3 + f^2\Theta_{,u}\right) \tag{57}$$

Using the lowest order solution for $\Phi(u)$, namely $\Phi_{,u} = -\mu B_0$, and neglecting all but the lowest order terms in the equation for Θ, we get

$$\Theta_{,u} = \sin(\Theta)\frac{2\mu^2}{3}\left((\mu B_0)^2 + f^2)\mu B_0\right) \tag{58}$$

which has solution

$$\cos(\Theta) = -tanh\left(\frac{2\mu^2}{3}((\mu B_0)^2 + f^2)\mu B_0(t-t_0)\right) \tag{59}$$

However, retaining the higher order terms leads to runaway solutions, where in particular Φ diverges exponentially. Ie, as usual, the electromagnetic radiation reaction is useful only in providing the lowest order corrections to the solution, and cannot be taken seriously as a complete solution.

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[7] Peter Musgrave, Dennis Pollney, Kayll Lake, **GRTensorII** The GRTensorII package for Maple and Mathematica is available for free from http://astro.queensu.ca∼grtensor.