as the direction in which the field was applied during aging.

Figure 1 shows the maximum torque as a function of aging time for three temperatures. In every case, the torque resulting from field annealing rises slowly with aging time, long after the saturation has become constant. At 800°C, the torque rises to a maximum and falls again with increasing aging time in a field, a curious behavior that Miyahara and Mitui first found for longer times at a lower temperature.[1] Presumably the same behavior would have been observed after longer times at 700° and 650°C.

The maximum torque measured was about 10^5 d-cm per cm^3 of precipitate, corresponding to an anisotropy energy of at least 10^5 erg per cm^3 of precipitate.

Rotational hysteresis was measured at various fields. The distribution of particle anisotropies for a given aging treatment, that is, the way in which the rotational hysteresis varies with measuring field, is about the same whether or not a field was applied during aging. This means that the field does not *produce* the particle anisotropies but only causes them to line up. Also there were always a few particles present of remarkably high anisotropy. A tiny but definite fraction of the particles present after 30 min at 650°C had anisotropy fields in excess of 8000 oersteds.

SUMMARY OF EXPERIMENTAL RESULTS

The experimental results that were obtained are:

1. In an alloy of 2% cobalt in copper, solution treated and quenched to a nonmagnetic solid solution, then aged to precipitate the cobalt-rich phase, a magnetic field applied during aging produces a uniaxial magnetic anisotropy in the material, as shown by torque measurements.

2. The magnetic saturation reaches a constant value very early during aging, and is independent of whether a field was applied.

3. The effect of the field occurs only after the precipitation is complete, during resolution and growth of the particles.

4. The torque produced by the field reaches a maximum and decreases again with increasing aging times in the field.

5. The maximum effect produced by aging in a field corresponds to an anisotropy energy of at least 10^5 erg per cubic cm of precipitate.

6. The torque maximum occurs before the maximum in intrinsic coercive force, H_{CI}.

7. The remanence and intrinsic coercive force are greater in the annealing field direction.

8. The coercive force to reduce the remanence to zero, H_{CR}, is greater transverse to the annealing field direction than parallel to it.

9. The distribution of particle anisotropies present is not a function of whether a field was on during aging.

10. A very small amount of precipitate has been observed having rotational hysteresis corresponding to an anisotropy field of more than 8000 oersteds.

The remainder of the experimental results, and their interpretation in terms of the variation of magnetic properties with particle size, are discussed more fully in a forthcoming publication.[2]

[2] J. J. Becker, J. Metals (to be published).

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Ferromagnetic Resonance: Line Structures

Resonant Modes of Ferromagnetic Spheroids

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The spectrum of modes of oscillation of a ferromagnetic spheroid situated in a uniform dc magnetic field is discussed. It is shown that, for the size of sample usually used in microwave experiments, a part of the spectrum consists of magnetostatic modes for which exchange and propagation may be ignored. The behavior of these modes is discussed at some length and it is shown that the analysis gives a satisfactory description of the observed absorption curves.

A SATURATED ferromagnetic body in a uniform dc magnetic field possesses a spectrum of modes of free oscillation. Our knowledge of the form of this spectrum has become somewhat clarified in the last few years and this survey will summarize present views. Attention will be confined exclusively to spheroids whose axis of symmetry is along the dc field, since the analysis may be carried through fairly completely in this case. We are concerned, further, only with small oscillations about the lined-up equilibrium state and discuss no high level effects. Also, for our purposes, it is not necessary to include losses of any kind in treating the frequency spectrum.

In a classical ferromagnetic resonance experiment a spheroid is placed with its symmetry axis along a uniform dc magnetic field strong enough to produce saturation; an rf magnetic field is applied at right angles to the dc field and the absorption of rf power is studied as a function of dc magnetic field at constant frequency. Because the sample is saturated, all the dipoles participate equally in the absorption and the latter is, in consequence, intense. This has the effect of making it desirable to use small samples—of the order of a few mils in the microwave range. This, in turn, implies that unless some deliberate effort is made the applied rf magnetic field will be uniform across the sample. One may expect that in such an experiment the dipoles will precess in phase together about the demagnetized internal dc magnetic field, driven by the demagnetized rf field. When a suitable relation exists between the dc field and the frequency, resonance and an enhanced absorption of power will occur. This will give rise to a single, symmetrical peak in the absorbed power as a function of dc field, when losses are included. The connection between field and frequency was given by Kittel. For spheroids it may be written in the form

$$\Omega-\Omega_H=(1-N_z)/2. \tag{1}$$

Ω is a reduced angular frequency, where $\Omega=\omega/\gamma 4\pi M_0$, ω being the actual angular frequency, γ the gyromagnetic ratio of the spins, and $4\pi M_0$ the saturation magnetization. Ω_H is a reduced internal field given by $H_i/4\pi M_0$, where $H_i=H_0-4\pi M_0 N_z$ is the true internal field, H_0 being the applied dc magnetic field, and N_z the demagnetizing factor of the spheroid along the applied field (which is also to be the z axis). It is to be noted that this formula depends upon the shape and magnetization of the sample, but, not upon its size. Observation of a single peak at the Kittel field was the only type of behavior observed for several years in experiments on the ferromagnetic insulators.[1]

In 1956, White, Solt, and Mercereau[2] and Dillon[3] observed much more complicated absorption curves. Figure 1 shows an absorption curve for a disk of manganese ferrite taken by Dillon. There are a large number of peaks, quite comparable in intensity, the most intense in this case being at the appropriate Kittel field. The distinction between all these experiments and earlier ones was that the sample was now placed at a point in the cavity where the rf magnetic field was sufficiently inhomogeneous to vary even over the small sample. It was further found that by moving the sample to different parts of the cavity, the relative intensity of the absorptions could be varied and, indeed, series of peaks could be obtained from which the Kittel or uniform mode was absent (see Fig. 2).

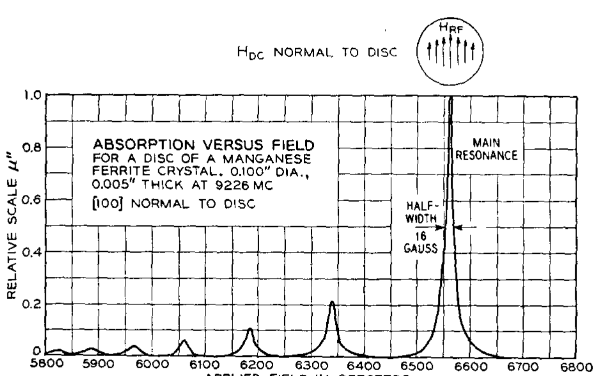

FIG. 1. Multiple absorption peaks in a disk of manganese ferrite. The rf field variation across the disk is indicated.

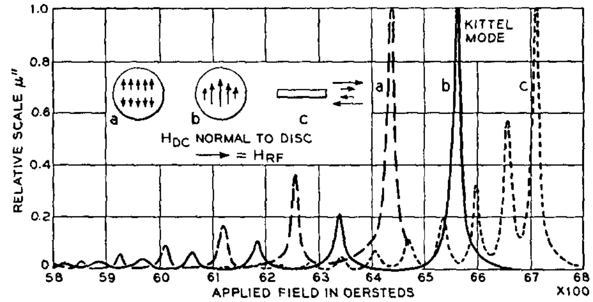

FIG. 2. Three series of absorption peaks in a manganese ferrite disk. The exciting rf field configuration of the different series is indicated. Note that the Kittel or uniform mode is a member of only one series. The maxima are normalized to unity.

The fact that an inhomogeneous rf exciting field was necessary to produce the new lines made it clear that dipoles in different parts of the sample must be precessing out of phase. The possibility of spatially varying modes was not at all a new idea. It had been remarked quite soon after the discovery of ferromagnetic resonance, that, at least in the ferrites with their dielectric constant of 10, some combinations of field and frequency might make the effective permeability of the material so high as to reduce the electromagnetic wavelength in the sample very markedly. This was expected to give rise to "dimensional" resonances and the desire to avoid such complications was an additional stimulus to the use of small samples. Further work on the new absorptions showed that they had a number of properties which would not ordinarily be associated with an electromagnetic or cavity resonance. The salient facts are that the positions of the peaks are substantially independent of sample size, provided that this is sufficiently small and that geometrically similar samples are compared. The positions, on the other hand, depend markedly upon sample shape. By a comparison of different materials and of the same material at different temperatures they depend also upon saturation magnetization. Very interesting series of lines can be found in spheres and in disks whose mutual separations are independent of size and of frequency and directly proportional to the magnetization. Such series may include the Kittel mode (see Fig. 2).

[1] Single peaks, but of a somewhat less simple shape, occur in metals.
[2] White, Solt, and Mercereau, Bull. Am. Phys. Soc. Ser. II, 1, 12 (1956); R. L. White and I. H. Solt, Phys. Rev. 104, 56 (1956).
[3] J. F. Dillon, Jr., Bull Am. Phys. Soc. Ser. II, 1, 125 (1956).

The qualitative explanation of these effects is readily found from a consideration of the resonant system involved. This consists of an array of dipoles which interact directly with a uniform dc magnetic field and mutually through exchange forces and through electromagnetic forces. It will be assumed that only hard and easy directions of magnetization are considered, for which the only effect of crystalline anisotropy should be to modify the internal dc magnetic field. Exchange interactions are of short range and, very roughly, if a substantial change takes place in the direction of the transverse magnetization over a distance of x cm, the exchange fields prevailing will be about $10^{-8}/x^2$ oe. If the plausible assumption is made that in the present case the distance x is comparable with the size of the sample the exchange fields will be only a few oersteds. But, while the samples ordinarily used are large enough for this to be true, they are sufficiently small for propagation to be ignored, except perhaps in certain narrow ranges of field and frequency. Thus, the electromagnetic forces are effectively magnetostatic and the resonances observed are just the free modes of oscillation of a spheroidal array of magnetic dipoles in a uniform field. The frequencies of such modes will indeed be size-independent since there is no longer a scale of length in the system, but, they will certainly be shape-dependent since they depend upon the dipolar configuration. They will be dependent upon the magnetization, since this in conjunction with the applied field determines the strength of the torques acting on the spins. Qualitatively, then, such modes of oscillation would resemble those observed. It is to be noted that from what was said about the Kittel formula the uniform mode of precession can be considered as a special instance of such a magnetostatic mode. It is also true that modes of this nature are not specific to ferromagnets. The condition for their existence are present in paramagnets and it has been pointed out by Geschwind that at sufficiently low temperatures, where the magnetization is sufficiently great for the separation of the lines to exceed their widths, they may be observable in such materials.

The formal problem of calculating the frequencies of these modes is straightforward for the case being considered here of spheroids aligned along the dc field.[4] Since the magnetization $\mathbf{M}$ is supposed to depart only slightly from line-up the equation of motion

$$d\mathbf{M}/dt=\gamma(\mathbf{M}\times\mathbf{H}) \tag{2}$$

may be linearized and solved to give the transverse rf magnetization, $\mathbf{m}$, in terms of the rf magnetic field, $\mathbf{h}$, in the form

$$\begin{aligned} 4\pi m_x &= \kappa h_x - i\nu hy, \\ 4\pi m_y &= i\nu hx + \kappa hy, \end{aligned} \tag{3}$$

(factors of $e^{i\omega t}$ are suppressed here and elsewhere). The quantities κ and ν are defined by

$$\kappa=\frac{\Omega_H}{\Omega_H{}^2-\Omega^2} \quad \text{and} \quad \nu=\frac{\Omega}{\Omega_H{}^2-\Omega^2},$$

where Ω and Ω_H have their earlier significance. Maxwell's equations in the absence of the propagation are just those of magnetostatics:

$$\begin{aligned} &\text{curl}\mathbf{h}=0, \\ &\text{div}\mathbf{b}=\text{div}(\mathbf{h}+4\pi\mathbf{m})=0. \end{aligned} \tag{4}$$

If a magnetic potential, Ψ, is introduced through $\mathbf{h}=\text{grad}\,\Psi$ it is found that Ψ satisfies the equation

$$(1+\kappa)\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)+\frac{\partial^2\Psi}{\partial z^2}=0 \tag{5}$$

within the spheroid and, of course, Laplace's equation without. It is interesting to note that (5) may be elliptic or hyperbolic according as $1+\kappa$ is positive or negative. Since the parameter ν is involved in the expression for $\mathbf{m}$ it re-enters the problem, when the usual matching conditions of magnetostatics are applied at the surface of the spheroid. One assumes that the spheroid is situated in empty space (it is supposed to be far from the cavity walls) and an eigenvalue problem arises when the fields are required to go to zero at large distances. What is actually found is a relation between Ω and Ω_H, either one of which can be considered as the eigenvalue for a given value of α, the aspect ratio of the spheroid. Since the problem is three dimensional the modes or eigenvalues may be expected to require three labels. For aligned spheroids (and it is this property which gives a manageable problem in contrast to all other cases) the boundaries have cylindrical symmetry, as the field equations plainly have. Solutions for Ψ now turn out to vary as $\exp jm\Phi$, where Φ is the angle measured about the z axis: no solutions of the form $\cos m\Phi$ or $\sin m\Phi$ are admissible. It appears that, as the resemblance of (5) to Laplace's equation would suggest, the solutions in the material are polynomials in x, y, and z. If the degree of the polynomial be n the characteristic equation of the problem is

$$\frac{d\log Q_n{}^m(i\xi_0)}{d\log i\xi_0}-\frac{d\log P_n{}^m(i\bar{\xi}_0)}{d\log i\bar{\xi}_0}=m\nu\alpha^2, \tag{6}$$

where

$$\xi_0{}^2=\alpha^2/(1-\alpha^2), \quad \bar{\xi}_0{}^2=\frac{(1+\kappa)\alpha^2}{1-(1+\kappa)\alpha^2},$$

and $P_n{}^m$, $Q_n{}^m$ are the associated Legendre functions. This equation has $1+\frac{1}{2}[n-|m|]$ solutions, where $[x]$ is the largest integer in x. These may be arranged in order of decreasing $\Omega-\Omega_H$ and labeled by the integer r, where $r+1$ is the order of the root in this sequence. The notation used to label the modes is to give the associated integers n, m, and r, in that order, writing (n,m,r).

[4] L. R. Walker, Phys. Rev. **105**, 390 (1957); T. E. Mercereau and R. P. Feynman, Phys. Rev. **104**, 63 (1956).

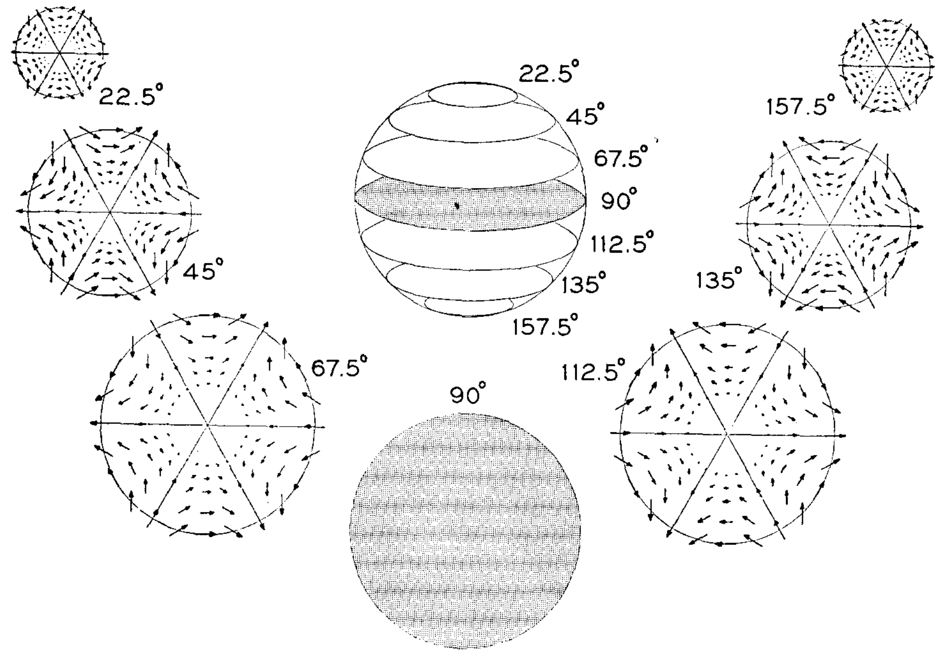

FIG. 3. The configuration of transverse magnetization seen in a rotating system for the (4,3,0) mode in a sphere. All the vectors precess together in this system. This mode is degenerate with the uniform mode in a sphere.

Since the field configurations involve the factor, $\exp jm\Phi$, they are circularly polarized in general and must rotate as a whole. With certain notable exceptions the field configuration of a given mode will be field and frequency dependent. The unusual cases are those in which $n=m$ or $n=m+1$. Here we have $\Psi\sim(x+jy)^m$ and $\Psi=z(x+jy)^m$, respectively, under all circumstances. Some idea of the field configurations may be obtained by going into the suitable rotating system and then showing the instantaneous positions of the transverse magnetization vectors, which must then precess with the same angular velocity. Figure 3 shows such a diagram for the (4,3,0) mode in a sphere. This is of the fixed-pattern type.

The mode spectrum can be discussed quite fully from the characteristic equation without numerical solution. Two general aspects may be commented upon here. One is the effect of field or frequency on a given mode at fixed axial ratio. The other is the effect of axial ratio at fixed field. The two are fortunately separable. For the first case it is convenient to plot $\Omega-\Omega_H$ as a function of Ω_H, the reduced internal field and this is shown in Fig. 4 for the case of the sphere. The qualitative features are independent of axial ratio. The modes with the simple patterns $(m,m,0)$ and $(m+1, m, 0)$ are seen to have $\Omega-\Omega_H$ independent of Ω_H. The Kittel mode in the present notation is (1,1,0); it belongs to this class of "simple" modes and, in fact, from Eq. (1) $\Omega-\Omega_H$ has the constant value, $(1-N_z)/2$. The analysis, therefore, predicts a number of modes, including the uniform one, whose mutual separations are independent of field and frequency and proportional directly to $4\pi M_0$. As mentioned earlier such series are found in practice. For the sphere, the characteristic equation allows of an explicit solution for these series:

$$\begin{aligned}\Omega-\Omega_H&=m/2m+1 \quad & n=m\\ &=m/2m+3 \quad & n=m+1.\end{aligned} \tag{7}$$

This implies that the modes $(m,m,0)$ and $(3m+1, 3m, 0)$ are permanently degenerate in the sphere [A particular example is (4,3,0) shown in Fig. 3 and the uniform mode, (1,1,0)]. This type of degeneracy is confined to the sphere.

For modes other than these simple ones it can be shown that $\Omega-\Omega_H$ increases steadily with Ω_H to a finite limit in high fields. Accidental degeneracies may occur at specific fields and frequencies, either by these modes crossing the simple ones with fixed $\Omega-\Omega_H$ or by crossing one another. A significant result which may be proved is that for all modes and for all axial ratios, $\Omega-\Omega_H$ lies between 0 and $\frac{1}{2}$, so that the complete magnetostatic spectrum lies in a frequency band between γH_i and $\gamma(H_i+2\pi M_0)$.

The effect of axial ratio is indicated in Fig. 5 which shows the course of $\Omega-\Omega_H$ for several of the simple modes. The modes are crowded together at $\Omega-\Omega_H=0$ for disks; it may be noted that the $(m,m,0)$ series which is even in z approaches $\Omega-\Omega_H=0$ differently from the $(m+1, m, 0)$ series which is odd in z. The modes become dispersed as α increases to 1 and then crowd together again at $\Omega-\Omega_H=\frac{1}{2}$ for needles. The crowding arises from the one-dimensional nature of the system in the limiting cases. One may show generally that for any mode at any fixed Ω_H, the quantity $\Omega-\Omega_H$ climbs monotonically from 0 to $\frac{1}{2}$ as α increases from zero to infinity.

It is now of some interest to fit the magnetostatic modes into the general frequency spectrum of the

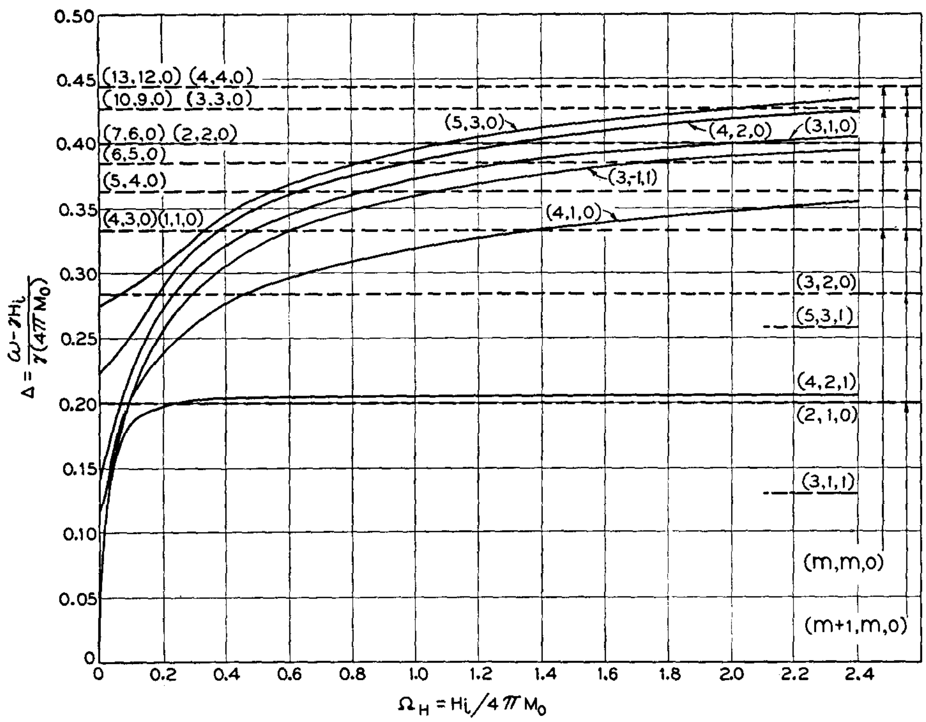

FIG. 4. The course of some of the modes in a sphere as a function of reduced internal field.

spheroid. Clearly for disturbances with a sufficiently rapid spatial variation exchange will become important. The exchange fields will be comparable with the magnetostatic ones when the period is about 10^{-5} cm. If the disturbances have a periodicity short compared to the sample size, the boundary shape or curvature cannot much affect the character of the modes. It is sufficient to assume that they are plane waves described by a wave vector, $\mathbf{k}$, for which the spectrum is described by a dispersion relation, $\omega=\omega(\mathbf{k})$. It is well known that for short wavelengths where exchange is dominant, this dispersion relation has the simple form, $\omega\sim c|\mathbf{k}|^2$, where c is a constant. As was pointed out by Anderson and Suhl,[5] the shape of the spheroid still plays a role by determining the rf demagnetizing fields, which, when they become comparable with the exchange, resolve the frequencies of plane waves of given $|\mathbf{k}|$, making different angles θ between $\mathbf{k}$ and the dc field, into a band. The complete dispersion relation expressed in the present notation is then

$$\Omega^2=\left(\Omega_H+\frac{|\mathbf{k}|^2}{k_0^2}\right)\left(\Omega_H+\sin^2\theta+\frac{|\mathbf{k}|^2}{k_0^2}\right), \qquad (8)$$

where $k_0^2=4\pi M_0/H_e a^2$, a is the lattice spacing and H_e is an interatomic exchange field of the order of 10^6 or 10^7 oe. For large $|\mathbf{k}|$, exchange is dominant, $\Omega\sim|\mathbf{k}|^2/k_0^2$ and the band becomes very narrow. As $|\mathbf{k}|$ decreases, the relative importance of the demagnetizing terms increases and, eventually, when $|\mathbf{k}|^2\ll k_0^2$ and exchange is negligible, the width of the band, having steadily increased, approaches a definite limit which depends upon Ω_H alone. If the sample is sufficiently large this regime will be entered for $|\mathbf{k}|$ values which are still large enough for the boundary effects to be insignificant. Finally, when $|\mathbf{k}|$ becomes so small that the wavelength of the plane waves is perhaps one-tenth of the sample dimensions, the exact boundary value problem must be solved without exchange and one is in the region of magnetostatic modes. The situation is illustrated in Fig. 6.

It is evident that the magnetostatic modes should fuse with the plane waves in the low $-|\mathbf{k}|$ region where the band is of fixed width. This may be demonstrated in the following manner. If the characteristic equation is examined for large n and m, m/n remaining finite, one may show that its solutions are of the form:

$$\Omega^2=\Omega_H^2+\Omega_H\frac{\alpha^2(1-z_r^2)}{\alpha^2+(1-\alpha^2)z_r^2},$$

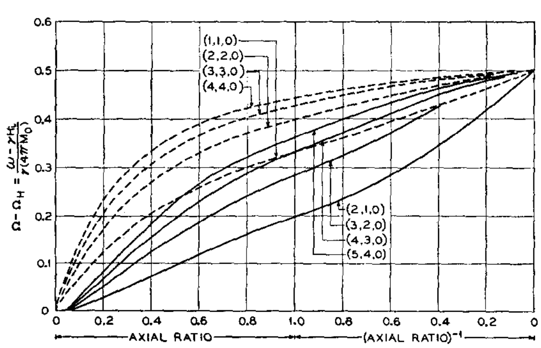

FIG. 5. The course of some of the simple modes as a function of axial ratio.

[5] P. W. Anderson and H. Suhl, Phys. Rev. 100, 1788 (1955).

where $P_n{}^m(z_r)=0$. In the exchange-free region, Eq. (8) becomes

$$\Omega^2=\Omega_H{}^2+\Omega_H \sin^2\theta.$$

Thus, each magnetostatic mode with large m and n, m/n finite, depends upon field and frequency in the same way as a plane wave with θ given by

$$\sin^2\theta=\frac{\alpha^2(1-z_r{}^2)}{\alpha^2+(1-\alpha^2)z_r{}^2}.$$

By a W.K.B. treatment of the differential equation for $P_n{}^m(z)$ one may find the density of its zeros for large m and n. Integrating this over all m and n one has an expression for the density of magnetostatic modes corresponding to plane waves in a definite angular interval, θ to $\theta+d\theta$. This may be shown to be actually the same as the density of plane waves of all $|\mathbf{k}|$ in the same angular range. It was mentioned that the width of the plane wave band depended upon Ω_H. As might be expected the band always falls within the limits of $\Omega-\Omega_H=0$ and $\frac{1}{2}$ which circumscribed the magnetostatic modes. Naturally, at a given field magnetostatic modes of low order may fall outside the plane wave band; in fact, the uniform mode does so for $\Omega_H<(1-N_z)^2/4N_z$. As m and n become large the bulk of the magnetostatic modes however will fall within the plane wave band.

It is now clear that the magnetostatic modes provide the appropriate extension of the spectrum into the region where the periodicity of the disturbances is comparable with sample size. In problems where this portion of the spectrum is excited they give the correct description of affairs. It is possible to prove orthogonality relations for them and to give a formal expansion theorem allowing arbitrary distributions of transverse magnetization to be expressed in terms of them. Only fields with curl $h=0$ can be so expressed, but, they are the only ones involved in this regime. In practice, one finds that the mode fields are, of course, considerably more awkward to work with than plane waves.

As has been shown the qualitative predictions of the analysis are in agreement with the facts. Quantitative comparison is also very satisfactory. The series of simple modes excited in Dillon's experiments (Fig. 2) can be identified by the symmetry of the exciting of field and the field separations compared with those

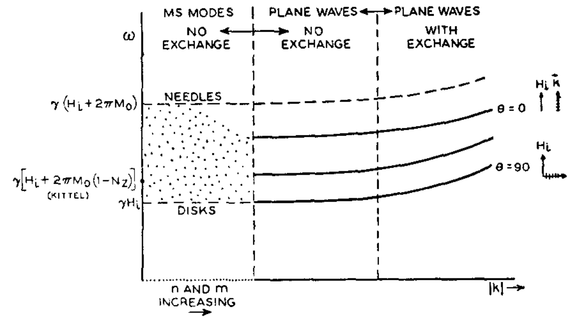

FIG. 6. A schematic picture of the complete resonance spectrum of a spheroid. The k scale is broken off where the plane wave approximation becomes invalid.

predicted. Agreement is good to a few percent for all lines provided that one assumes an aspect ratio for the spheroid treated by the theory substantially greater than that of the disks actually used. P. C. Fletcher[6] has recently made extensive measurements on modes in spheres where there is less ambiguity and has found agreement with the analysis to about two or three oersteds for the low-order simple modes falling off to 20 oe for some of the higher nonsimple modes. He has also shown that the relative intensities are in good agreement with those calculated assuming a driving rf field given by the undisturbed cavity field at the sample.

It may be expected that the magnetostatic modes will furnish some information relevant to the discussion of line width in ferromagnetic resonance. All observations seem to show that the line width of the different modes is essentially the same when certain obvious corrections have been included and this must be accounted for by any adequate theory. It has been suggested that the uniform mode of precession (and the argument would extend to the other magnetostatic modes) relaxes first to other spin-wave disturbances with the same frequency.[7] It is possible, but not inevitable, that if this were the case, the line width of any particular mode would show an anomaly at any field and frequency at which it was degenerate with another magnetostatic mode. So far no evidence has been found to support this belief, but no thorough investigation has been made.

[6] Unpublished. I am indebted to Dr. Fletcher and the Hughes Research Laboratories for making these results available.

[7] Clogston, Suhl, Walker, and Anderson, J. Phys. Chem. Solids. 1, 129 (1956).