Magnetostatic Modes in Ferromagnetic Resonance

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It has been found recently that in ferromagnetic resonance experiments performed in inhomogeneous rf exciting fields at a fixed frequency, absorption of power takes place at a number of distinct magnetic fields. This is ascribed to the existence of long-wavelength modes of oscillation of the ferromagnetic sample. The mode spectrum of spheroids is examined for the case, which may often hold in practice, where exchange and electromagnetic propagation can be ignored simultaneously.

INTRODUCTION

RECENT experiments on ferromagnetic resonance in ferrites[1,2] have shown that under suitable conditions the power absorption at a fixed frequency may pass through a number of maxima as the dc magnetic field is varied. A necessary condition for the excitation of these multiple absorptions is that the rf magnetic exciting field at the sample be inhomogeneous. It seems to be characteristic of the extra peaks that their field separation from the familiar one obeying the Kittel formula, usually excited by uniform fields, is substantially size-independent for sufficiently small samples. At the same time the separations depend markedly upon the saturation magnetization of the ferrite and have a temperature variation which may be ascribed to this effect. It is consistent with the conditions of excitation to suppose that these new absorptions are associated with modes of oscillation of the assembly of spins in which the phase varies throughout the sample.[1,3] Since in the experiments the rf exciting fields do not vary very rapidly over the sample the wavelength of these modes must be comparable with the dimensions of the body.

The ellipsoids which are used in these resonance experiments generally have no dimension less than a few mils, so that it is likely that the shortest wavelengths involved here will be of this order. This implies that the exchange and pseudo-dipolar forces can be ignored; the effective exchange field for a wavelength λ (cm) is about $10^{-8}/\lambda^2$ oersteds, so that λ must be about 10^{-5} cm before exchange becomes comparable with the usual applied magnetic fields of a few kilogauss. The pseudo-dipolar term will be substantially smaller than that due to exchange. Electromagnetic propagation within the ferrite will always be significant when the frequency and applied field are such as to make the effective permeability of the medium very large. The range of such frequencies and fields will, however, be the narrower the smaller the sample, since propagation is important when the dimensions of the body are comparable with the wavelength within it. For ellipsoids whose dimensions are of the order of mils one may expect to find propagation unimportant in a wide range of experimental conditions. For such samples, large enough to ignore exchange and small enough to neglect propagation, except in critical regions of field and frequency, there should be modes which are essentially size-independent. For now the forces are purely magnetostatic, each spin moves in the external dc magnetic field and in the resultant dipolar field of the other spins; since there is no characteristic length the size-independence follows. The usual uniform precession of the spins is clearly a mode of this type and it might be expected that the Kittel formula with its dependence upon ac and dc demagnetizing factors would be characteristic.

We shall consider here only this relatively simple problem in which propagation is set aside. The results may always be examined to see whether they are consistent with this assumption in any specific case. The boundary value problem for determining the frequencies of the modes is readily formulated. It is assumed that the spins deviate only slightly from the direction of the applied dc magnetic field and the equations of motion may then be linearized. Their solution yields a relation between the transverse components of magnetization and those of the rf magnetic field, with all rf quantities assumed to vary with time as $e^{i\omega t}$, where ω is the angular frequency. Thus, a connection is given between rf **B** and rf **H** in the medium. With the neglect of propagation Maxwell's equations reduce to those of magnetostatics:

$$\mathrm{div}\mathbf{B}=0, \quad \mathrm{curl}\mathbf{H}=0.$$

Since the sample is always small compared to the cavity in which it is placed, it is reasonable to find the frequencies of the modes of the sample when it is situated in empty space. Then **B** and **H** must satisfy the usual magnetostatic boundary conditions at the surface of the body and tend to zero at large distances from it. Imposition of these boundary conditions leads to a characteristic equation for the mode frequencies.

It is of some interest to consider the relation of these modes to the complete spectrum of the system. In the main this will consist of short-wavelength disturbances for which exchange is important. As has already been pointed out, when exchange is significant the wave-

[1] White, Solt, and Mercereau, Bull. Am. Phys. Soc. Ser. II, **1**, 12 (1956); R. L. White and I. H. Solt, Phys. Rev. **104**, 56 (1956).
[2] J. F. Dillon, Jr., Bull. Am. Phys. Soc. Ser. II, **1**, 125 (1956).
[3] L. R. Walker, Bull. Am. Phys. Soc. Ser. II, **1**, 125 (1956); J. E. Mercereau and R. P. Feynman, Phys. Rev. **104**, 63 (1956).

length is very small compared to the samples in which we are interested and it is not essential to fit boundary conditions at the surface. One may take the disturbances to be plane waves for which a dispersion relation is readily obtained. It has been pointed out by Anderson and Suhl[4] that, for samples of finite size, the dispersion relation is shape-dependent because of the effects of demagnetizing fields. For a spheroid the dispersion relation is in fact, of the form

$$\omega(\mathbf{k})=\gamma[(H_0-4\pi N_zM_0+H_ea^2k^2)(H_0-4\pi N_zM_0+H_ea^2k^2+4\pi M_0\sin^2\theta)]^{\frac{1}{2}}, \quad (1)$$

where $\mathbf{k}$=wave number of the spin wave, γ=gyromagnetic ratio, H_0=applied dc magnetic field, M_0=saturation magnetization, N_z=demagnetizing factor along the applied field, H_e=an exchange field, a=lattice spacing, and θ=angle between $\mathbf{k}$ and the applied field $k^2=\mathbf{k}\cdot\mathbf{k}$, whereas the Kittel formula for the mode of uniform precession is

$$\omega=\gamma[H_0-\tfrac{1}{2}(3N_z-1)4\pi M_0]. \quad (2)$$

The formula (1), based upon the plane wave assumption, will be applicable until the wavelength is perhaps one-tenth the size of the spheroid, beyond which the effect of the boundaries becomes significant. Since the exchange term in (1) will by that time have become quite negligible (again assuming the spheroid dimensions to be of the order of a few mils), $\omega(\mathbf{k})$ will have assumed an essentially constant value. This value is obtained by setting $k^2=0$ in (1) and depends upon θ. The spectrum will now be completed by the magnetostatic or long wavelength modes which we are discussing, whose location may be expected to be closely related to the $k=0$ limits of (1). One of the significant consequences of the shape-dependent dispersion relation (1) is a recognition of the fact that the frequency of the uniform mode may be degenerate with that of a number of spin waves of moderate k number, roughly $k\sim(4\pi M_0/H_ea^2)^{\frac{1}{2}}$. This is significant for the absorption line width since it provides a possibility for the transfer of power from the uniform mode to these higher k numbers.[5] An important question then to be examined about the long wavelength modes is whether or not they can provide further degeneracies with the uniform mode. The answer appears to be that they do.

CHARACTERISTIC EQUATION

The sample will be assumed a spheroid of arbitrary axial ratio whose axis of symmetry lies along the applied dc magnetic field. The ratio of the longitudinal axis, b, to the transverse axis, a, is denoted by α. The internal, demagnetized, dc magnetic field in the spheroid, $H_0-4\pi M_0N_z$, will be called H_i. The magnetization is written as

$$\mathbf{M}=M_0\mathbf{1}_z+\mathbf{m}e^{i\omega t},$$

where $\mathbf{1}_z$ is a unit vector in the z direction (that of the applied field) and $\mathbf{m}$, which is small compared to M_0, lies in the $x-y$ plane. Similarly the magnetic field is

$$\mathbf{H}=H_i\mathbf{1}_z+\mathbf{h}e^{i\omega t},$$

where $\mathbf{h}$ may also have a z component. The equation of motion is taken to be the Landau-Lifshitz equation without loss

$$d\mathbf{M}/dt=\gamma(\mathbf{M}\times\mathbf{H}).$$

In the linear approximation this becomes

$$i\omega\mathbf{m}=\gamma[\mathbf{1}_z\times(M_0\mathbf{h}-H_i\mathbf{m})]. \quad (3)$$

If it is assumed that the sample is a single crystal with the applied field along an easy or a hard direction, the effect of crystalline anisotropy will, in the linear approximation, be to modify the applied field by an additive anisotropy field; this will be absorbed then into H_0. The quantities, $\mathbf{h}$ and $\mathbf{m}$, must satisfy the equations

$$\text{curl}\mathbf{h}=0, \quad (4a)$$

$$\text{div}(\mathbf{h}+4\pi\mathbf{m})=0, \quad (4b)$$

when propagation is ignored. According to (4a) a magnetic potential, ψ, may be introduced such that $\mathbf{h}=\text{grad}\psi$, and (4b) then becomes

$$\nabla^2\psi+4\pi\,\text{div}\mathbf{m}=0. \quad (5)$$

Equation (3) may be written in component form as

$$\begin{aligned}i\omega m_x&=\gamma(H_im_y-M_0h_y),\\ i\omega m_y&=\gamma(-H_im_x+M_0h_x),\end{aligned} \quad (6)$$

from which the components of magnetization may be found to be

$$\begin{aligned}4\pi m_x&=\kappa\frac{\partial\psi}{\partial x}-i\nu\frac{\partial\psi}{\partial y},\\ 4\pi m_y&=i\nu\frac{\partial\psi}{\partial x}+\kappa\frac{\partial\psi}{\partial y},\end{aligned} \quad (7)$$

where

$$\kappa=\Omega_H/(\Omega_H^2-\Omega^2), \quad \nu=\Omega/(\Omega_H^2-\Omega^2), \quad (8a)$$

with

$$\Omega=\omega/4\pi\gamma M_0, \quad \Omega_H=H_i/4\pi M_0. \quad (8b)$$

With these expressions for m_x and m_y, the Eq. (5) for ψ inside the spheroid becomes

$$(1+\kappa)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi+\frac{\partial^2\psi}{\partial z^2}=0. \quad (9)$$

Outside the spheroid

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi=0. \quad (10)$$

[4] P. W. Anderson and H. Suhl, Phys. Rev. **100**, 1788 (1955); H. Suhl, Proc. Inst. Radio Engrs. (to be published).

[5] Clogston, Suhl, Walker, and Anderson, Intern. J. Chem. Phys. Solids (to be published).

It is worth noting that $1+\kappa$ may, in general, have either sign, so that (9) may be of elliptic or of hyperbolic type. The boundary conditions on ψ at the surface of the spheroid are the continuity of ψ and of the normal component of $\mathbf{h}+4\pi\mathbf{m}$. Further, $\psi\to 0$ at infinity.

We introduce oblate spheroidal coordinates ξ, η, Φ, defined by

$$\begin{aligned} x &= (a^2-b^2)^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\cos\Phi, \\ y &= (a^2-b^2)^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\Phi, \\ z &= (a^2-b^2)^{\frac{1}{2}}\xi\eta, \end{aligned} \tag{11}$$

in terms of which, the surface of the given spheroid,

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1,$$

is given by

$$\xi=\xi_0, \quad \text{with} \quad \xi_0^2=\alpha^2/(1-\alpha^2).$$

Rather than define a separate set of prolate spheroidal coordinates for the case $\alpha>1$, we shall assume that ξ^2 in (11) may be negative. For prolate spheroids, ξ^2 will lie in the range $(-\infty, -1)$; for oblate spheroids, ξ^2 is in the range $(0,\infty)$. Acceptable solutions ψ_e, of Laplace's equation, bounded at infinity, are now of the form

$$\psi_e=Q_n{}^m(i\xi)P_n{}^m(\eta)e^{im\Phi}, \tag{12}$$

where $P_n{}^m$ and $Q_n{}^m$ are the associated Legendre functions of the first and second kind. It will be understood that when m is used as a suffix it indicates $|m|$. The form (12) is satisfactory for both oblate and prolate cases. On the surface, $\xi=\xi_0$, (12) reduces to $AP_n{}^m(\eta)e^{im\Phi}$, where A is a constant. A solution of (9), regular in the interior, is now required which reduces to a similar form on $\xi=\xi_0$. If ξ', η', Φ are some set of spheroidal coordinates, the expression $P_n{}^m(i\xi')P_n{}^m(\eta')e^{im\Phi}$ is a solution of Laplace's equation, which for integral n and m and all ξ', η', satisfies the identity[6]

$$P_n{}^m(i\xi')P_n{}^m(\eta')e^{im\Phi} = A_{n,m}e^{im\Phi}\int_{-\pi}^{+\pi} P_n[i\xi'\eta'+(1+\xi'^2)^{\frac{1}{2}}(1-\eta'^2)^{\frac{1}{2}}\cos\alpha] \times\cos m\alpha d\alpha, \tag{13}$$

where $A_{n,m}$ is a numerical constant. If c' is the constant which replaces $(a^2-b^2)^{\frac{1}{2}}$ in the equations analogous to (11) defining the ξ', η', Φ, the right-hand side of (13) is

$$A_{n,m}e^{im\Phi}\int_{-\pi}^{+\pi} P_n\left(\frac{iz+(x^2+y^2)^{\frac{1}{2}}\cos\alpha}{c'}\right)\cos m\alpha d\alpha.$$

Since (9) becomes Laplace's equation if z is replaced by $(1+\kappa)^{\frac{1}{2}}z$, it follows that

$$A_{n,m}e^{im\Phi}\int_{-\pi}^{\pi} P_n\left(\frac{i(1+\kappa)^{\frac{1}{2}}z+(x^2+y^2)^{\frac{1}{2}}\cos\alpha}{c'}\right) \times\cos m\alpha d\alpha \tag{14}$$

is a solution of (9), which, since it is a polynomial in x, y, and z, must be regular. On the surface, $\xi=\xi_0$, we have $x^2+y^2=a^2(1-\eta^2)$ and $z=b\eta$ and if c' is now chosen to satisfy $c'^2=a^2-(1+\kappa)b^2$, then (14) is just of the form of (13) with $\eta'=\eta$ and

$$\xi'=\bar{\xi}_0=\left(\frac{(1+\kappa)^{\frac{1}{2}}b}{[a^2-(1+\kappa)b^2]^{\frac{1}{2}}}\right)=\left(\frac{(1+\kappa)^{\frac{1}{2}}\alpha}{[1-(1+\kappa)\alpha^2]^{\frac{1}{2}}}\right).$$

Equation (14) thus reduces to

$$P_n{}^m(i\bar{\xi}_0)P_n{}^m(\eta)e^{im\Phi}$$

on the surface and is of the required form. An acceptable interior solution, ψ^i, matched to (12) on $\xi=\xi_0$ is, therefore,

$$\psi^i=A_{n,m}\frac{Q_n{}^m(i\xi_0)}{P_n{}^m(i\bar{\xi}_0)}e^{im\Phi} \times\int_{-\pi}^{+\pi} P_n\left(\frac{i(1+\kappa)^{\frac{1}{2}}z+(x^2+y^2)^{\frac{1}{2}}\cos\alpha}{[a^2-(1+\kappa)b^2]^{\frac{1}{2}}}\right) \times\cos m\alpha d\alpha. \tag{15}$$

It remains to satisfy the condition on the normal component of $\mathbf{h}+4\pi\mathbf{m}$ at the surface. Externally,

$$(\mathbf{h}+4\pi\mathbf{m})_{\text{normal}} =\text{grad}_\xi\psi^e=\frac{1}{(a^2-b^2)^{\frac{1}{2}}}\left(\frac{1+\xi_0^2}{\eta^2+\xi_0^2}\right)^{\frac{1}{2}}\left(\frac{\partial\psi^e}{\partial\xi}\right)_{\xi_0} =\frac{a/b}{(a^2-b^2)^{\frac{1}{2}}(\eta^2+\xi_0^2)}[i\xi_0 Q_n{}^{m\prime}(i\xi_0)]P_n{}^m(\eta)e^{im\Phi}. \tag{16}$$

Internally, the Cartesian components of $\mathbf{h}+4\pi\mathbf{m}$ are found from (7) to be

$$(1+\kappa)\frac{\partial\psi^i}{\partial x}-i\nu\frac{\partial\psi^i}{\partial y},$$

$$(1+\kappa)\frac{\partial\psi^i}{\partial y}+i\nu\frac{\partial\psi^i}{\partial x}, \quad \text{and} \quad \frac{\partial\psi^i}{\partial z}.$$

The direction cosines of the normal are

$$\left(\frac{1-\eta^2}{a^2}+\frac{\eta^2}{b^2}\right)^{-\frac{1}{2}}\left[\frac{(1-\eta^2)^{\frac{1}{2}}}{a}\cos\Phi, \frac{(1-\eta^2)^{\frac{1}{2}}}{a}\sin\Phi, \frac{\eta}{b}\right].$$

The normal component of $\mathbf{h}+4\pi\mathbf{m}$ is now found from

[6] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, Cambridge, 1931).

(14) to be

$$(\mathbf{h}+4\pi\mathbf{m})_{\text{norm}} = A_{n,m}\frac{Q_n^m(i\xi_0)}{P_n^m(i\bar{\xi}_0)}\frac{abe^{im\varphi}}{(a^2-b^2)^{\frac{1}{2}}(\xi_0^2+\eta^2)^{\frac{1}{2}}} \times\left[\frac{m\nu}{a^2}\int_{-\pi}^{+\pi}P_n(i\bar{\xi}_0\eta+(1+\bar{\xi}_0^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\cos\alpha)\cos m\alpha d\alpha + \frac{(1+\kappa)^{\frac{1}{2}}}{b[a^2-(1+\kappa)b^2]^{\frac{1}{2}}}\int_{-\pi}^{+\pi}P_n'(i\bar{\xi}_0\eta+(1+\bar{\xi}_0^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\cos\alpha) \times\left(i\eta+(1+\kappa)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\frac{b}{a}\cos\alpha\right)\cos m\alpha d\alpha\right],$$

and making use of (13), this becomes finally

$$(\mathbf{h}+4\pi\mathbf{m})_{\text{norm}}=\frac{Q_n^m(i\xi_0)}{P_n^m(i\bar{\xi}_0)}\frac{(a/b)e^{im\varphi}P_n^m(\eta)}{(a^2-b^2)^{\frac{1}{2}}(\xi_0^2+\eta^2)} \times\left(m\nu\frac{b^2}{a^2}P_n^m(i\bar{\xi}_0)+i\bar{\xi}_0P_n^{m\prime}(i\bar{\xi}_0)\right). \quad (17)$$

From (16) and (17) the characteristic equation is obtained in the form

$$i\xi_0\frac{Q_n^{m\prime}(i\xi_0)}{Q_n^m(i\xi_0)}-i\bar{\xi}_0\frac{P_n^{m\prime}(i\bar{\xi}_0)}{P_n^m(i\bar{\xi}_0)}=m\nu\frac{b^2}{a^2}=m\nu\alpha^2. \quad (18)$$

SPECTRUM

The nature of the solutions of (18) is more easily examined if the latter is rearranged. The associated Legendre polynomials have only real zeros lying in $(-1, 1)$ and may then be written in the form[7]

$$P_n^m(z)=\text{constant}\times(1-z^2)^{\frac{1}{2}|m|}z^{0,1}\prod_{r=1}^{[\frac{1}{2}(n-|m|)]}(z^2-z_r^2)$$

taking 0 or 1 as $n-|m|$ is even or odd and $[\frac{1}{2}(n-|m|)]$ is the greatest integer in $\frac{1}{2}(n-|m|)$. We shall write, in future, p for $[\frac{1}{2}(n-|m|)]$. The z's are the zeros of $P_n^m(z)$, other than zero, and we set $1>z_1^2>z_2^2>\cdots>0$. It follows that

$$\frac{i\bar{\xi}_0P_n^{m\prime}(i\bar{\xi}_0)}{P_n^m(i\bar{\xi}_0)}=\frac{|m|\bar{\xi}_0^2}{1+\bar{\xi}_0^2}+(0,1)+2\sum_{r=1}^{p}\frac{\bar{\xi}_0^2}{\bar{\xi}_0^2+z_r^2}. \quad (19)$$

From the definitions of ξ_0 and $\bar{\xi}_0$, we have $\bar{\xi}_0^2=(1+\kappa)\xi_0^2/(1-\kappa\xi_0^2)=(1+\kappa)\alpha^2/[1-(1+\kappa)\alpha^2]$; substituting this in (19), one has

$$i\bar{\xi}_0\frac{P_n^{m\prime}(i\bar{\xi}_0)}{P_n^m(i\bar{\xi}_0)}=|m|(1+\kappa)\alpha^2+(0,1) +2(1+\kappa)\alpha^2\sum_{r=1}^{p}\frac{1}{z_r^2+(1+\kappa)\alpha^2(1-z_r^2)},$$

which may be combined with (18) to give

$$|m|\kappa+m\nu=-\left[\frac{1+\xi_0^2}{i\xi_0}\frac{Q_n^{m\prime}(i\xi_0)}{Q_n^m(i\xi_0)}+|m|+\frac{(0,1)}{\alpha^2} +2(1+\kappa)\sum_{r=1}^{p}\frac{1}{z_r^2+(1+\kappa)\alpha^2(1-z_r^2)}\right] =\frac{|m|}{\Omega_H-\Omega\ \text{sgn}m}, \quad (20)$$

where the definitions of κ and ν have been used. It is shown in Appendix I that $[(1+\xi_0^{2n})/i\xi_0][Q_n^{m\prime}(i\xi_0)/Q_n^m(i\xi_0)]$ is positive for all α and it then follows immediately that (20) can have no solutions with $\Omega_H>\Omega$. For since Ω and Ω_H are both positive, if $\Omega_H>\Omega$, then $\kappa>0$ and the right hand side of (20) is negative, whereas the left is positive for all m. We, therefore, consider (20) in the variables Ω_H and $\Delta=\Omega-\Omega_H$, where $\Delta>0$. Introducing the notation $G_{n,m}(\alpha)$ for $[(1+\xi_0^2/i\xi_0][Q_n^{m\prime}(i\xi_0)/Q_n^m(i\xi_0)]+|m|+[(0,1)/\alpha^2]$, we rearrange (20) in final form:

$$F\left(2\Delta+\frac{\Delta^2}{\Omega_H}\right)=|m|\left(G_{n,m}(\alpha)+2\sum_{r=1}^{p}\frac{1-(2\Delta+\Delta^2/\Omega_H)}{\alpha^2(1-z_r^2)-[z_r^2+\alpha_2(1-z_r^2)](2\Delta+\Delta^2/\Omega_H)}\right)^{-1} =\Delta \quad \text{for} \quad m>0 =-(\Delta+2\Omega_H) \quad \text{for} \quad m<0. \quad (21)$$

Equation (21), in spite of its clumsy appearance, permits a straightforward discussion of the behavior of the solutions for a given n and m.

The denominator in $F(2\Delta+\Delta^2/\Omega_H)$, considered as a function of Δ, has simple poles at

$$2\Delta_r+\frac{\Delta_r^2}{\Omega_H}=\frac{\alpha^2(1-z_r^2)}{\alpha^2(1-z_r^2)+z_r^2};$$

for all α these lie in the interval $0<2\Delta_r+(\Delta_r^2/\Omega_H)<1$; as α increases from 0 to ∞ (from the disk to the needle), each pole moves from $\Delta_r=0$ to $2\Delta_r+(\Delta_r^2/\Omega_H)=1$, while the order of the Δ_r, which is the reverse of that of the z_r^2, remains unchanged. Each term in the sum is easily shown to be steadily increasing with Δ; it starts from $1/[\alpha^2(1-z_r^2)]$, which is positive, at $\Delta=0$, runs through the pole at $\Delta=\Delta_r$, increases from $-\infty$ to 0 at $2\Delta+(\Delta^2/\Omega_H)=1$ and finally tends to $1/[\alpha(1-z_r^2)+z_r^2]$ as $\Delta\to\infty$. It follows that the denominator of F also increases steadily from a positive value at $\Delta=0$ to

[7] It would clearly be better to write the zeros, z_r, in a more explicit, but cumbersome way, as $z_{n,m,r}$. It should always be kept in mind that the z_r depend upon the n and m of any equations in which they occur.

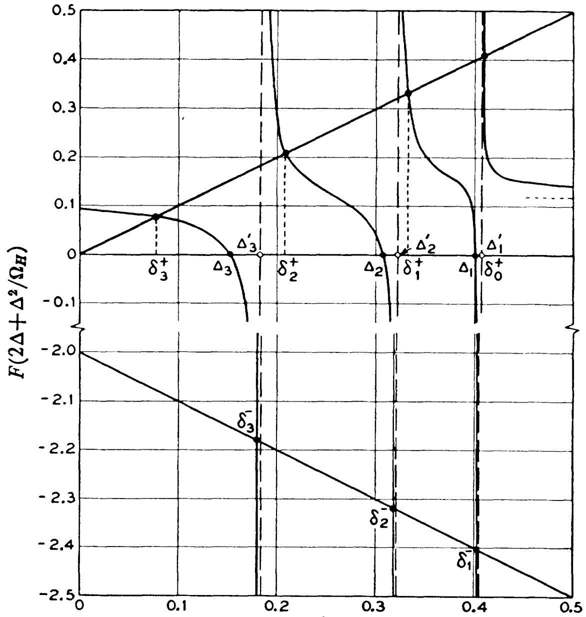

FIG. 1. $F(2\Delta+\Delta^2/\Omega_H)$ as a function of Δ and the lines Δ and $-(\Delta+2\Omega_H)$ for the case $\Omega_H=1$, $n=8$ and $m=2[p=3]$.

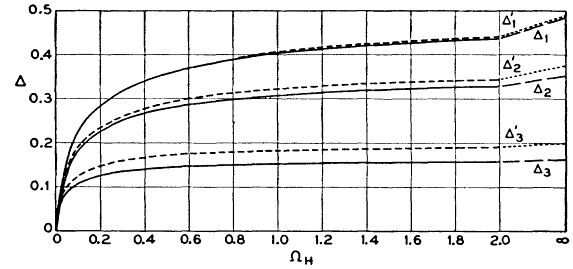

FIG. 2. Variation of the poles, Δ_r', and zeros, Δ_r, of $F(2\Delta+\Delta^2/\Omega_H)$ as a function of Ω_H for the case $n=8$, $m=2[p=3]$.

infinity at Δ_p, then runs $p-1$ times from $-\infty$ to $+\infty$ between the successive poles, Δ_r, finally increasing from $-\infty$ at Δ_1 to a positive value as $\Delta\to\infty$. Between each pair of poles will be a zero which we label Δ_r', with $\Delta_p>\Delta_p'>\Delta_{p-1}>\cdots$. Since $G_{n,m}(\alpha)$ is positive, the denominator is positive at $2\Delta+(\Delta^2/\Omega_H)=1$ and the last zero, Δ_1', must occur for a lower value of Δ than this. F consequently starts positive at $\Delta=0$, decreases steadily everywhere, going through zero at $\Delta=\Delta_r$ and to $+\infty$ at $\Delta=\Delta_r'$; it finally decreases from $+\infty$ at Δ_1' to a positive value as $\Delta\to\infty$. It should be noted that if $2\Delta+(\Delta^2/\Omega_H)=1$, then $0<\Delta<\frac{1}{2}$, so that it is always true that Δ_r, $\Delta_r'<\frac{1}{2}$.

Equation (21) may now be examined graphically by plotting the two sides as functions of Δ. Figure 1 illustrates the procedure for the case $n=8$, $m=2$. We shall suppose temporarily that $p\neq 0$ and return to the very simple case, $p=0$, later. Guided by Fig. 1 and considering first the case, $m>0$, it is clear that for any Ω_H, Eq. (21) has a solution in each of the $p+1$ ranges of Δ in which F is positive and that as Ω_H varies each solution remains in the same positive interval. We introduce, then, in addition to n and m, a third integer, r, to specify these different solutions and write (n,m,r) to label any one of the modes. The solution with maximum Δ is called $r=0$, the others are assigned the r value of that zero of F, Δ_r, which lies on the boundary of the region in question. Again, for $m<0$, (21) has a solution in each of the p ranges in which F is negative, which stays in this range of Δ as Ω_H varies. Values of r are assigned to these roots in the same way as for $m>0$; however, in this case there is no $r=0$ solution.

It is clear from Fig. 1 that the solutions for positive and negative m are, with $|m|$ fixed, always interlaced. If the notation, δ_r^+ and δ_r^-, is introduced to indicate the values of Δ for the rth solution with positive and negative m, we have for all Ω_H

$$\delta_0^+>\delta_1^->\delta_1^+>\delta_2^->\cdots>\delta_p^->\delta_p^+,$$

and similarly for the corresponding frequencies

$$\Omega_0^+>\Omega_1^->\Omega_1^+>\Omega_2^->\cdots>\Omega_p^->\Omega_p^+.$$

Considering now the solutions for $m>0$, as Ω_H increases, the curve $F(2\Delta+\Delta^2/\Omega_H)$ is shifted everywhere to the right and the solutions of (21) move to larger Δ. Ω is evidently a steadily increasing function of Ω_H for all r. Since all the Δ_r' and Δ_r tend to zero with Ω_H, so must all the δ_r^+, except for $r=0$. Figure 2 shows the variation of Δ_r' and Δr with Ω_H for the $n=8$, $m=2$ case. It is easy to verify that for small Ω_H the solutions start out according to

$$\Omega_r^+\sim\delta_r^+\sim\left(\frac{\alpha^2(1-z_r^2)\Omega_H}{(1-z_r^2)\alpha^2+z_r^2}\right)^{\frac{1}{2}},\quad r\neq 0.$$

The intersection for the $r=0$ root lies beyond all the Δ_r and Δ_r' and leads to a finite value as Ω_H becomes small

$$\Delta=F(\infty)$$
$$=|m|\left[G_{n,m}(\alpha)+2\sum_1^p\left(\frac{1}{\alpha^2(1-z_r^2)+z_r^2}\right)\right]^{-1}=\Omega_0^+.\quad(22)$$

The $r=0$ modes evidently have a finite frequency even when the internal applied field vanishes and only the dipolar forces remain. For all modes with $m>0$, as Ω_H increases, the separation, δ, will tend to a finite limit, since $2\Delta+(\Delta^2/\Omega_H)\to 2\Delta$. These limiting values are the solutions of

$$F(2\delta)=\delta\quad\text{for}\quad m>0,\quad\text{all } r.\qquad(23)$$

In a similar way, when $m<0$, the solutions again have δ increasing steadily with Ω_H, for the F curves move to the right and the line $-(\Delta+2\Omega_H)$ moves downward with increasing Ω_H. The behavior near $\Omega_H=0$ is identical with that of the corresponding solution with $m>0$, to terms of order $\Omega_H^{\frac{1}{2}}$. When Ω_H becomes very large, δ_r^- tends to the same limit as Δ_r', given by

$$G_{n,m}(\alpha)+2\sum_1^p \frac{1-2\delta_r^-}{\alpha^2(1-z_r^2)-[\alpha^2(1-z_r^2)+z_r^2]2\delta_r^-}=0 \quad \text{for} \quad m<0, \quad r\neq 0. \quad (24)$$

Finally, when $m=0$, (21) has solutions only if the denominator in F vanishes. Thus, $\delta_r^0=\Delta_r'$ and, since $2\Delta_r'+(\Delta_r'^2/\Omega_H)$ is a constant, δ_r^0 again increases steadily from zero at $\Omega_H=0$ to a finite limit in high fields. There is no $r=0$ solution.

Returning now to the case $p=0$, in which $n=m$ or $n=m+1$, the situation is particularly simple since F now reduces to the constant, $|m|/G_{n,m}(\alpha)$. There will be no modes with $m<0$ and the only solution for $m>0$ is that with $r=0$. We have exactly

$$\Omega-\Omega_H=|m|/G_{n,m}(\alpha). \quad (25)$$

This solution differs from other $r=0$ solutions only in the respect that $\Omega-\Omega_H$ is constant rather than steadily increasing. For all spheroids the pattern of these modes is of the form $\psi\sim(x+jy)^m$ with no z variation for the $(m,m,0)$ modes and $\psi\sim z(x+jy)^m$ for the $(m+1, m, 0)$ set. This shows that the mode, (1,1,0), is the familiar uniform mode and, in fact, (25) becomes in this case the usual Kittel formula for a spheroid. This is somewhat obscured because the demagnetizing factors are here expressed in Legendre functions. It is a consequence of (25) that, for either of these two series of modes, the separation of the modes in frequency (or field) from the uniform mode is field (or frequency) independent and directly proportional to the saturation magnetization.

Summarizing, when $p\neq 0$, all modes have Ω increasing steadily with Ω_H, with the separation $\Omega-\Omega_H$ increasing steadily to a finite limit in high fields; the positive and negative m solutions are interlaced and for fixed $|m|$ all solutions except for that $m>0$, $r=0$ have Ω going to zero with Ω_H. When $p=0$, only $r=0$ is admitted and the interval $\Omega-\Omega_H$ is field independent. Consideration of Fig. 1 shows that for fixed Ω the modes will be interlaced in Ω_H for $p\neq 0$ and that $\Omega-\Omega_H$ will also increase steadily with Ω to a finite limit. The $r=0$ modes will disappear below the frequency given by (22), but all others will persist to arbitrarily low frequencies.

A connection with the rest of the spin wave spectrum as given by (1) may be established in the following way. The maximum value of δ which can occur for any Ω_H, n and m is always the largest solution of

$$\delta=F(2\delta). \quad (23)$$

It may be shown that this root cannot exceed $\frac{1}{2}$. For $F(1)$ is equal to $|m|/G_{n,m}(\alpha)$ and in Appendix I, $G_{n,m}(\alpha)$ is found to be greater than $2m$. Thus, $F(1)<1$, which implies that the root of (23) is less than $\frac{1}{2}$. It follows that for all modes, with any n, m, and r,

$$\tfrac{1}{2}\geqslant\Omega-\Omega_H\geqslant 0.$$

If the formula (1) is written in the present notation and k^2 put equal to zero, it becomes

$$\Omega^2=\Omega_H^2+\Omega_H\sin^2\theta,$$

where θ is the angle made by the direction of propagation with the z axis. This gives

$$\begin{aligned}\Omega-\Omega_H&=(\Omega_H^2+\Omega_H\sin^2\theta)^{\frac{1}{2}}-\Omega_H\\&\leq(\Omega_H^2+\Omega_H)^{\frac{1}{2}}-\Omega_H\\&<\tfrac{1}{2}.\end{aligned}$$

Thus all of the magnetostatic or long-wavelength modes lie within the same limits as do the longest spin waves for which the dispersion relation is applicable.

In Fig. 3, the course of $\Omega-\Omega_H$ as a function of Ω_H is shown for a number of representative modes for the case of a sphere ($\alpha=1$, $\xi_0=\infty$). The sphere is a quite typical case except for the existence of a permanent degenerace between two series of modes which will be discussed in the next section. While it has been noted above that for a given n the modes of various m and r preserve their order as Ω_H increases, no such rule can be expected to apply to modes of different n. In fact, one sees from Fig. 3 that the crossing of modes or their accidental frequency degeneracy at particular values of Ω_H is of common occurrence. In particular, the uniform mode, (1,1,0), shows such degeneracies. One would expect to find this reflected in effects upon the line profile at magnetic fields where such degeneracies occur with modes to which the coupling of the (1,1,0) mode by nonlinearities or by imperfections is appreciable.

EFFECT OF SHAPE

A general observation may be made about the effect of varying the axial ratio by considering how $F(2\Delta+\Delta^2/\Omega_H)$ varies when Δ, Ω_H, n, and m are held fixed and α is changed. Consider first any term in the sum which occurs in the denominator and write this as

$$\left[\alpha^2(1-z_r^2)-z_r^2\left(2\Delta+\frac{\Delta^2}{\Omega_H}\right)\Big/\left(1-2\Delta-\frac{\Delta^2}{\Omega_H}\right)\right]^{-1}.$$

If $2\Delta+\Delta^2/\Omega_H$ is greater than 1, this term is always positive and decreases steadily with α. This is true of every term and it is shown in Appendix I to be true also of $G_{n,m}(\alpha)$, which, in addition, is infinite at $\alpha=0$; it follows that F is positive and increases with α from the value zero, at $\alpha=0$. If $2\Delta+(\Delta^2/\Omega_H)<1$ each term is negative at $\Delta=0$, decreases through a simple pole and decreases again from infinity to zero for large α. Recalling again that $G_{n,m}(0)$ is infinite, the denominator will be steadily decreasing, with a pole at $\alpha=0$ and p additional poles, finally tending to $G_{n,m}(\infty)$ which is given by $2m$ (see Appendix I). F is clearly an increasing

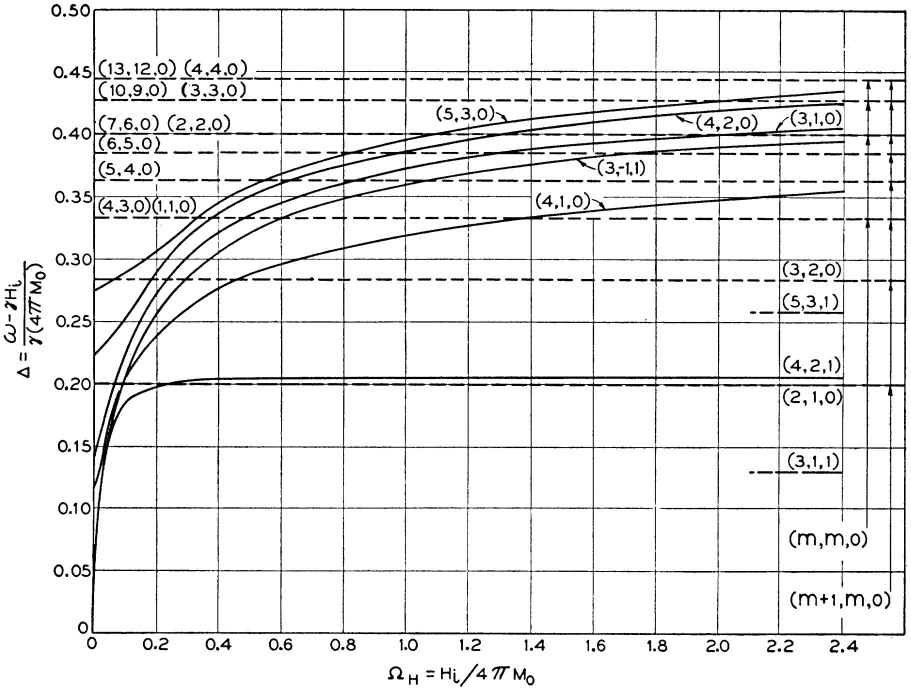

FIG. 3. $\Omega-\Omega_H$ as a function of Ω_H for some typical modes in the spherical case.

function of α, again starting from 0 at $\alpha=0$, with p poles and tending to the limit $\frac{1}{2}$ for large α. Returning to Fig. 1 it is clear that, since, for any Δ and Ω_H, F decreases with α (or the whole F curve is depressed by an increase in α) the intersections with the lines Δ and $-\Delta-2\Omega_H$ must move to higher Δ. Thus, *all* modes, including those with $r=0$, start with $\Delta=0$ when $\alpha=0$, have Δ steadily increasing with α and tend to $\Delta=\frac{1}{2}$ as $\alpha\rightarrow\infty$. In Fig. 4 the course of some of the $(m,m,0)$ and $(m+1, m, 0)$ modes is shown as a function of axial ratio. A consideration of (21) for large α shows that, approximately,

$$2\delta_r+(\delta_r^2/\Omega_H)=1-O(1/\alpha^2) \quad \text{for all } m, \quad r\neq 0.$$

Thus the modes of different n, $|m|$ and r differ only in order α^{-2} and the $+m$ and $-m$ are degenerate to this order. For $r=0$, one finds from the expression given in Appendix I for $G_{n,m}(\alpha)$ with α large, that

$$\begin{aligned}\delta_0&=\tfrac{1}{2}-O(1/\alpha^2) \quad &\text{for} \quad m\neq 1\\ &=\tfrac{1}{2}-O(\ln\alpha/\alpha^2) \quad &\text{for} \quad m=1.\end{aligned}$$

The singular behavior of the $r=0$, $m=1$ mode in very prolate spheroids may be ascribed to the presence of fields with axial symmetry about the z axis.

When α is small, all δ's are also small under all circumstances, but not necessarily of the same order in α. A distinction arises between the two cases, $n-|m|$ even and odd. The modes with $n-|m|$ even are symmetric in z, those with $n-|m|$ odd are antisymmetric in z. Clearly in a very thin disk a splitting of the two types is to be expected. One finds the following behavior: for $m>0$, $r\neq 0$, δ is of order α^2; for $m>0$, $r=0$, $n-|m|$ odd, δ is of order α^2; for $m>0$, $r=0$, $n-|m|$ even, δ is of order α; for $m\leq 0$, $r\neq 1$, δ is of order α^2; for $m\leq 0$, $r=1$, δ is of order α^2 when $n-|m|$ is odd and of order α when $n-|m|$ is even.

The sphere is distinguished from other shapes by the fact that here some of the modes are degenerate at all fields. For $\alpha=1$, $G_{n,m}$ becomes $n+1+m+(0,1)$ and thus the expression (25) for the modes with $p=0$ is seen to reduce to the remarkably simple form

$$\begin{aligned}\Omega-\Omega_H&=m/(2m+1) \quad &\text{for} \quad n=m\\ &=m/(2m+3) \quad &\text{for} \quad n=m+1.\end{aligned}$$

Clearly the modes $(m,m,0)$ and $(3m+1, 3m, 0)$ are permanently degenerate. It is interesting to note that amongst such pairs are the (1,1,0)-mode, the uniform mode usually excited in experiments and the (4,3,0)-mode. The latter has $z=0$ as a nodal plane and three nodal planes 120° apart through the z axis.

It is of occasional interest to know the field configuration of the modes and this can be readily found from the expression (15) for the internal ψ. For the cases $p=0$ and $p=1$, the ψ (not normalized in any way) are given by

$$p=0, \quad n=m, \quad \psi\sim(x+jy)^m;$$

$$p=0, \quad n=m+1, \quad \psi\sim z(x+jy)^m;$$

$$p=1, \quad n=m+2, \quad \psi\sim(x+jy)^m \times\left\{\frac{x^2+y^2}{2(m+1)}-\left(1+\frac{\Omega_H}{\Omega_H^2-\Omega^2}\right)z^2 -\left[a^2-\left(1+\frac{\Omega_H}{\Omega_H^2-\Omega^2}\right)b^2\right]\Big/(2m+3)\right\};$$

$p=1,\quad n=m+3,\quad \psi\sim z(x+jy)^m$

$$\times\left\{\frac{x^2+y^2}{2(m+1)}-\frac{1}{3}\left(1+\frac{\Omega_H}{\Omega_H{}^2-\Omega^2}\right)z^2 -\left[a^2-\left(1+\frac{\Omega_H}{\Omega_H{}^2-\Omega^2}\right)b^2\right]\Big/(2m+5)\right\}.$$

Plainly the azimuthal symmetry is governed by m; functions with p even are even about $z=0$, those with p odd are odd about $z=0$. Modes with the same n and m, but different r will have dissimilar patterns in view of the different values of Ω for a given Ω_H. The patterns of a mode of given n, m, and r will vary with field or frequency. Whether a mode with particular n and m is excited by a given external field depends upon the presence or absence in the expansions of the latter in spherical harmonics of the term $P_n{}^m(\theta)e^{\pm jm\phi}$.

DISCUSSION

We have examined qualitatively the mode structure to be expected when the assumptions hold which would put us into a purely magnetostatic regime. Looking back to Eq. (8) one sees that the diagonal and off-diagonal components of permeability are $\Omega_H/(\Omega_H{}^2-\Omega^2)$ and $\Omega/(\Omega_H{}^2-\Omega^2)$ times the free-space permeability, respectively. These factors will be roughly $1/2\Delta$ when Δ is small. The wavelength in the ferrite is reduced by about a factor of $(2\Delta/\epsilon)^{\frac{1}{2}}$, where $\epsilon\sim 10$ is the dielectric constant of the ferrite. If we require that the largest dimension of the sample, D, be less than 1/10 of the wavelength in the medium we have a crude restriction on Δ, namely,

$$\Delta>500(D/\lambda_0)^2.$$

As we have seen, Δ is small in weak fields for $r\neq 0$ and for small axial ratios. The first case causes no trouble because $\Delta\sim\Omega=O(\Omega_H{}^{\frac{1}{2}})$ for small Ω_H and therefore $1/\lambda_0{}^2\propto\Delta_H$, so that the restriction actually becomes less acute as the field diminishes. For flat spheroids, however, the situation is difficult, for the condition on Δ is restrictive and at the same time the practical problem of making small disks of small axial ratio is rather severe. In view of this fact, it is somewhat unfortunate that the most extensive data available on multiple resonance at present have been taken on rather large disks.[2]

Dillon[8] has studied disks of manganese ferrite single crystals with axial ratios of 1/20 and 1/15 in inhomogeneous magnetic fields of various types and has observed as many as twenty absorption lines at a fixed frequency in a given exciting field. These lines are spaced in a systematic way in field and clearly represent series of modes in which n, m, and r vary in some simple manner. It is consistent with the data to suppose that they are the series $(m,m,0)$ and $(m+1, m, 0)$. In the first place, the separations of the lines are independent of frequency over a wide range and have a temperature dependence which suggests that they are proportional to saturation magnetization. These were the features which, according to (25) characterize these series.

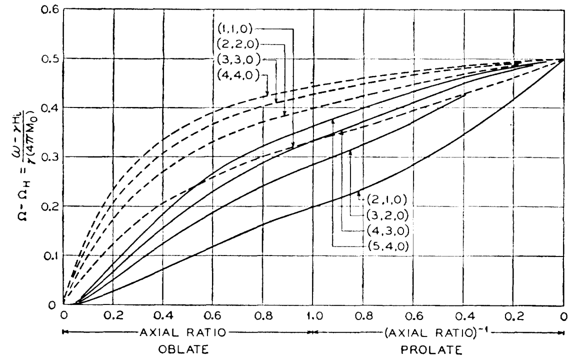

FIG. 4. $\Omega-\Omega_H$ as a function of axial ratio for some of the $(m,m,0)$ and $(m+1, m, 0)$ modes. $\Omega-\Omega_H$ is independent of Ω_H for these modes.

Further, Dillon has used fields with three distinct symmetries; (a) even in z, even about a plane through the z axis; (b) even in z, odd about a plane through the z axis; (c) odd in z. In the first case, he finds a series of modes of which the one with the largest field coincides with the value for the uniform mode and all the others occur at lower fields. In the second case, a series is found which fall between the successive members of the first series. These two series may be identified as $(m,m,0)$ with m odd and with m even. The nature of the exciting field and the fact that the (1,1,0) mode has the largest field agree with this interpretation. In the third case, a series is found with three members at higher fields than the (1,1,0) and several at lower fields. This is consistent with (25) for disks of this axial ratio and for the series $(m+1, m, 0)$ which one might expect to find excited in a field odd in z. When one attempts to compare the actual spacings observed by Dillon with those given by the analysis it is found that a fair qualitative agreement is found if it is assumed that the effective axial ratio of the disk considered as a spheroid is substantially greater than the thickness to diameter ratio. We observe again that Dillon's disks are about 0.100 in. in diameter which is certainly too large for the assumptions of this analysis to be valid.

In spheres, White, Solt, and Mercereau[9] have made measurements on samples of manganese ferrite and manganese-zinc ferrite placed at various positions in a rectangular cavity. The saturation was not measured, but the positions of the peaks was observed as a function of temperature from around room temperature to the Curie point. The ratio of the separations of two of the lines from the uniform mode remained essentially

[8] The author is indebted to Dr. Dillon for permission to discuss his results before publication.

[9] The writer wishes to thank these authors for permission to quote their results.

constant with temperature in the case of the manganese ferrite (the manganese-zinc data shows some puzzling deviations); and the separations themselves were essentially unchanged between X and K band. This would indicate that these two modes belong to the $(m,m,0)$ or $(m, m+1, 0)$ series and if the saturation magnetization is taken to be $3300/4\pi$ gauss at the lowest temperature (about 15°C), the two lines may be identified as (1,2,0) and (2,2,0) with an error in the calculated separations of about 2%. The presence of these lines is consistent with the field symmetry used to excite them. Two other modes were excited whose separation was frequency-sensitive and apparently not proportional to the magnetization. If one uses the same value for the saturation at 15°C and assumes that the separation of the two earlier modes provides a magnetization curve, the extra modes appear to be (1,3,0) and either (0,2,0) or (2,5,0) (these are barely resolved in a sphere over the range of Ω_H covered in these experiments). The presence of these modes is consistent with the excitation and the calculated separations are within 3% of the observed values. White *et al.* have given a somewhat similar identification for their lines. It would be interesting to have further data on spheres taken in fields of definite symmetry.

The problem of excitation for the different modes is not resolved. The intensities observed by Dillon are such that the absorption at resonance falls off by a factor of roughly two between successive resonances; this is too slow if one simply considers the known inhomogeneity of the exciting field and computes the amplitudes which should be excited. It is also true that the excitation of the modes seems to depend somewhat upon the material examined. This makes it plausible to suppose that some internal mechanism, such as inhomogeneities or variations in dc magnetic field, such as exist in disks, may cause excitation.

APPENDIX I

The function, $G_{n,m}(\alpha)$ was defined as

$$G_{n,m}(\alpha)=\frac{1+\xi_0^2}{i\xi_0}\frac{Q_n^{m\prime}(i\xi_0)}{Q_n^m(i\xi_0)}+|m|+\frac{(0.1)}{\alpha^2},$$

where $\xi_0^2=\alpha^2/(1-\alpha^2)$. It is possible, of course, to deduce the properties of $G_{n,m}$ by expressing the Q_n^m explicitly in hypergeometric functions, but enough for our purposes can be found directly. We write

$$f_{n,m}(\alpha)=\frac{1+\xi_0^2}{i\xi_0}\frac{Q_n^{m\prime}(i\xi_0)}{Q_n^m(i\xi_0)}. \qquad \text{(A1)}$$

Using Legendre's equation and changing the independent variable to α, it is found that $f_{n,m}$ satisfies

$$\alpha\frac{df}{d\alpha}=m^2-f^2+\frac{(f+n)(f-n-1)}{1-\alpha^2}. \qquad \text{(A2)}$$

Then, using the properties of Q_n^m for large values of its argument, $f_{n,m}(1)=n+1$, which is sufficient to identity the solutions of (A2). Near $\alpha=1$, we put $f_{n,m}=(n+1)+c_1(\alpha^2-1)+\cdots$, and find that

$$c_1=-[(n+1)^2-m^2]/(2n+3)<0,$$

so that $f_{n,m}$ is less than $n+1$ and decreasing, for α slightly above 1. Since for any $\alpha>1$, $df/d\alpha$ changes from positive to negative once in going from $f=m$ to $f=n+1$, then $df_{n,m}/d\alpha$ remains negative for $\alpha>1$ and tends to m as $\alpha\rightarrow\infty$. For large α, we put

$$f_{n,m}=m+d_1\alpha^{-s}+\cdots$$

in (A2) and obtain, for $m\neq 1$, 0,

$$2(m-1)d_1=(n+m)(n-m+1), \quad s=2. \qquad \text{(A3)}$$

For $m=1$, we put

$$f_{n,1}=1+\ln\alpha[d_2\alpha^{-s}+\cdots]$$

and find

$$d_2=n(n+1), \quad s=2, \qquad \text{(A4)}$$

while for $m=0$,

$$f_{n,0}\rightarrow(\ln\alpha)^{-1}.$$

When α is just below 1, $f_{n,m}$ is greater than $n+1$ and decreasing with α. Again, since $df/d\alpha$ changes once from negative to positive as f goes from $n+1$ to ∞ for any $\alpha<1$, $f_{n,m}$ must always lie below the line on which $df/d\alpha$ vanishes. It is therefore a steadily decreasing function of α. For small α, we put

$$f_{n,m}=e_1\alpha^{-s}+\cdots$$

and find

$$s=1.$$

The constant e_1 cannot be determined in this manner but is not essential.

It is now established that $G_{n,m}(\alpha)$ decreases steadily with α. It is infinite for α small, going as α^{-1} for $n-m$ even, and as α^{-2} for $n-m$ odd. Its value is $n+1+|m|+(0,1)$ at $\alpha=1$. As $\alpha\rightarrow\infty$, $G_{n,m}$ tends to $2|m|$; for $m\neq 1,0$ it goes as $2|m|+O(\alpha^{-2})$, for $m=1$ as $2|m|+O(\alpha^{-2}\ln\alpha)$ and for $m=0$ as $(\ln\alpha)^{-1}$.

APPENDIX II

For some purposes it may be useful to have orthogonality relations between the different modes. We let the subscripts λ and μ refer to two modes with distinct frequencies at some field Ω_H. Then

$$\nabla^2\psi_\lambda+\mathrm{div}4\pi\mathbf{m}_\lambda=0,$$
$$\nabla^2\psi_\mu+\mathrm{div}4\pi\mathbf{m}_\mu=0,$$

and we have

$$\int_{\text{space}}[\psi_\lambda\nabla^2\psi_\mu-\psi_\mu\nabla^2\psi_\lambda]d\tau + \int_{\text{spheroid}}[\psi_\lambda\,\mathrm{div}4\pi\mathbf{m}_\mu-\psi_\mu\,\mathrm{div}4\pi\mathbf{m}_\lambda]d\tau=0,$$

which yields

$$\int_{\text{spheroid}}[\text{grad}\psi_\lambda\cdot\mathbf{m}_\mu-\text{grad}\psi_\mu\cdot\mathbf{m}_\lambda]d\tau=0.$$

Substituting in this expression the components of gradψ found from Eq. (7), one has

$$(\Omega_\lambda+\Omega_\mu)\int_{\text{spheroid}}[m_{\mu x}m_{\lambda y}-m_{\lambda x}m_{\mu y}]d\tau=0, \quad \text{(B1)}$$

and, therefore, the integral must vanish. The orthogonality relation may equally well be written as

$$\int_{\text{spheroid}}[m_\mu^- m_\lambda^+ - m_\lambda^- m_\mu^+]d\tau, \quad \text{(B2)}$$

where

$$m^\pm = m_x \pm j m_y.$$

If one starts from the complex conjugate equation

$$\nabla^2\psi_\lambda^* + \text{div}4\pi\mathbf{m}_\lambda^* = 0,$$

and proceeds in a similar manner, a second orthogonality relation is found in the form

$$\int_{\text{spheroid}}[m_{\mu x}m_{\lambda y}^* - m_{\mu y}m_{\lambda x}^*]d\tau=0, \quad \text{(B3)}$$

or

$$\int_{\text{spheroid}}[m_\mu^+ m_\lambda^{+*} - m_\mu^- m_\lambda^{-*}]d\tau=0. \quad \text{(B4)}$$

One use which can be made of the orthogonality relations is to verify that the magnetostatic modes actually diagonalize the total magnetic energy of the system when the latter is expanded to quadratic terms in the deviations from line-up.

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Nonradiative Transitions of Trapped Electrons in Polar Crystals

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An algebraic formula is derived which represents the integral expression of Huang-Rhys for nonradiative transitions of trapped electrons in polar crystals. The formula is a good representation over a very wide range of trap depths and coupling constants.

A THEORY of nonradiative transitions of trapped electrons in polar crystals involving multiphonon processes has been given by Huang and Rhys.[1] Using the same physical ideas but a different formal approach, Vasileff[2] has investigated the Huang-Rhys formula in some detail and has produced, in the course of his work, a relatively simple formula for the probability of thermal ionization at low temperatures. It is the purpose of this letter to give a high-temperature approximation for the Huang-Rhys formula, and it will be found that the approximation may in fact be valid down to very low temperatures depending on the value of a certain interaction constant which arises in the theory.

Huang and Rhys find that the probability for a nonradiative multiphonon transition involving p phonons between a trapped state μ which has an energy W below the bottom of the conduction band and a state in the conduction band which is approximated to a free-electron state of wave vector $\mathbf{k}$, is given by

$$\frac{16\pi^2\hbar\omega_l^2}{3v_a}\left(\frac{1}{\epsilon_\infty}-\frac{1}{\epsilon_0}\right)(\bar{n}+\tfrac{1}{2})\frac{|\langle\mathbf{k}|e\mathbf{x}|\mu\rangle|^2}{(W+\hbar k^2/2m)^2}R_{\pm p}, \quad (1)$$

where ω_l is the circular frequency of longitudinal polarization waves, v_a is the volume of the unit cell of the crystal, ϵ_0 and ϵ_∞ are the static and high-frequency dielectric constants, $\bar{n}$ is the average number of phonons ω_l excited at the temperature considered, $\langle\mathbf{k}|e\mathbf{x}|\mu\rangle$ is the matrix element of the electric moment between the conduction state $\mathbf{k}$ and the trapped state μ, and

$$R_p=\exp[-(2\bar{n}+1)S]\left(\frac{\bar{n}+1}{\bar{n}}\right)^{p/2}I_p(2S[\bar{n}(\bar{n}+1)]^{\frac{1}{2}}), \quad (2)$$

where S is an interaction constant and I is the Bessel function of imaginary argument. The positive sign of p in (1) refers to recombination of the electron into the trap with the emission of phonons, and the negative sign corresponds to thermal ionization by absorption of phonons.

[1] K. Huang and A. Rhys. Proc. Roy. Soc. (London) **A204**, 406 (1950).
[2] H. D. Vasileff, Phys. Rev. **96**, 603 (1954); **97**, 891 (1955).