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Strong focusing and the radiofrequency quadrupole accelerator

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I. SCOPE

These problems and solutions use basic principles of electromagnetism to explain an ingenious, strong-focusing accelerator known as the radiofrequency quadrupole (RFQ). This device has been a major advance in accelerator technology throughout the world during the past 15 years.

The problems are suitable for a course in electromagnetism. Their solutions use Laplace's equation and require the extraction of information from boundary conditions. They also introduce the quasistatic approximation and show how under appropriate conditions Laplace's equation can be used to solve time-dependent problems. The approximate solution of the Mathieu–Hill equation used to present the basic principles of alternating-gradient focusing in the RFQ may be of interest for a course in classical mechanics.

II. INTRODUCTION

The RFQ is a new type of rf accelerator, and its recent development has been a significant breakthrough in the field of low-velocity accelerators. Because the RFQ is especially suited for the acceleration of nonrelativistic charged particle beams, it is an important acceleration technique for ions. The RFQ can be used alone to accelerate high-current ion beams or to inject such beams into other accelerators that produce even higher energies. The RFQ built at Los Alamos Laboratory for the now defunct Superconducting Supercollider has typical RFQ parameters. It will bunch and accelerate a beam of negative hydrogen ions from 35 keV to 2.5 MeV. It operates at a frequency of 428 MHz, is 2.2 m long, and has a cross-sectional diameter of 17 cm. Figure 1 shows a schematic drawing of this device.

Kapchinskiy and Tepliakov, the inventors of the RFQ, described its principles of operation in 1970.[1] They proposed to modify the shapes of the four poles of a rf quadrupole so that its rf electric fields would accelerate as well as focus. Using a potential function description, they showed what shape of poles and pole spacing would produce the required fields. The achievement of practical velocity-independent electric focusing in a low-velocity accelerator gives the RFQ a significant advantage over conventional accelerators that use velocity-dependent magnetic lenses. The RFQ extends the practical range of operation of ion linacs to low velocities, and thus eliminates the need for large, high-voltage dc accelerators for injecting a beam into a linac.

We can begin to understand the RFQ[2] most easily by first examining how transverse focusing is accomplished. (The word "transverse" here refers to a Cartesian reference frame

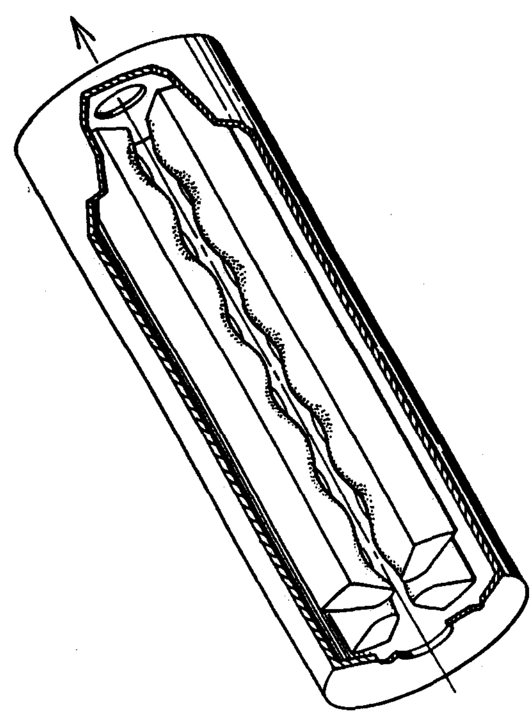


Fig. 1. A schematic diagram showing a 4-vane RFQ cavity with modulated vane tips. The charged ion beam is accelerated and confined near the central axis.

in which the direction of the accelerated beam is the z axis, and the x–z and the y–z planes are called the "transverse planes.") Consider four equally spaced conducting poles symmetrically placed about the beam axis. Figure 1 shows schematically how these poles are provided in the rf cavity by the shaped edges of four vanes running the length of the cavity. Suppose we apply an ac voltage such that its instantaneous polarity alternates sign from pole to pole. This will result in a time-varying electric-quadrupole field that exerts no force on ions at the axis, but produces strong, alternating-gradient focusing[3] of off-axis particles.

The objective of focusing within a particle accelerator is usually not to compress the beam to the smallest possible spot, as might be the purpose of focusing in a microscope, but rather to confine the beam so that it can propagate through the limited aperture of a long accelerator. Particles always have velocity components transverse to the beam axis. These may arise from the random thermal velocities in the source, or from outward forces within the accelerator resulting from the applied fields or from the self-fields associated with space-charge repulsion within the beam itself. Whatever the cause, the focusing system must provide an adequate restoring force to deflect the off-axis particles back toward the axis.

The nonzero longitudinal electric-field component necessary for particle acceleration is produced by making the separation between the vertical poles different from the separation between the horizontal poles. As a result, the axial potential will be nonzero, because it is influenced more by the poles that are nearer to the axis than by those of opposite polarity that are further away. Then if the spacings between pairs of electrodes are made to vary along the axis, the potential at the axis will change as a function of longitudinal position, and there will be an electric field along the axis.

For sustained acceleration of the beam particles, this axial electric field must have a pattern that synchronizes the accelerating fields and the particle motion. Such synchronism is achieved by appropriate modulation of the electrode separations. If during the time that the electrode voltages change sign, the pattern of the electrode separations changes through half a period, the axial voltage will maintain the correct sign for sustained acceleration, and there will be synchronous acceleration. This means that the period of the electrode modulation should match the axial distance traversed by a synchronous particle during one rf period. Such a match is produced by the geometry shown in Fig. 1, where the pattern of spatial modulation of the horizontal pole tips is out of phase with the pattern of the vertical pole tips.

The design of the RFQ and the solutions of the problems presented here assume the validity of the quasistatic approximation. In this approximation we can derive the time-dependent electric-field components from a potential that satisfies Laplace's equation. The powerful advantage of this approach is that one can then use a numerically controlled milling machine to make the four pole tips precisely the shape of the equipotential surfaces of the potential function. Once constructed and installed in a properly tuned cavity, the electric fields seen by the beam correspond accurately to those derived from the potential function. The quasistatic approximation is reviewed in Sec. IV.

The problems in Sec. IV progressively introduce the basic principles of the RFQ, both as a transverse focusing device and as a linear accelerator. Problem 1 shows that the electrostatic quadrupole lens is astigmatic, focusing in one transverse plane while defocusing in the other orthogonal plane. Problem 2 introduces the sinusoidal time dependence of the potential. An approximate solution to the equation of motion shows a time-dependent alternating focusing effect that produces a net transverse focusing in both planes. Problem 3 shows that unequal separations between pairs of pole tips introduce a nonzero axial potential, and if the pole–tip separations are varied, the resulting axial electric field can accelerate the beam. Finally, problem 4 introduces the full time-dependent RFQ potential function that specifies the desired geometry for varying the displacements of the pole tips.

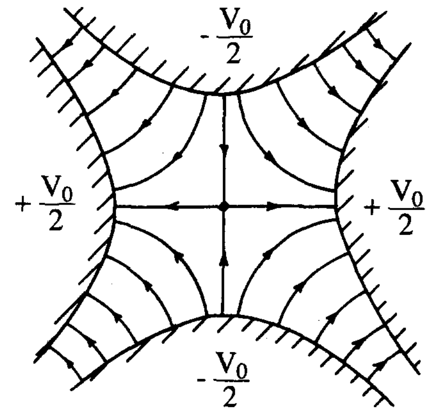


Fig. 2. A cross-sectional view of an electrostatic quadrupole lens. The beam axis is at the center, perpendicular to the page. The potentials on the four electrodes and the electric-field lines are shown.

III. PROBLEMS

A. Electrostatic quadrupole lens

(1) Show that the electric quadrupole potential function

$$U(x,y)=\frac{V_0}{2}\left[\frac{x^2-y^2}{a^2}\right] \tag{1}$$

satisfies Laplace's equation.

(2) Suppose that to focus a beam of charged particles, we build an electric quadrupole lens by installing four electrodes with potentials like those shown in Fig. 2. Obtain an expression for the shape of the electrodes.

(3) Write the x- and y-electric field components and the equations of motion in the x–z and y–z planes, and show that the beam is focused in one plane and defocused in the other orthogonal plane.

B. Time-dependent electric quadrupole focusing

Although the static quadrupole electric field cannot simultaneously focus in both planes, a time varying electric quadrupole field can. The time averaged effect produces net focusing in both planes. This is the principle of alternating gradient focusing.

Suppose that the time-dependent electric fields for the electric quadrupole are derivable from a scalar potential with a harmonic dependence given by

$$U(x,y)=\frac{V_0}{2}\left[\frac{x^2-y^2}{a^2}\right]\sin\omega t, \tag{2}$$

where ω is the rf frequency. The time-dependent factor produces a time-dependent variation of the field polarity which results in net focusing in both planes.

(1) Show that the equation of motion for the y coordinate in the y–z plane is

$$\ddot{y}-\frac{qV_0 y\sin(\omega t)}{ma^2}=0,$$

which describes a periodic force with period $\tau=2\pi/\omega$.

(2) The above equation is an example of the Mathieu–Hill equation. Floquet's theorem[4] tells us that we can always find a solution of the Mathieu–Hill equation of the form $y=P(t)\cos[\phi(t)+\theta]$, where θ is a constant, $P(t)$ is a periodic function of t with period $\tau=2\pi/\omega$, and $\phi(t)$ satisfies the condition that $\phi(t+\tau)-\phi(t)=\sigma$, where σ is called the phase-advance per period, and must satisfy the condition $0<\sigma<\pi$ for bounded motion. Guided by the Floquet theorem, assume a simple trial solution

$$y=C[1+\epsilon\sin\omega t]\sin\Omega t, \tag{3}$$

where C is a constant, $\phi(t)$ is assumed to be a linear function of t, $\phi(t)=\Omega t$ where $\Omega=\sigma/\tau$ and $P(t)$ has been chosen to have a sinusoidal modulation of the form $P(t)=1+\epsilon\sin\omega t$. ϵ is called the "flutter amplitude" of the rapidly oscillating rf factor, C is a constant, and Ω is the frequency of the slowly varying average trajectory. Because $\Omega/\omega\ll1$, and $\epsilon\ll1$, you can take this trial solution to be approximately valid if it satisfies Laplace's equation when you neglect terms of order $\epsilon\Omega/\omega$ and Ω^2/ω^2 and smaller. By substituting this solution into the equation of motion, identify in lowest order the value of ϵ that makes the trial solution approximately valid.

(3) Show that the averaged trajectory is confined by the time-dependent focusing system. You can do this by considering the equation of motion averaged over a rf period, and then showing that the averaged trajectory satisfies the equation of simple harmonic motion. Calculate the frequency of this average motion.

C. Electrostatic quadrupole with axial potential

A RFQ's focusing geometry can be modified to provide longitudinal acceleration by configuring its electrodes to produce a longitudinal electric field. In other words, it is possible to create a potential that varies spatially along the axis.

(1) Suppose that the x-pole tips are at voltage $+V_0/2$ and are displaced from the axis a distance a, while the y-pole tips are at voltage $-V_0/2$ and are displaced a distance ma where $m\geq1$, as shown in Fig. 3. Consider the potential function given in Eq. (4). Find the values of X and A expressed in terms of the parameter m, that make Eq. (4) satisfy the boundary conditions of this geometry,

$$U(x,y)=\frac{V_0}{2}\left[X\frac{(x^2-y^2)}{a^2}+A\right]. \tag{4}$$

(2) How do X and A change as functions of m if the displacements of the x- and y- pole tips are interchanged to ma and a, respectively, while their voltages remain the same?

D. The radio frequency quadrupole

Consider the three-dimensional, time-dependent potential function with a spatially oscillating longitudinal term

$$U(x,y)=\frac{V_0}{2}\left[X\frac{(x^2-y^2)}{a^2}+A\ I_0(kr)\cos kz\right]\times\sin(\omega t+\phi), \tag{5}$$

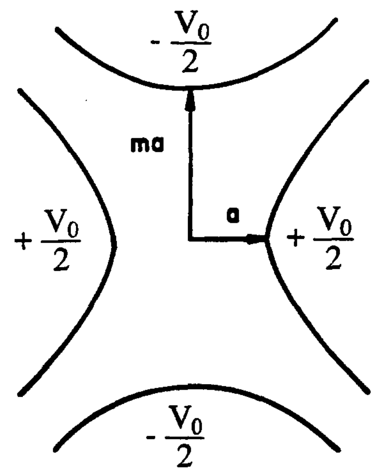


Fig. 3. The RFQ transverse geometry with unequal horizontal and vertical displacements, a and ma, of the pole tips from the axis.

which, by direct substitution, can be shown to be a solution of Laplace's equation. The parameters X and A are constants, I_0 is called the modified Bessel function of order zero,[5] $r=\sqrt{x^2+y^2}$, and $k=2\pi/\beta\lambda$, where βc is the velocity of the synchronous beam particle (c is the speed of light), $\lambda=2\pi c/\omega$ is the rf wavelength, and $\beta\lambda$ is equal to the distance traversed by a synchronous particle in one rf period. (The modified Bessel functions are just the familiar Bessel functions of the first kind with a pure imaginary argument. They appear as radial solutions of Laplace's equation in cylindrical coordinates whenever the axial solution is a harmonic function.) The voltage of the horizontal poles is $(V_0/2)\sin(\omega t+\phi)$ and that of the vertical poles is $-(V_0/2)\times\sin(\omega t+\phi)$. The pole–tip geometry in the x–z plane is shown in Fig. 4. The geometry in the y–z plane is shifted axially by half a wavelength. At $z=0$ the horizontal pole–tip coordinates are $x=a, y=0$, and the vertical pole–tip coordinates are $x=0$, $y=ma$, where we assume that $m\geq1$.

(1) Derive expressions for X and A as functions of the geometry parameters a, m, and k, by requiring that the boundary conditions on the pole tips are satisfied at $z=0$.
(2) Derive expressions for the shapes of the pole tips in the x–z and y–z planes.

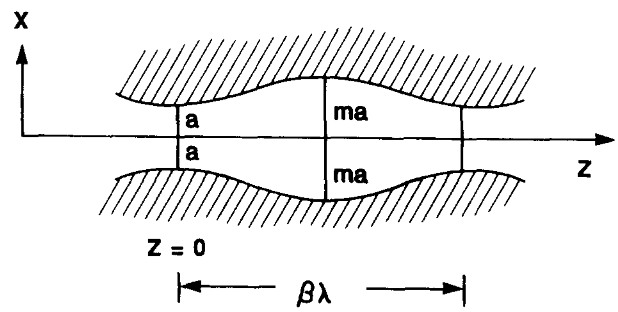


Fig. 4. A view of the RFQ pole–tip geometry in the x–z plane through one period, showing the minimum displacement a and the maximum displacement ma. The y pole tip is a distance ma from the axis at $z=0$. The spatial period is chosen to equal the distance $\beta\lambda$ traveled by a synchronous particle in one rf period, where βc is the synchronous velocity and λ is the rf wavelength.

(3) Obtain expressions for the three electric-field components.

(4) In the approximation that the radial position and the velocity of a particle within a period are constant, calculate the energy gain ΔW for a synchronous particle with velocity βc, by replacing ωt with $2\pi z/\beta\lambda$ and integrating to obtain the average work done on the particle by the axial electric field over one period.

IV. QUASI STATIC APPROXIMATION

The analytic treatment of the time-dependent RFQ fields is based on the use of a quasistatic approximation. For some applications involving time-dependent electromagnetic fields, the spatial distribution of the fields is nearly the same as for the static problem, even though the time variation is rapid. In these cases the electric-field components in charge-free space can be derived from a time-dependent scalar potential that satisfies Laplace's equation. This method is called the quasistatic approximation, and it is usually valid when the geometrical sizes of the elements are small compared to the free-space wavelength.

This approximation follows from consideration of the wave equation in charge-free space, which for the case of a harmonic time dependence becomes the Helmholtz equation

$$\nabla^2\mathbf{E}+k^2\mathbf{E}=0,$$

where the wave number is expressed in terms of the frequency ω, the speed of light c, and the free-space wavelength λ as $k=\omega/c=2\pi/\lambda$. Expanding the Helmholtz equation into its three components gives

$$\frac{d^2\mathbf{E}}{dx^2}+\frac{d^2\mathbf{E}}{dy^2}+\frac{d^2\mathbf{E}}{dz^2}+\left[\frac{2\pi}{\lambda}\right]^2\mathbf{E}=0.$$

When field variations develop on a scale determined by the locations of elements which have small spacings compared to the wavelength, the values of derivative terms near those elements will dominate the last term, and the equation then reduces approximately to $\nabla^2\mathbf{E}\cong 0$. This is the quasistatic approximation. The electric field must also satisfy the basic vector relationship $\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$. Since the last term is approximately zero, and in charge-free space $\nabla\cdot\mathbf{E}=0$, the relationship implies that $\nabla\times(\nabla\times\mathbf{E})=0$. This equation is satisfied if $\mathbf{E}=-\nabla U$, where U is a scalar potential because the vector identity $\nabla\times\nabla U=0$ guarantees that $(\nabla\times\mathbf{E})=0$. Therefore, in the quasistatic approximation, the time-dependent electric-field components in charge-free space are derivable from a scalar potential U, which, since $\nabla\cdot\mathbf{E}=\nabla^2U=0$, must satisfy Laplace's equation. Notice that the quasistatic approximation decouples the magnetic and electric fields in Maxwell's equations, which implies that the contribution to the electric fields from Faraday's law is negligible. For the RFQ the quasistatic approximation applies very accurately to the region near the four poles, including the vicinity of the beam because the spacings of the poles in that region are typically only a few thousandths of the wavelength.

V. SOLUTIONS

A. Electrostatic quadrupole lens

(1) Laplace's equation for the electric-quadrupole potential is satisfied because

$$\frac{d^2U(x,y)}{dx^2}+\frac{d^2U(x,y)}{dy^2}=-\frac{V_0}{a^2}+\frac{V_0}{a^2}=0.$$

(2) If we set $U(x,y)$ equal to $\pm V_0/2$, we obtain equipotentials in the shapes of rectangular hyperbolas given by $x^2-y^2=a^2$. The curves for the four poles are symmetrically spaced about the axis as shown in Fig. 2.

(3) The electric-field components are $E_x=-V_0x/a^2$ and $E_y=V_0y/a^2$. These are linear functions of the transverse displacements x and y. A positively charged particle that is off axis will experience an inward restoring force in the $x-z$ plane, but will experience a force outward from the axis in the $y-z$ plane. The nonrelativistic equations of motion correspond to focusing in x and defocusing in y, and are given by

$$\ddot{x}+\frac{qV_0x}{ma^2}=0,$$

and

$$\ddot{y}-\frac{qV_0y}{ma^2}=0,$$

where the notation $\ddot{x}$ has the usual meaning of d^2x/dt^2 and similarly for y. If we change the sign of the potential function, the polarities of the $x-z$ and $y-z$ planes are reversed: $y-z$ becomes the focusing plane, and $x-z$ becomes the defocusing plane. We can obtain the desired focusing effect in both planes by introducing time-varying fields that produce alternating-gradient focusing.

B. Time-dependent electric-quadrupole focusing

(1) The electric field in the $y-z$ plane is $E_y=V_0y/a^2\sin\omega t$, and the equation of motion for y is

$$\ddot{y}-\frac{qV_0y\sin\omega t}{ma^2}=0,$$

which describes a periodic force with period $\tau=2\pi/\omega$.

(2) Guided by the Floquet theorem, you were to assume a simple trial solution

$$y=C[1+\epsilon\sin\omega t]\sin\Omega t. \tag{6}$$

To find whether the trial solution is approximately valid, differentiate the trial solution twice and neglect small terms of order $\epsilon\Omega/\omega$ and Ω^2/ω^2. This gives $\ddot{y}\cong-\epsilon\omega^2\sin\Omega t\sin\omega t$. Then, substitute this result and the solution for y into the equation of motion and solve for the flutter amplitude ϵ, to obtain in the lowest order of approximation

$$\epsilon\cong\frac{qV_0}{m\omega^2a^2}. \tag{7}$$

(3) To obtain an approximate expression for Ω, return to the equation of motion, and again substitute Eq. (6), the trial solution for y. This gives

$$\ddot{y}-\frac{qV_0C\sin\Omega t[1+\epsilon\sin\omega t]\sin\omega t}{ma^2}=0.$$

Averaging over the rf period τ and using the fact that $\overline{\sin^2\omega t}=1/2$, yields

$$\bar{\ddot{y}}-\frac{qV_0\epsilon C\sin\Omega t}{2ma^2}=0, \tag{8}$$

where the bar represents an average over the rf period. From Eq. (6) it follows that $\bar{y}=C \sin \Omega t$. Substituting this result into Eq. (8), gives

$$\ddot{\bar{y}}-\frac{qV_0\epsilon\bar{y}}{2ma^2}=0. \tag{9}$$

This result implies that the trajectory for the average motion is the solution of this simple harmonic oscillator equation and, therefore, that the squared frequency is $\Omega^2=qV_0\epsilon/2ma^2$. Substituting the result of Eq. (7) for ϵ yields

$$\Omega^2=\frac{1}{2}\left[\frac{qV_0}{m\omega a^2}\right]^2. \tag{10}$$

The frequency Ω is called the "betatron frequency" in usual accelerator terminology, and σ the corresponding phase advance per focusing period is given by

$$\sigma^2=\left[\frac{2\pi\Omega}{\omega}\right]^2=\frac{1}{2}\left[\frac{2\pi qV_0}{m\omega^2a^2}\right]^2=\frac{1}{8\pi^2}\left[\frac{qV_0\lambda^2}{mc^2a^2}\right]^2. \tag{11}$$

We find that the motion of the mean trajectory averaged over a rf period is the same as if a continuous linear restoring force were acting on the beam. This result illustrates the general principle of alternating gradient focusing, a technique used in almost every particle accelerator. In the RFQ the polarity varies as a function of time, rather than spatially as is the case in most particle accelerators. The mechanism of alternating gradient focusing can be understood simply by considering the example of two fixed quadrupole lenses of equal strength and opposite polarity in series along the beam axis.[3] If a particle that is off axis horizontally first enters the horizontally focusing lens, it will be bent toward the axis, and will then enter the second lens closer to the axis. Because of the smaller displacement in the second lens, which is horizontally defocusing, the outward force from the second lens is less than the inward force from the first lens, and there is a net focusing force. A particle that is off axis vertically will be defocused vertically in the first lens, and enters the second lens with an increased displacement. As a result it will be focused more strongly here than in the first lens. Again, the effect is a net focusing force. The RFQ is different only in that the polarity of the focusing gradient varies in time instead of in space.

C. Electrostatic quadrupole with axial potential

(1) To find the values of X and A that satisfy the boundary conditions at the pole tips, first evaluate the potential at $x=a$ and $y=0$ and equate it to $+V_0/2$. This gives $1=X+A$. From the condition that at $x=0$ and $y=ma$ the potential is $-V_0/2$, we find $-1=-Xm^2+A$. Solving these two simultaneous equations for A and X gives

$$X=\frac{2}{m^2+1}$$

and

$$A=\frac{m^2-1}{m^2+1}.$$

Therefore, with these values of X and A the potential function does satisfy the boundary conditions. When $m=1$, we obtain $A=0$ and $X=1$, which recovers the result of Eq. (2) for the time-dependent electric quadrupole in which the pole tips are the same distance apart.

Note that as m increases, A increases and X decreases, and the potential on axis is no longer zero, but is $U(0,0)=V_0A/2$. Thus we are able to control the on-axis potential by adjusting the parameter m. For this example the axial electric field is zero because the axial potential is constant, but the result suggests that if the displacements of the pole tips varied along the axial direction, the axial potential would also vary, and the axial electric field would be nonzero. This case is treated in problem 4.

(2) If the displacement of the x- and y-pole tips are interchanged to ma and a, respectively, but their voltages are kept the same, the result for $X(m)$ is unchanged, but the value of $A(m)$ changes sign. This is reasonable physically because the pole tip at potential $-V_0/2$ is now closer to the axis and has more influence on the axial potential than the pole tip at positive potential.

D. The radio frequency quadrupole

(1) At $x=a$, $y=0$, and $z=0$, the potential evaluated at the tip of the horizontal vane is

$$1=X+AI_0(ka),$$

and at $x=0$, $y=ma$, and $z=0$, the potential at the tip of the vertical vane is

$$-1=-X\,m^2+A\,I_0(kma).$$

Solving these two equations for X and A yields

$$X=\frac{I_0(ka)+I_0(kma)}{m^2I_0(ka)+I_0(kma)},$$

and

$$A=\frac{m^2-1}{m^2I_0(ka)+I_0(kma)}.$$

(2) It is necessary to deviate from the ideal equipotential pole surfaces to control the peak surface electric field and for ease in machining. However, the contours for horizontal pole tips in the $x-z$ plane at $y=0$ and for the vertical pole tips in the $y-z$ plane at $x=0$ can be made to correspond to the exact equipotential curves of the potential function. From the expression for the potential function, the x tip at $y=0$ is given by

$$1=\frac{X}{a^2}x^2+AI_0(kx)\cos kz$$

and the y tip at $x=0$ is given by

$$-1=-\frac{X}{a^2}y^2+A\,I_0(ky)\cos kz.$$

(3) The particle dynamics is determined from the electric-field components, which may now be obtained from the gradient of the potential function as $\mathbf{E}=-\nabla U$. The components are obtained in Cartesian coordinates (where $r=\sqrt{x^2+y^2}$) as

$$E_x=-\frac{XV_0}{a^2}x-\frac{kAV_0}{2}\frac{I_1(kr)x}{r}\cos kz,$$

$$E_y=\frac{XV_0}{a^2}y-\frac{kAV_0}{2}\frac{I_1(kr)y}{r}\cos kz,$$

$$E_z = \frac{kV_0A}{2} I_0(kr)\sin(kz),$$

where $I_0(kr)$ and $I_1(kr)$ are modified Bessel functions, and each field component is multiplied by $\sin(\omega t + \phi)$. We will usually be interested in the lowest order approximation for the Bessel functions, $I_0(\nu) \cong 1 + \nu^2/4$, and $I_1(\nu) \cong \nu/2$.

We call A the acceleration efficiency and X the focusing efficiency. When $m = 1$, the acceleration efficiency $A = 0$, the focusing efficiency $X = 1$, and the RFQ becomes a pure rf electric-quadrupole transport channel with no acceleration and transverse electric-field components only. As m increases, A increases and X decreases, and the longitudinal field component increases. The terms that contain X represent the electric quadrupole field, and XV_0/a^2 is the quadrupole gradient. If the amplitude m of the modulation is increased to raise the accelerating fields, the focusing fields are decreased.

(4) In the approximation that the radial position and the velocity of a particle within a period are constant, the energy gain ΔW for a synchronous particle with a velocity βc can be calculated. From $t = z/\beta c$ we obtain $\omega t = 2\pi z/\beta\lambda$. Substituting for ωt and integrating over the period to find the work done by the electric field on the particle, $E_z \sin(\omega t + \phi)$, gives

$$\Delta W = \frac{qkAV_0I_0(kr)}{2} \int_0^{\beta\lambda} \sin kz \sin(kz + \phi) dz$$

$$= \frac{\pi qAV_0I_0(kr)\cos\phi}{2}.$$

Thus the synchronous particle gains energy if $\cos\phi$ is positive. When $\phi = 0$, the electric fields are at the peak value, and the energy gain is maximum. In practice the phase is chosen off the peak because then there is longitudinal focusing.

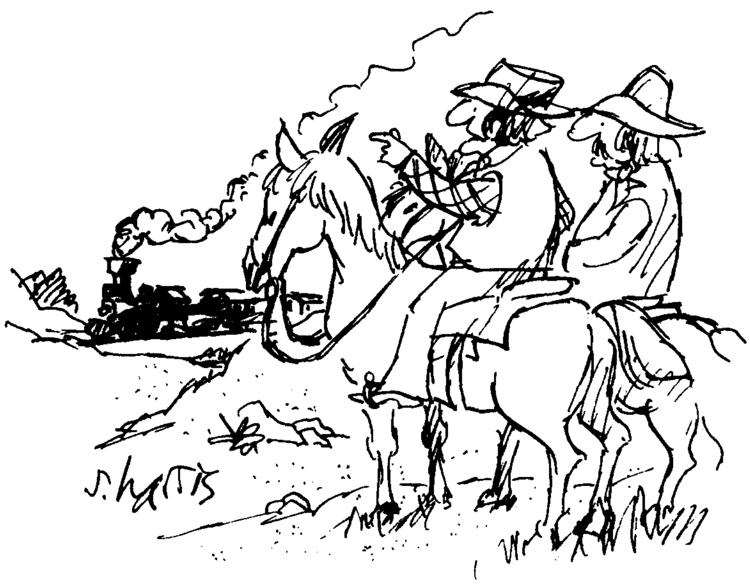
[1]I. M. Kapchinsky and V. A. Teplyakov, "Linear ion accelerator with spatially homogeneous strong focusing, "Pribory i Tekhnika Eksperimenta **2**, 19–22 (1970); English translation is Instrum Exp. Tech. No. **2**, 322–326 (1970).

[2]Richard H. Stokes and Thomas P. Wangler, "Radio frequency quadrupole accelerators and their applications," Annu. Rev. Nucl. Sci. **38**, 97–118 (1988).

[3]Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA 1964). Sec. 29-7 discusses alternating-gradient focusing.

[4]I. M. Kapchinsky, *Theory of Resonance Linear Accelerators*, (Harwood Academic, Chur, Switzerland, 1985). Section 2.3 discusses the Floquet theorem and Floquet functions.

[5]For example, see George Arfken, *Mathematical Methods for Physicists*, (Academic, New York, 1985), 3rd ed., p. 610.



"I love hearing that lonesome wail of the train whistle as the magnitude of the frequency of the wave changes due to the Doppler effect."