Electromagnetic Forces on Point Dipoles[1]

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Introduction

The forces exerted on classical electric and magnetic dipoles by externally applied electromagnetic fields are derived from first principles. It is found, in accordance with Penfield and Haus [1, pp. 214–216 and 244] and De Groot and Suttorp [2, pp. 195–196], that the force on a magnetic dipole is the same for a perfectly conducting electric-current (Amperian) model and for a magnetic-charge model of the magnetic dipole, provided the sources of the externally applied field lie outside the dipole. However, if the dipoles lie within the polarization densities of the externally applied field, the force they experience depends on the model chosen for the dipoles (and for the polarization densities of the externally applied fields). The difference between the forces on point Amperian and magnetic-charge magnetic dipoles within the polarization densities of an external field has been used to demonstrate experimentally that neutrons scattered by the fields in ferromagnetic materials act like Amperian rather than magnetic-charge magnetic dipoles [3], [4, p. 191].

Force on Amperian dipoles

Let us determine from first principles the electromagnetic force caused by external fields, $\mathbf{E}_{ext}(\mathbf{r},t)$ and $\mathbf{B}_{ext}(\mathbf{r},t)$, applied to an infinitesimal region V_b of "bound" charge and current characterized by electric and magnetic dipole moments $\mathbf{p}(t)$ and $\mathbf{m}(t)$. The total Lorentz force on the bound electric charge and current distribution, that is, on the Amperian electric and magnetic dipoles, is given by

$$\mathbf{F}_{el}^{d}(t) = \int\limits_{V_b} \{\rho_b(\mathbf{r},t)[\mathbf{E}_{ext}(\mathbf{r},t) + \mathbf{E}_b(\mathbf{r},t)] + \mathbf{J}_b(\mathbf{r},t) \times [\mathbf{B}_{ext}(\mathbf{r},t) + \mathbf{B}_b(\mathbf{r},t)]\} dV \quad (1)$$

where the integration over V_b covers the bound charge and current, ρ_b and $\mathbf{J}_b$. The internal fields $\mathbf{E}_b$ and $\mathbf{B}_b$ are produced by the bound charge and current. The bound charge and current may be induced in part or entirely by the externally applied fields. The applied fields $\mathbf{E}_{ext}$ and $\mathbf{B}_{ext}$ are produced by sources (ρ_{ext}, $\mathbf{J}_{ext}$, $\mathbf{P}_{ext}$, $\mathbf{M}_{ext}$) other than ρ_b and $\mathbf{J}_b$. First, consider the "external part" of the force

$$\mathbf{F}_{el}^{ext}(t) = \int\limits_{V_b} [\rho_b(\mathbf{r},t)\mathbf{E}_{ext}(\mathbf{r},t) + \mathbf{J}_b(\mathbf{r},t) \times \mathbf{B}_{ext}(\mathbf{r},t)]\, dV. \quad (2)$$

Let the bound charge-current be distributed about the point $\mathbf{r}_0$ and expand $\mathbf{E}_{ext}(\mathbf{r},t)$ and $\mathbf{B}_{ext}(\mathbf{r},t)$ in a power series about the point $\mathbf{r}_0$ to get

$$\mathbf{E}_{ext}(\mathbf{r},t) = \mathbf{E}_{ext}(\mathbf{r}_0,t) + (\mathbf{r}-\mathbf{r}_0)\cdot\nabla_0\mathbf{E}_{ext}(\mathbf{r}_0,t) + O(|\mathbf{r}-\mathbf{r}_0|^2) \quad (3)$$

$$\mathbf{B}_{ext}(\mathbf{r},t) = \mathbf{B}_{ext}(\mathbf{r}_0,t) + (\mathbf{r}-\mathbf{r}_0)\cdot\nabla_0\mathbf{B}_{ext}(\mathbf{r}_0,t) + O(|\mathbf{r}-\mathbf{r}_0|^2) \quad (4)$$

where ∇_0 denotes the del operator with respect to the $\mathbf{r}_0$ coordinates. To simplify the following derivation, we shall initially choose $\mathbf{r}_0 = 0$, that is, the origin of the

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coordinate system is chosen within V_b.

Substituting (3) and (4) into the integrand of (2), then using the results that the total bound charge is zero, $\int_{V_b} \rho_b(\mathbf{r},t)dV = 0$, and that

$$\int\limits_{V_b} \mathbf{J}_b(\mathbf{r},t)dV = -\int\limits_{V_b} [\nabla \cdot \mathbf{J}_b(\mathbf{r},t)]\mathbf{r}dV = \int\limits_{V_b} \frac{\partial \rho_b}{\partial t}(\mathbf{r},t)\mathbf{r}dV = \frac{d\mathbf{p}(t)}{dt} \tag{5}$$

which follows from the identity $\nabla \cdot (\mathbf{J}_b\mathbf{r}) = (\nabla \cdot \mathbf{J}_b)\mathbf{r} + \mathbf{J}_b$ and the definition of the electric dipole moment

$$\mathbf{p}(t) = \int\limits_{V_b} \rho_b(\mathbf{r},t)\mathbf{r}dV \tag{6}$$

one obtains for the direct external part of the force

$$\mathbf{F}_{el}^{ext}(t) = \mathbf{p}(t)\cdot\nabla\mathbf{E}_{ext}(0,t) + \frac{d\mathbf{p}(t)}{dt}\times\mathbf{B}_{ext}(0,t) + \int\limits_{V_b} \mathbf{J}_b(\mathbf{r},t)\times[\mathbf{r}\cdot\nabla_0\mathbf{B}_{ext}(0,t)]dV. \tag{7}$$

All higher order terms in (7) have been omitted because they vanish as the size of the region V_b in which the bound charge-current distribution is located approaches zero (assuming the multipole moments of higher order than dipole moments are zero). With the identities

$$\mathbf{J}_b \times (\mathbf{r}\cdot\nabla_0\mathbf{B}_{ext}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}} = -\hat{\mathbf{a}} \times (\mathbf{J}_b\mathbf{r})\cdot\nabla_0\mathbf{B}_{ext}\cdot\hat{\mathbf{a}} \tag{8}$$

$$\nabla \cdot (\mathbf{J}_b\mathbf{r}\mathbf{r}) = (\nabla\cdot\mathbf{J}_b)\mathbf{r}\mathbf{r} + \mathbf{J}_b\mathbf{r} + \mathbf{r}\mathbf{J}_b \tag{9}$$

$$\mathbf{J}_b(\mathbf{r}\cdot\nabla_0\mathbf{B}_{ext}) - \mathbf{r}(\mathbf{J}_b\cdot\nabla_0\mathbf{B}_{ext}) = (\mathbf{r}\times\mathbf{J}_b)\times\nabla_0\mathbf{B}_{ext} \tag{10}$$

the integral in (7) can be written as

$$\begin{aligned}\int\limits_{V_b} \mathbf{J}_b(\mathbf{r},t)\times[\mathbf{r}\cdot\nabla_0\mathbf{B}_{ext}(0,t)]dV &= \nabla_0\mathbf{B}_{ext}(0,t)\cdot\mathbf{m}(t)\\ &= \mathbf{m}(t)\cdot\nabla_0\mathbf{B}_{ext}(0,t) + \mathbf{m}(t)\times[\nabla_0\times\mathbf{B}_{ext}(0,t)]\end{aligned} \tag{11}$$

in which the Amperian magnetic dipole moment is defined as

$$\mathbf{m}(t) = \frac{1}{2}\int\limits_{V_b} \mathbf{r}\times\mathbf{J}_b(\mathbf{r},t)dV. \tag{12}$$

Inserting (11) into (7) gives the final expression for the direct external part of the force exerted on the point Amperian electric and magnetic dipoles

$$\begin{aligned}\mathbf{F}_{el}^{ext}(t) \;=\; & \mathbf{p}(t)\cdot\nabla_0\mathbf{E}_{ext}(\mathbf{r}_0,t) + \frac{d\mathbf{p}(t)}{dt}\times\mathbf{B}_{ext}(\mathbf{r}_0,t) + \mathbf{m}(t)\cdot\nabla_0\mathbf{B}_{ext}(\mathbf{r}_0,t)\\ & +\mathbf{m}(t)\times[\nabla_0\times\mathbf{B}_{ext}(\mathbf{r}_0,t)]\end{aligned} \tag{13}$$

where $\mathbf{r}_0$, which is the position of the point electric and magnetic dipoles with dipole moments $\mathbf{p}(t)$ and $\mathbf{m}(t)$, can be an arbitrary point in space.

The foregoing derivations allowed the bound electric charge and current densities, ρ_b and $\mathbf{J}_b$, to vary arbitrarily in space and time within their infinitesimal region as long as they could be characterized by electric and magnetic dipole moments $\mathbf{p}(t)$

and $\mathbf{m}(t)$. To the force in (13) must be added the internal force found from (1) as

$$\mathbf{F}_{el}^{int}(t) = \int_{V_b} [\rho_b(\mathbf{r},t)\mathbf{E}_b(\mathbf{r},t) + \mathbf{J}_b(\mathbf{r},t) \times \mathbf{B}_b(\mathbf{r},t)]\, dV. \tag{14}$$

The fields $\mathbf{E}_b(\mathbf{r},t)$ and $\mathbf{B}_b(\mathbf{r},t)$ produced by the bound charge and current $\rho_b(\mathbf{r},t)$ and $\mathbf{J}_b(\mathbf{r},t)$ vary rapidly (like ρ_b and $\mathbf{J}_b$) about the point $\mathbf{r}_0$ where the infinitesimal region V_b of bound charge and current is located. Therefore, $\mathbf{E}_b(\mathbf{r},t)$ and $\mathbf{B}_b(\mathbf{r},t)$ cannot be represented near $\mathbf{r}_0$ by two terms of a power series, as were $\mathbf{E}_{ext}(\mathbf{r},t)$ and $\mathbf{B}_{ext}(\mathbf{r},t)$ in (3) and (4).

In order to simplify the evaluation of (14), assume the Amperian electric dipole is formed by equal and opposite electric charges at the ends of a fixed infinitesimal nonconducting rod. The charges are stationary and the equal and opposite forces exerted by these charges on each other cancel. Thus, this static Amperian electric dipole will not contribute to the electromagnetic force in (14).

Assume the Amperian magnetic dipole consists of magnetostatic current on a perfectly conducting sphere placed in the external field ($\mathbf{E}_{ext}(\mathbf{r},t)$, $\mathbf{B}_{ext}(\mathbf{r},t)$), and take the Fourier transform of this time-domain external field to obtain the frequency-domain external field. At each frequency, the frequency-domain external field will induce surface charge and current densities on the sphere that can be determined exactly, along with their fields, from the Mie series solution [5, ch. 9]. An asymptotic expansion of the Mie series solution [6], as the radius of the sphere approaches zero about $\mathbf{r}_0$, reveals (after a considerable amount of algebra followed by transforming back to the time domain) that $\mathbf{F}_{el}^{int}$ for this sphere with static magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{F}_{el}^{int}(t) = -\epsilon\mu \frac{\partial}{\partial t}[\mathbf{m} \times \mathbf{E}_{ext}(\mathbf{r}_0,t)]. \tag{15}$$

Moreover, as the radius of the sphere approaches zero, the radius to which the induced fields extend approaches zero. In other words, the induced fields of the sphere produce the nonzero internal force (on the charge-current of the sphere) given in (15), but these induced fields are undetectable externally as the radius of the sphere approaches zero. Furthermore, the electric and magnetic dipole moments of the induced charge-current become zero as the radius of the sphere approaches zero.

Although (15) can be derived, as just outlined, from a perfectly conducting spherical model of a static Amperian magnetic dipole, Penfield and Haus also derive this term for a perfectly conducting current-loop model of an arbitrarily time varying Amperian magnetic dipole [1, pp. 214-216 and 244], [7]. Also, De Groot and Suttorp [2, pp.195-196] derive the term in (15) for an Amperian model consisting of charged point particles (electrons and nuclei) grouped into stable "atoms." If one abandons the idea of a passive physically realizable model (such as current on a perfect conductor or electrons orbiting a nucleus) for the Amperian magnetic dipole, and simply postulates that the electric current forming the magnetic dipole is constrained to flow in free space without the external field generating extra charge-current (for example, a battery connected to a highly resistive wire loop), then the terms in (15) would not arise [8].

Adding the internal force (15) to the direct external force in (13) produces the total force exerted by the fields ($\mathbf{E}_{ext}$, $\mathbf{B}_{ext}$) (directly, as well as indirectly through the induced current on the perfectly conducting model of the Amperian magnetic dipole)

$$\mathbf{F}_{el}^{d}(t) = \mathbf{p}\cdot\nabla_0\mathbf{E}_{ext}+\frac{d\mathbf{p}}{dt}\times\mathbf{B}_{ext}+\mathbf{m}\cdot\nabla_0\mathbf{B}_{ext}+\mathbf{m}\times(\nabla_0\times\mathbf{B}_{ext})-\epsilon\mu\frac{\partial}{\partial t}(\mathbf{m}\times\mathbf{E}_{ext}). \quad (16)$$

Force on magnetic-charge model of magnetic dipole

If the magnetic dipole moment is produced by magnetic charges rather than electric currents, the internal force in (15) vanishes (for the magnetic dipole formed by equal and opposite magnetic charges at the ends of a fixed rod), and the total Lorentz force is given by just the external part [9, sec. 10.1]

$$\mathbf{F}_{el}^{md}(t) = \int\limits_{V_b}(\rho_b\mathbf{E}_{ext} + \mu\mathbf{J}_b \times \mathbf{H}_{ext} + \rho_{mb}\mathbf{H}_{ext} - \epsilon\mathbf{J}_{mb} \times \mathbf{E}_{ext})dV \quad (17)$$

in which the magnetic dipole moment of the electric bound charge-current $(\rho_b, \mathbf{J}_b)$ is zero, and the electric dipole moment of the magnetic bound charge-current $(\rho_{mb}, \mathbf{J}_{mb})$ is zero. Evaluating the integral in (17), like we evaluated the integral in (2), yields

$$\mathbf{F}_{el}^{md}(t) = \mathbf{p}\cdot\nabla_0\mathbf{E}_{ext} + \mu\frac{d\mathbf{p}}{dt}\times\mathbf{H}_{ext} + \mu\mathbf{m}\cdot\nabla_0\mathbf{H}_{ext} - \epsilon\mu\frac{d\mathbf{m}}{dt}\times\mathbf{E}_{ext}. \quad (18)$$

Comparing (18) with (16) reveals that the force on point dipoles for Amperian and magnetic-charge models of magnetic polarization (and electric-charge (Amperian) electric polarization) is the same if the sources of the external field lie outside the point dipoles. However, the force, on point dipoles located within the sources of the external field, predicted by (16) for an Amperian model of magnetic polarization differs, in general, from the force predicted by (18) for a magnetic-charge model of magnetic polarization. (In both cases the electric polarization is created by electric charge.) In other words, the force needed to keep the dipoles from moving, when they lie within the sources of an external field, depends on the model of the dipoles.

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