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- *How one of the most useful tools of actuaries came to be named after the renowned and elegant international capital of gambling.*

# Monte Carlo

By Thomas N. Herzog



MONTE CARLO METHODS may be broadly defined as that area of applied mathematics concerned with experiments using random numbers. Monte Carlo methods are useful in solving a wide range of financial and insurance problems, both deterministic and stochastic, which can't easily be solved using analytic methods.

The actuarial applications of this technique include: (1) model offices of life insurance and annuities; (2) analysis of investment and asset allocation strategies (e.g., bond call properties); (3) asset-liability management; (4) product design and pricing studies; (5) dynamic solvency testing of insurance company (or pension fund) solidity and resilience; (6) collective risk models in general; and (7) aggregate loss distributions in particular.

Today, the nearly universal availability of high-speed electronic computers makes Monte Carlo methods a cheap and effective means of solving a wide variety of complex, practical problems. We begin by describing the solution to a sample problem as illustrated in the figure at left.

To find the area under an irregular curve,  $f(x)$ , we surround the shaded area with a rectangle and instruct the computer to generate a large number of random points within the rectangle. We then count the number of points lying under the curve and divide it by the total number of points generated. This gives us the proportion of points lying within the region of interest. We obtain our desired result by multiplying this proportion by the area of the rectangle. To increase the reliability of our results, we generate a larger number of random points.

We investigate three issues:

- How does the computer generate the random numbers?
- Why is the name of the city of Monte Carlo associated with these procedures?
- What sorts of problems are amenable to such procedures?

As science is the work of individuals, we relate our tale through the work of six such individuals: John von Neumann; Stanislaw Ulam; Marin Mersenne; Leonhard Euler; William Sealy Gossett; and Georges Louis Leclerc, Comte de Buffon.

John von Neumann



Stanislaw Ulam



Marin Mersenne



Leonhard Euler



William Sealy Gossett



Georges Louis Leclerc, Comte de Buffon



# in the Game

Monte Carlo methods go back to at least 1777 when Georges Louis Leclerc, Comte de Buffon, published his *Essai d'arithmétique morale*. In that work, Buffon described an experiment that would later be called “Buffon’s needle problem.”

In 1820, Pierre-Simon, Marquis de Laplace, suggested that Buffon's problem could be employed to obtain an estimate of  $\pi$ .

Gossett was working on agricultural experiments being performed to improve the yield of the crops used in the production of Guinness beer. Because the brewing company wanted to keep its research confidential for competitive purposes, Gossett had to publish his research under an alias and also sanitize the work. So Gossett, writing under the pseudonym of “Student,” described his experiments with random numbers that he used to help determine the sampling distribution of (1) the correlation coefficient and (2) the Student’s “t” distribution.

But the origins of the Monte Carlo method go back even further.

Marin Mersenne was born on Sept. 8, 1588, in Oize Maine, France. He studied first at the College of Mans, and then at the Jesuit College in La Flèche from 1604 to 1609. Mersenne went on to study theology at the Sorbonne from 1609 to 1611, at which time he joined the religious order of the Minims. He subsequently completed additional theology study within the order at Nigeon and Meaux.

Mersenne had frequent contact with other eminent European mathematicians, and, in the absence of mathematical journals, he facilitated the communication of mathematical knowledge among them.

 $2^p - 1$ 

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Mersenne defended a number of scientists/philosophers, such as Descartes and Galileo, whom the established religious hierarchy had criticized. He published a number of important works, including *Cogitata Physico-Mathematica* in 1644. Mersenne died on Sept. 1, 1648, in Paris.

Born in Basel, Switzerland, in 1707, Leonhard Euler was the son and grandson of Protestant ministers. Although Euler's father taught him some elementary mathematics as well as some other subjects, his father wanted Euler to follow him into the ministry. At age 14, Euler was sent to the University of Basel to embark on such a course of study.

Euler completed his mathematics studies in 1726 and went on to a highly distinguished career in various prestigious academic positions in Europe, including ones in Berlin and St. Petersburg. During his 25 years in Berlin, Euler wrote about 380 articles. After his death in 1783, the St. Petersburg Academy continued to publish his unpublished work for almost 50 years. One of his discoveries of 1772 would have an important practical application almost 200 years later.

John von Neumann was born in Budapest, Hungary, in 1903, into an affluent family. He entered the Lutheran gymnasium in 1911 and continued his studies there until 1921. Von Neumann demonstrated mathematical genius at an early age, but his father felt his son could make a better living as a chemist than as a mathematician, and von Neumann went off to the University of Berlin to study chemistry. He remained there until 1923 when





he moved to Zurich.

Although he didn't attend any mathematics classes, he got high scores on the mathematics examinations at the University of Budapest. In 1926, von Neumann received both a Ph.D. in chemistry from the Technische Hochschule in Zurich and a Ph.D. in mathematics from the University of Budapest.

He was a Rockefeller fellow at the University of Götting from 1925 to 1926, taught mathematics at the University of Berlin from 1927 to 1929 as well as at the University of Hamburg during 1929, took a position as a visiting professor of mathematical physics at Princeton University in 1930, and was named a professor there in 1931. In 1933, he became one of the six original mathematicians at the newly created Institute for Advanced Studies at Princeton. (Albert Einstein was one of the other mathematicians.) Von Neumann was a member of the staff of the institute for the remainder of his life.

Early in his career, his research interests included mathematical logic, set theory, measure theory, and quantum mechanics. During and after World War II, von Neumann was a consultant to the U.S. military where, among other things, he was involved with the development of both the atomic and hydrogen bombs. He was, along with Oskar Morgenstern, a co-founder of game theory.

Von Neumann was also one of the pioneers in computer science and, after World War II, worked on the development of large-scale electronic computers. He died an early death in 1957, perhaps in part because of radiation exposure during tests of nuclear weapons.

## Ulam

Stanislaw Ulam was born in Lemberg, Poland, in 1909. (Lemberg is now known as Lvov and is part of the Ukraine.) He enrolled in the gymnasium in Lvov at the age of 10 and had an early interest in astronomy and then physics. At age 14, he began to study mathematics texts in order to enhance his understanding of Einstein's special theory of relativity.

In 1927, he began his studies at the Polytechnic Institute in Lvov, and obtained his doctorate there in 1933. In 1935, Ulam visited the Institute for Advanced Studies at Princeton

University at the invitation of John von Neumann. There, he met the mathematician G.D. Birkhoff, who invited Ulam to teach at Harvard University.

From Harvard, he went to the University of Wisconsin in 1940 as an assistant professor in the Department of Mathematics. Ulam became a U.S. citizen in 1943, the same year von Neumann invited him to take part in some work on secret weapons for the U.S. government at the Los Alamos National Laboratory in New Mexico. In 1976, Ulam related how he and von Neumann arranged to meet

*"... in Chicago in some railroad station to learn a little bit more about it. I went there and he could not tell me where he was going. There were two guys, sort of guards, looking like gorillas, with him. He discussed with me some mathematics, some interesting physics, and the importance of this work. And that was Los Alamos at the very start."*

The work was the Manhattan Project—the building of the first atomic bomb. After World War II, many of the physicists and mathematicians from the Manhattan Project left Los Alamos and took positions at various universities and research institutes. Ulam accepted an associate professorship in mathematics at the University of Southern California. According to Richard Rhodes, writing in *The Making of the Atomic Bomb*:

*"Shortly after his arrival there, he [Ulam] became seriously ill and spent some time on medical leave from the University. Resting at home during his extended recovery, Ulam amused himself playing solitaire. Sensitivity to patterns was part of his gift. He realized that he could estimate how a game would turn out if he laid down a few cards and then noted what proportion of his tries were successful, rather than attempting to work out all of the combinations in his head. (Here Ulam was thinking about Canfield, or other versions of solitaire where the skill of the player is not important.) 'It occurred to me,' he remembers, 'that this could be equally true of all processes involving branching of events.' Fission with its exponential spread of reactions was a branching process; so would the propagation of thermonuclear reaction be. 'At each stage of the [fission] process, there are many possibilities determining the fate of the neutron. It can scatter to one angle, change its velocity, be absorbed, or produce more neutrons by a fission of the target nucleus, and so on.' Instead of trying to derive the expected outcomes of these processes with complex mathematics, Ulam saw that it would be possible to follow a few thousand individual sample particles, selecting a range for each particle's fate at each step of the way by throwing in a random number, and take the outcomes as an approximate answer—a useful result. This iterative process was something a computer could do."*

In April of 1946, Ulam was invited to return to Los Alamos and subsequently told von Neumann about his solitaire dis-

covery. Ulam and von Neumann then worked out the mathematics in tandem and named the procedure the Monte Carlo method.

The name itself was suggested by Nicholas C. Metropolis, a physicist at Los Alamos. According to Ulam, "It was named for Monte Carlo because of the element of chance, the production of random numbers with which to play the suitable games." Ulam stated that "the name Monte Carlo contributed very much to the popularization of this procedure."

Von Neumann had access to the University of Pennsylvania's Electronic Numerical Integrator and Computer (ENIAC), a multipurpose electronic computer completed in 1945. The ENIAC was the first digital computer to be controlled by vacuum tubes. The ENIAC was housed in a 30'-by-50' room, weighed 30 tons, and had 19,000 vacuum tubes. Von Neumann used the ENIAC to generate random numbers, thereby successfully achieving the first computer application of a Monte Carlo method.

### The Middle-Square Method

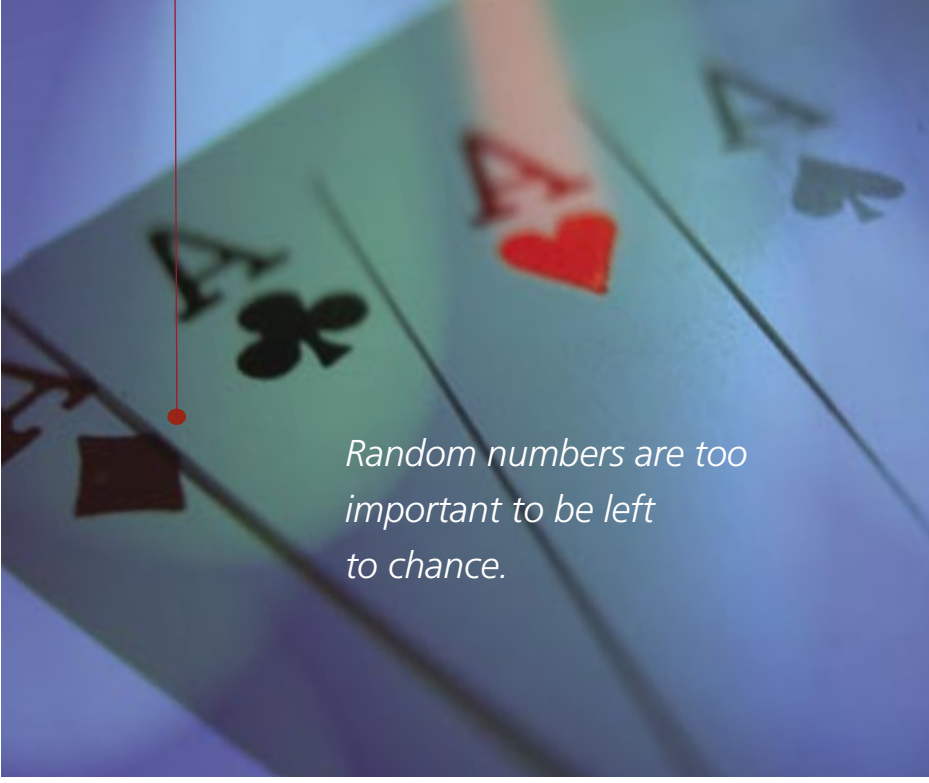
The generation of random numbers is frequently a key component in the applications of Monte Carlo methods. Unfortunately, there's not a single scheme for generating random numbers that's optimal for every possible application.

Nevertheless, some random generators are clearly better than others. Two individuals—Marin Mersenne and Leonhard Euler—facilitated the development of a modern random number generator known as GGL that was developed by IBM. Their work was completed about 200 years before its first widely publicized, practical application.

In 1951, von Neumann proposed one of the first schemes for generating random numbers on an electronic computer. This artificial procedure, called the *middle-square method*, generated random numbers  $x_0, x_1, \dots$ , each composed of  $n$  or  $n+1$  digits. This algorithm starts with a number,  $x_0$ ,  $n$  digits long, squares it, and then sets  $x_1$  to be the middle  $n$  or  $n+1$  digits of  $x_0^2$ . The algorithm continues with  $x_2$  being the middle  $n$  or  $n+1$  digits of  $x_1^2$ , and so on.

For example, if we choose  $x_0 = 157$ , a number having three digits, then  $x_1 = 464$  because the middle three digits of  $(157)^2 = 24,649$  are 464. Similarly, because  $(464)^2 = 215,296$ , we obtain  $x_2 = 1,529$ .

Unfortunately, this method has been found to be a poor source of random numbers, flawed by imprudent choices of the starting value  $x_0$ . Starting with the four-digit number 3,792 and squaring it, for example, we obtain the number 14,379,264 with the middle four digits being 3,792, the same four-digit number with which we started.



*Random numbers are too important to be left to chance.*

### IBM'S Random-Number Generator

To do better than the middle-square method, in the late 1960s IBM developed the following random-number generator known as GGL:

$$X_{n+1} = 16,807 \cdot X_n \bmod 2^{31} - 1$$

which means that  $X_{n+1}$  is the remainder when  $(16,807 \cdot X_n)$  is divided by  $2^{31} - 1$ . If the user doesn't specify an initial value,  $X_0$ , for this random-number generator, it's assumed to be 16,807.

GGL has a cycle length of  $2^{31} - 2$  or approximately 2 billion numbers. This is the maximum possible length. Before its implementation, GGL passed a wide range of statistical tests of randomness. Moreover, GGL is still the random number generator employed as the "?" operator in IBM's version of the APL computer language.

Because IBM had 31 bits available for computation in the 32-bit general register of the IBM System/360 Computer (one bit is a sign bit), according to the mathematical theory, the researchers needed to find the smallest prime number less than  $2^{31}$  in order to attain the maximal cycle length. Amazingly, the crucial piece of the puzzle goes back to the conjecture of Mersenne in the 1600s that  $2^{31} - 1$  is indeed a prime number and its resolution by Euler in 1772—a result Euler obtained after a number of unsuccessful attempts in prior years.

So that's how the Monte Carlo method developed and how it got its name. We have presented two random-number generators including one—GGL—that was formulated in the late 1960s and is still widely used today. No single random number generator is optimal for every possible application. Random numbers are too important to be left to chance. ☒

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