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How to Define Complexity in Physics, and Why

Various notions of complexity are listed and discussed. The advantage of having a definition of complexity that is rigorous and yet in accord with intuitive notions is that it allows certain complexity-related questions in statistical physics and the theory of computation to be posed well enough to be amenable to proof or refutation.

INTRODUCTION

Natural irreversible processes are nowadays thought to have a propensity for self-organization—the spontaneous generation of complexity (Figure 1). One may attempt to understand the origin of complexity in several ways. One can attempt to elucidate the actual course of galactic, solar, terrestrial, biological, and even cultural evolution. One can attempt to make progress on epistemological questions such as the anthropic principle[3]—the ways in which the complexity of the universe is conditioned by the existence of sentient observers—and the question often raised in connection with interpretations of quantum mechanics of what, if any, distinction

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science should make between the world that did happen and the possible worlds that might have happened. One can seek a cosmological "theory of everything" without which it would seem no truly general theory of natural history can be built. Finally, at an intermediate level of humility, one can attempt to discover general principles governing the creation and destruction of complexity in the standard mathematical models of many-body systems, e.g., stochastic cellular automata such as the Ising model, and partial differential equations such as those of hydrodynamics or chemical reaction-diffusion. An important part of this latter endeavor is the formulation of suitable definitions of complexity: definitions that on the one hand adequately capture intuitive notions of complexity, and on the other hand are sufficiently objective and mathematical to prove theorems about. Below we list and comment on several candidates for a complexity measure in physics, advocating one, "logical depth," as most suitable for the development of a general theory of complexity in many-body systems. Further details can be found in Bennett.[6]

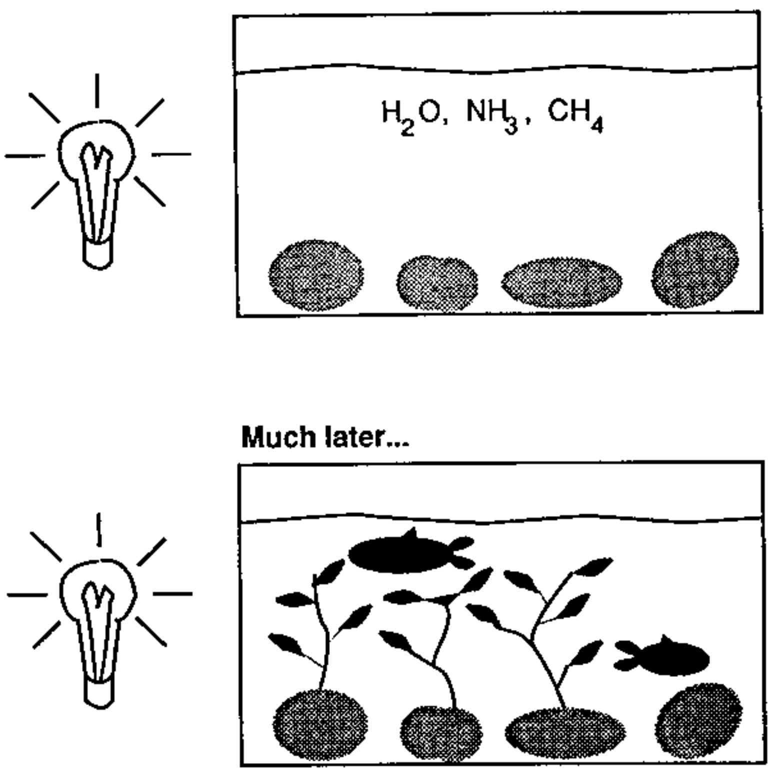

FIGURE 1 What is complexity? What causes it to increase? Is there a limit to its increase?

HOW: CANDIDATES FOR A SATISFACTORY FORMAL MEASURE OF COMPLEXITY

LIFE-LIKE PROPERTIES

Life-like properties (e.g., growth, reproduction, adaptation) are very hard to define rigorously, and also are too dependent on function, as opposed to structure. Intuitively, a dead human body is still complex, though it is functionally inert.

THERMODYNAMIC POTENTIALS

Thermodynamic potentials (entropy, free energy) measure a system's capacity for irreversible change, but do not agree with intuitive notions of complexity. For example, a bottle of sterile nutrient solution (Figure 2) has higher free energy, but lower subjective complexity, than the bacterial culture it would turn into if inocculated with a single bacterium. The rapid growth of bacteria following introduction of a seed bacterium is a thermodynamically irreversible process analogous to crystalization of a supersaturated solution following introduction of a seed crystal. Even without the seed either of these processes is vastly more probable than its reverse: spontaneous melting of crystal into supersaturated solution, or transformation of bacteria into high-free-energy nutrient. The unlikelihood of a bottle of sterile nutrient transforming itself into bacteria is therefore not a manifestation of the second law, but rather of a putative new "slow growth" law that complexity, however defined, ought to obey: complexity ought not to increase quickly, except with low probability, but can increase slowly, e.g., over geological time as suggested in Figure 1.

COMPUTATIONAL UNIVERSALITY

The ability of a system to be programmed through its initial conditions to simulate any digital computation. Computational universality, while it is an eminently mathematical property, is still too functional to be a good measure of complexity of physical states: it does not distinguish between a system capable of complex behavior and one in which the complex behavior has actually occurred. As a concrete example, it is known that classical billiard balls,[10] moving in a simple periodic potential, can be prepared in an initial condition to perform any computation; but if such a special initial condition has not been prepared, or if it has been prepared but the computation has not yet been performed, then the billiard ball configuration does not deserve to be called complex. Much can be said about the theory of universal computers; here we note that their existence implies that the input-output relation of any one of them is a microcosm of all of deductive logic, and in particular of all axiomatizable physical theories; moreover the existence of *efficiently* universal computers, which can simulate other computers with at most additive increase in program size and typically polynomial increase in execution time, allows the development of nearly machine-independent (and thus authoritative and absolute) theories of algorithmic information and computational time/space complexity.

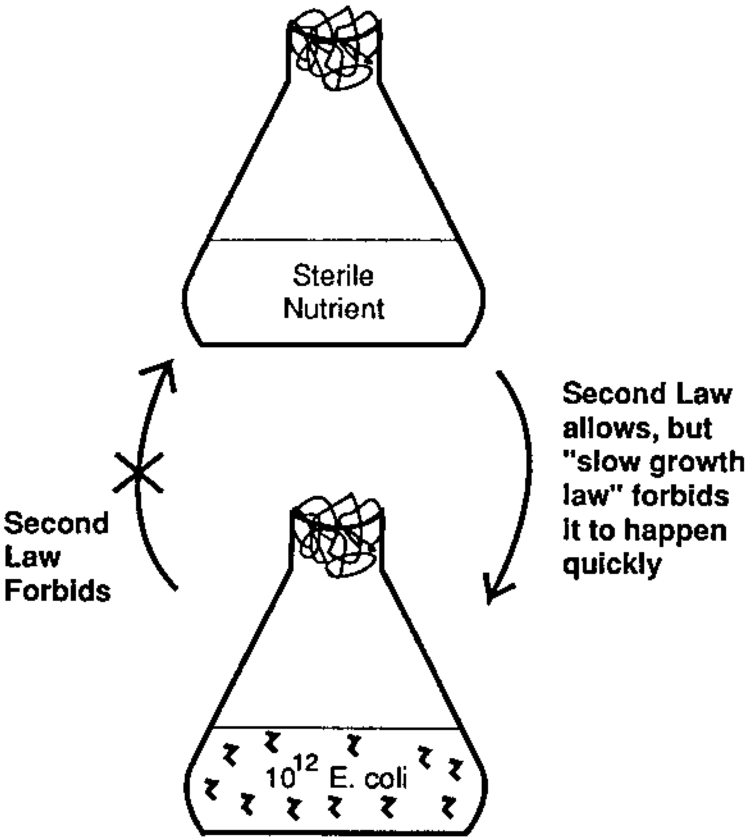

FIGURE 2 Complexity is not a thermodynamic potential like free energy. The second law allows a bottle of sterile nutrient solution (high free energy, low complexity) to turn into a bottle of bacteria (lower free energy, higher complexity), but a putative "slow growth law" forbids this to happen quickly, except with low probability.

COMPUTATIONAL TIME/SPACE COMPLEXITY

Computational time/space complexity is the asymptotic difficulty (e.g., polynomial vs. exponential time in the length of its argument) of computing a function.[13] By diagonal methods analogous to those used to show the existence of uncomputable functions, one can construct arbitrarily hard-to-compute computable functions. It is not immediately evident how a measure of the complexity of *functions* can be applied to *states* of physical models.

ALGORITHMIC INFORMATION

Algorithmic Information (also called Algorithmic Entropy or Solomonoff-Kolmogorov-Chaitin Complexity) is the size in bits of the most concise universal computer program to generate the object in question.[1,8,9,14,19,20] Algorithmic entropy is closely related to statistically defined entropy, the statistical entropy of an ensemble being, for any concisely describable ensemble, very nearly equal to the ensemble average of the algorithmic entropy of its members; but for this reason algorithmic entropy corresponds intuitively to randomness rather than to complexity. Just as the intuitively complex human body is intermediate in entropy between a crystal and a gas, so an intuitively complex genome or literary text is intermediate in algorithmic entropy between a random sequence and a perfectly orderly one.

LONG-RANGE ORDER

Long-Range Order, the existence of statistical correlations between arbitrarily remote parts of a body, is an unsatisfactory complexity measure, because it is present in such intuitively simple objects such as perfect crystals.

LONG-RANGE MUTUAL INFORMATION

Long-Range Mutual Information (Remote Non-Additive Entropy): is the amount by which the joint entropy of two remote parts of a body exceeds the sum of their individual entropies (Figure 3). In a body with long-range order it measures the amount, rather than the range, of correlations. Remote mutual information arises for rather different reasons in equilibrium and non-equilibrium systems, and much more of it is typically present in the latter.[6] In equilibrium systems, remote non-additivity of the entropy is at most a few dozen bits and is associated with the order parameters, e.g., magnetic or crystalline order in a solid. Correlations between remote parts of such a body are propagated via intervening portions of the body sharing the same value of the order parameter. By contrast, in nonequilibrium systems, much larger amounts of non-additive entropy may be present, and the correlations need not be propagated via the intervening medium. Thus the contents of two newspaper dispensers in the same city is typically highly correlated, but this correlation is not mediated by the state of the intervening air (except for weather news). Rather it reflects each newspaper's descent from a common causal origin in the past. Similar correlations exist between genomes and organisms in the biosphere, reflecting the shared frozen accidents of evolution. This sort of long-range mutual information, not mediated by the intervening medium, is an attractive complexity measure in many respects, but it fails to obey the putative slow-growth law mentioned above: quite trivial processes of randomization and redistribution, for example smashing a piece of glass and stirring up the pieces, or replicating and stirring a batch of random meaningless DNA, generate enormous amounts of remote non-additive entropy very quickly.

LOGICAL DEPTH

Logical Depth = Execution time required to generate the object in question by a near-incompressible universal computer program, i.e., one not itself computable as output of a significantly more concise program. Logical depth computerizes the Occam's razor paradigm, with programs representing hypotheses, outputs representing phenomena, and considers a hypothesis plausible only if it cannot be reduced to a simpler (more concise) hypothesis. Logically deep objects, in other words, contain internal evidence of having been the result of a long computation or slow-to-simulate dynamical process and could not plausibly have originated otherwise. Logical depth satisfies the slow-growth law by construction.

THERMODYNAMIC DEPTH

The amount of entropy produced during a state's actual evolution has been proposed as a measure of complexity by Lloyd and Pagels.[16] Thermodynamic depth can be very system-dependent: some systems arrive at very trivial states through much dissipation; others at very nontrivial states with little dissipation.

SELF-SIMILAR STRUCTURES AND CHAOTIC DYNAMICS

Self-similar structures are striking to look at, and some intuitively complex entities are self-similar or at least hierarchical in structure or function; but others are not. Moreover, some self-similar structures are rapidly computable, e.g., by deterministic cellular automaton rules. With regard to chaotic dynamics, Wolfram[18] distinguished between "homoplectic" processes which generate macroscopically random behavior by amplifing the noise in their initial and boundary conditions, and a more conjectural "autoplectic" type of processes which would generate macroscopically pseudorandom behavior autonomously in the absence of noise, and in the presence of noise, would persist in reproducing the same pseudorandom sequence despite the noise. Such a noise-resistant process would have the possibility of evolving toward a deep state, containing internal evidence of a long history. A homoplectic processes, on the other hand, should produce only shallow states, containing evidence of that portion of the history recent enough not to have been swamped by dynamically amplified environmental noise.

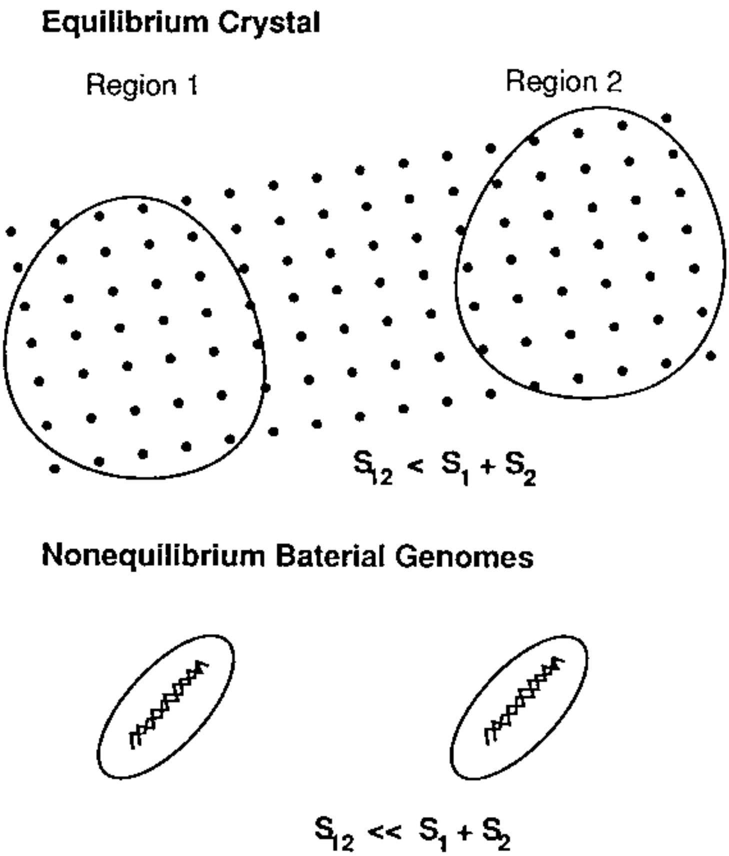

FIGURE 3 Remote Non-Additive Entropy. (a) Entropy of remote parts of an equilibrium crystal is non-additive by a few dozen bits due to correlations mediated by the order parameters of the intervening medium. (b) Entropy of two bacteria is non-additive by many thousands of bits due not to the intervening medium but to frozen accidents of a common evolutionary history.

WHY: USEFULNESS OF A FORMAL MEASURE OF COMPLEXITY

Aside from their non-specific usefulness in clarifying intuition, formal measures of complexity such as logical depth, as well as measures of randomness and correlation (e.g., algorithmic entropy and remote mutual information) raise a number of potentially decidable issues in statistical physics and the theory of computation.

THEORY OF COMPUTATION

The conjectured inequality of the complexity classes P and $PSPACE$ is a necessary condition, and the stronger conjecture of the existence of "one-way" functions[7,15] is a sufficient condition, for certain very idealized physical models (e.g., billiard balls) to generate logical depth efficiently.

COMPUTATIONALLY UNIVERSAL MODEL SYSTEMS

Which model systems in statistical mechanics are computationally universal? The billiard-ball model, consisting of hard spheres colliding with one another and with a periodic array of fixed mirrors in two dimensions, is computationally universal on a dynamically unstable set of trajectories measure zero. In this model, the number of degrees of freedom is proportional to the space requirement of the computation, since each billiard ball encodes one bit. Probably the mirrors could be replaced by a periodic wind of additional balls, moving in the third dimension so that one "wind" ball crosses the plane of computation at time and location of each potential mirror collision, and transfers the same momentum as the mirror would have done. This mirrorless model would have a number of degrees of freedom proportional to the time-space product of the computation being simulated. One might also ask whether a dynamical system with a fixed number of degrees of freedom, perhaps some version of the three-body problem, might be computationally universal. Such a model, if it exists, would not be expected to remain computationally universal in the presence of noise.

ERROR-CORRECTING COMPUTATION

What collective phenomena suffice to allow error-correcting computation and/or the generation of complexity to proceed despite the locally destructive effects of noise? In particular, how does dissipation favor the generation and maintenance of complexity in noisy systems?

- Dissipation allows error-correction, a many-to-one mapping in phase space.

- Dissipative systems are exempt from the Gibbs phase rule. In typical d-dimensional equilibrium systems with short-ranged interactions, barring symmetries or accidental degeneracy of parameters such as occurs on a coexistence line, there is a unique thermodynamic phase of lowest free energy.[4] This renders equilibrium systems ergodic and unable to store information reliably in the presence of "hostile" (i.e., symmetry-breaking) noise. Analogous dissipative systems, because they have no defined free energy in d dimensions, are exempt from this rule. A $(d+1)$-dimensional free energy can be defined, but varying the parameters of the d-dimensional model does not in general destabilize one phase relative to another.
- What other properties besides irreversibility does a system need to take advantage of the exemption from Gibbs phase rule? In general the problem is to correct erroneous regions, in which the data or computation locally differs from that originally stored or programmed into the system. These regions, which may be of any finite size, arise spontaneously due to noise and to subsequent propagation of errors through the system's normal dynamics. Local majority voting over a symmetric neighborhood, as in the Ising model at low temperature, is insufficient to suppress islands when the noise favors their growth. Instead of true stability, one has a metastable situation in which small islands are suppressed by surface tension, but large islands grow. Two methods are known for achieving absolute stability in the presence of symmetry-breaking noise.

Anisotropic Voting Rules[4,12,17] in two or more dimensions contrive to shrink arbitrarily large islands by differential motion of their boundaries. The rule is such that any island, while it may grow in some directions, shrinks in others; eventually the island becomes surrounded by shrinking facets only and disappears (Figure 4). The requisite anisotropy need not be present initially, but may arise through spontaneous symmetry breaking.

Hierarchical Voting Rules.[11] These complex rules, in one or more dimensions, correct errors by a programmed hierarchy of blockwise majority voting. The complexity arises from the need of the rule to maintain the hierarchical structure, which exists only in software.

SELF-ORGANIZATION

Is "self-organization," the spontaneous increase of complexity, an asymptotically qualitative phenomenon like phase transitions? In other words, are there reasonable models whose complexity, starting from a simple uniform initial state, not only spontaneously increases, but does so without bound in the limit of infinite space and time? Adopting logical depth as the criterion of complexity, this would mean that for arbitrarily large times t most parts of the system at time t would

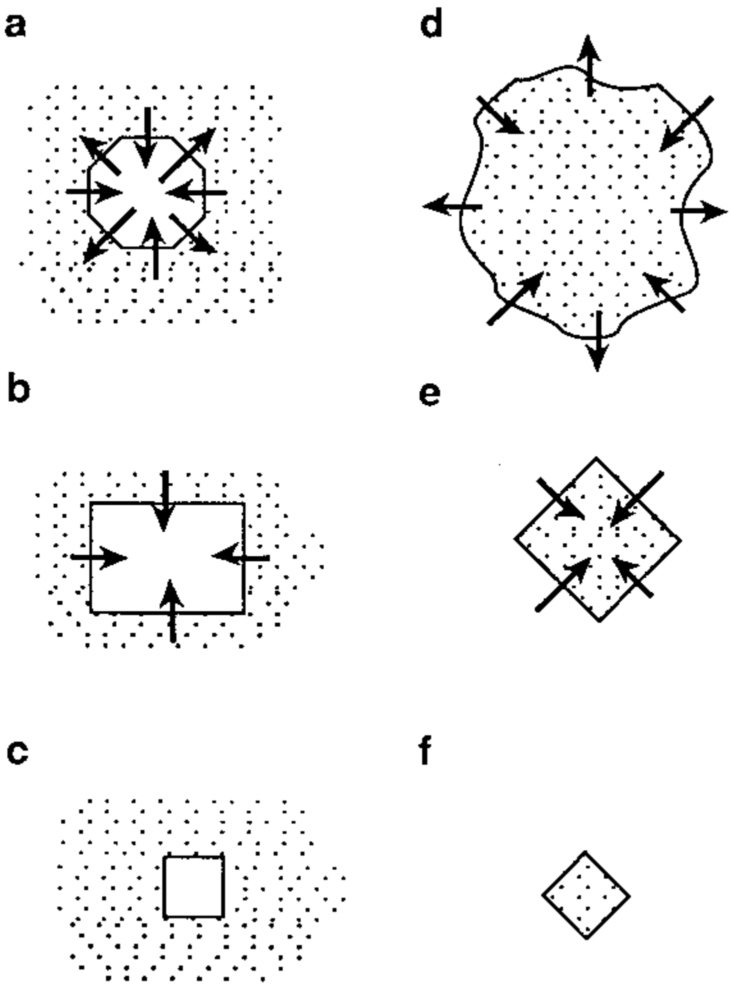

FIGURE 4 Anisotropic Voting Rules stabilize information against symmetry-breaking noise. It is not difficult to find irreversible voting models in which the growth velocity of a phase changes sign depending on boundary orientation (this is impossible in reversible models, where growth must always favor the phase of lowest bulk free energy). Here we show the fate of islands in an irreversible two-phase system in which growth favors one phase (stippled) at diagonal boundaries and the other phase (clear) at rectilinear boundaries. (a-c) An island of the clear phase becomes square and disappears. Similarly (d-f) an island of the stippled phase becomes diamond-shaped and disappears. Small perturbations of the noise perturb the boundary velocities slightly but leave the system still able to suppress arbitrarily large fluctuations of either phase.

contain structures that could not plausibly have been generated in time much less than t. A positive answer to this question would not explain the history of our finite world, but would suggest that its quantitative complexity can be legitimately viewed as an approximation to a well-defined property of infinite systems. On the other hand, a negative answer would suggest that our world should be compared to chemical reaction-diffusion systems that self-organize on a macroscopic but finite scale, or to hydrodynamic systems that self-organize on a scale determined by their boundary conditions, and that the observed complexity of our world may not be "spontaneous" but rather heavily conditioned by the anthropic requirement that it produce observers.

EQUILIBRIUM SYSTEMS

Which equilibrium systems (e.g., spin glasses, quasicrystals) have computationally complex ground states?

DISSIPATIVE PROCESSES

Do dissipative processes such as turbulence, that are not explicitly genetic or computational, still generate large amounts of remote non-additive entropy? Do they generate logical depth? Does a waterfall contain objective evidence, maintained despite environmental noise, of a nontrivial dynamical history leading to its present state, or is there no objective difference between a day-old waterfall and a year-old one? See Ahlers and Walden[1] for evidence of fairly long-term pseudorandom behavior near the onset of convective turbulence.

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