Copula Methods vs
Canonical Multivariate Distributions:
the multivariate Student T distribution with
general degrees of freedom

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In mathematical finance and other applications of statistics, the computation of expectations is often taken over a multi-dimensional probability distribution where there is no clear multivariate distribution. Copula theory has become increasingly popular as a means of gluing marginals together to circumvent this difficulty. There is then the issue of reconciling the distributions implied by various choices of copula and marginal with candidates for the *canonical* multivariate distribution when such candidates become available. This article looks at the copulae and candidate multivariate distributions for a general multivariate Student's T distribution when the marginals do not *necessarily* have the same degrees of freedom. We discuss the *grouped T copula* proposed recently by Demarta and McNeil, and Daul *et al* and other options, including one based on our own generalization of recent work by Jones, and a further proposal of our own. We compare these with the meta-elliptical distributions proposed as the canonical multivariate distribution by Fang *et al*. We argue that the natural appearance of *independence* in the zero-dependence case should take priority over preserving the elliptical structure commonplace in multivariate distribution theory. We are able to give several detailed and explicit representations for the bivariate case. For the bivariate case where one distribution is Normal we argue that there is indeed a *canonical* bivariate Student-Normal distribution with a naturally associated copula that arises simultaneously from several of the copula methods, and an elegant tractable density is available. For the Student-Student case there appears to be some genuine choice as to the canonical distribution, though the requirement of independence in the zero-correlation case appears to constrain us to just one choice. We also briefly discuss the inclusion of correlation data relevant to calibration.

Key Words: T Copula; Student Copula; bivariate Student; multivariate Student; degrees of freedom; elliptical; independence; correlation; dependence; Pearson; Spearman; Kendall. AMS Classifications: 60E05, 62E15, 62H20, 60-08

1. Introduction

The computation of quantities of interest in mathematical finance often involves expectations taking over complicated multivariate distributions. Such distributions are of interest for their intrinsic mathematical properties as well as for their use in applications of statistics generally. In mathematical finance one often has an idea as to the nature of marginal distributions. This may be analytical or numerical. Then one has the task of characterizing the links between the marginal distributions. The

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theory of copulas allows the links to be characterized separately from the marginals and has become a powerful tool. See for example the book by Cherubini *et al* [7].

The widespread use of copula techniques has not been universally embraced, to say the least. A critical and entertaining discussion has been given by Mikosch [28], prompting a vigorous defence by Genest and Rémillard [18]. I am not going to take sides on this particular collection of discussions. Rather, the purposes of this paper is (a) to try to exhibit the reconciliation of copula techniques with a classical multivariate analysis for a case of interest, and (b) to discuss the choices that may be available and when these choices evaporate, leaving a natural or *canonical* choice of copula and multivariate distribution.

1.1. *Why consider the Student distribution at all?*

The "Student", or "T" distribution was introduced by W. Gosset in 1908 [37], and has become well established in statistical theory, especially in the context of hypothesis testing for small samples. Fisher's elegant 1922 paper [15] on the distribution of the coefficients in a linear regression with Normal errors established its importance in model-fitting. Since this early work the distribution has become well established - the recent survey book on the multivariate T distribution alone, by Kotz and Nadarajah [25] cites about 400 references on the matter and the reader is encouraged to consult this comprehensive text for the background. While we shall explain later how this formula comes about, the density function for the univariate distribution is given by

$$f_n(t) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{1}{(1+t^2/n)^{\frac{n+1}{2}}} \tag{1.1}$$

The number n, often assumed to be an integer, is called the "degrees of freedom" of the distribution. A discussion of the univariate distribution and methods for its simulation using quantile functions was given in [34].

The Student distribution is enjoying new interest for is applications in mathematical finance, education and other applications of statistics. Recent work on T copulas and multivariate distribution theory will be considered shortly, but there are several other significant papers. The 2006 paper by Ferguson and Platen [14] suggests, for example, that the "T" (with $n \sim 4$) is a good model of index returns in a global setting. That returns have positive excess kurtosis has been known for over four decades – see e.g. the work in the 1960s by Mandelbrot [5] and Fama [13].

The book by Gencay *et al* [17] indicates that very short term returns exhibit power law decay in the PDF. For a T distribution with n degrees of freedom, we see from Equation (1.1) that the decay of the PDF is

$$O(t^{-n-1}) \tag{1.2}$$

and the decay of the CDF is

$$O(t^{-n}) \tag{1.3}$$

so that if the power decay index in the CDF is q we take a value of $n = q$. The values of q reported in [17] take values in the range 2 to 6. So this leads us to consider not only small integer values of $n : 2 \leq n \leq 6$ but also non-integer n. This idea of non-integer degrees of freedom, which was not part of Gosset's original plan, was also applied by Andreev and Kanto in their 2005 paper on the T applied to VaR estimation [4]. The use of the T in risk management has become widely established. See for example the recent work by Daul *et al* [8], Embrechts *et al* [10], Frey *et al* [16] and Schloegl & O'Kane [33].

The T distribution is also well known in educational applications. The case $n = 1$ is better known as the Cauchy distribution and is widely taught because of its extreme fat-tailed behaviour and pathological moments, and more recently the case $n = 2$ has been studied in more detail by Jones [24] and Nevzorov *et al* [32]. The cases $n = 1, 2$ and also $n = 4$ are also interesting because for these cases (at least) there are closed-form solutions for the quantile function. For other even n the quantile function can be found by solving a polynomial equation of degree $n - 1$ [34].

Much of the recent work focuses on simulation aspects. As well as considering here such approaches in the multivariate case, we shall also be concerned here with generating detailed expressions for the probability density functions, and analytical representations of correlation information. While the presentation of such distributions involves some detailed, and perhaps rather dry, representations in terms of "special functions", we consider this appropriate as it allows those doing simulation work to make explicit checks against analytical representations in the two-dimensional case, thereby facilitating the identification of errors or poor convergence in simulation work. One of our goals is to assist in the process of making the T as easy to work with as the Normal in multivariate applications. Likewise, the presentation of formulae for correlations is designed to make the calibration to observations on data more tractable, and this is very much in the same spirit as the work by Daul *et al* [8] for the Kendall coefficient, and we shall present here (we believe for the first time) some explicit formulae for the product-moment correlations for two types of T distribution.

1.2. *Recent work on T copulas and distributions*

As noted above, the Gaussian and T-copula are already in widespread use. This is no doubt in part due to the ease with which the Monte-Carlo simulation may be carried out compared to copulas in general, and also because of the ease with which a correlation structure may be introduced. In this context it should be noted that the T-copula in widespread use is that for which the associated marginal distributions all have *the same number of degrees of freedom.* There are a host of options for what else one might write down, as is discussed comprehensively in the recent book by Kotz and Nadarajah [25]. More recent surveys of the bivariate and multivariate cases have been by Nadarajah and co-workers [29,30], who emphasize, quite literally, that there are a *multitude* of possibilities. However, many of the variations cited in [29,30] focus on the introduction of non-centrality or skewness into the bi- and multivariate T, rather than the key concept of what should be meant by a "multivariate T" in the first place.

Recently, more attention has been paid to the case of unequal marginal degrees of freedom. This has been discussed to some extent in [25], but in terms of the relevance to mathematical finance, and an understanding of what might be a truly canonical approach, there seem to be (at least) three related threads of thought on the matter. These are[a]

(1) The *grouped t copula* approach, as discussed by Demarta and McNeil [9] and Daul *et al* [8];
(2) The bi- and multi-variate T distributions developed by Jones [23];
(3) The meta-elliptical distribution developed by Fang *et al* [11].

These papers take various approaches and have varying emphases on

(1) ease of Monte-Carlo simulation of the multivariate distribution;
(2) simulation of the associated copula;
(3) analytical representation of the copula;
(4) analytical representations of the density function and other properties.

We should also point out, however, that in terms of the fundamental mathematics involved, the first T copulas that cope with the case of unequal degrees of freedom in the marginals were in fact implicit in the much earlier work of Bulgren *et al* [6]. Their work introduced the idea of considering dependent Normal numerators divided by the square root of χ^2 variables of differing orders, and followed on from earlier work by Siddiqui [35] where the orders were the same. The work by Bulgren *et al* [6] dealt with the matter by exploiting the elegant additive properties of the χ^2 distribution within the context of statistical hypothesis testing. The additive properties are also the basis for the work by Jones [23]. Our own contribution to this particular thread of thought is largely to put it in the same setting as the others, but also to clarify how it all works in the case of non-zero correlation, and indeed to give an explicit formula for the (product-moment) correlation as well as a tractable representation of the density, albeit as a finite integral.

This paper is an attempt to characterize the precise relationship between these approaches, with the goal of trying to establish a truly canonical multi-variate Student T for the case of general marginals, with clear details about Monte Carlo and copula aspects with a good understanding of the associated probability density functions. This it is hoped will provide a combination of analytical and numerical tools that may be useful. The focus here will be on items 1,2 and 4 above. Financial practitioners are often interested in the simulation of the distribution and its

[a]Comments on other options which might be candidates for a canonical distribution or its Monte-Carlo or copula equivalent are welcome!

associated copula, while the density is useful for corresponding analytical or semi-analytical calculations. Analytical representations of the copula itself tend to be less useful. I will also tend to refer to simulation of the copula as being essentially done by covering item 1, since the copula can be simulated by just applying the marginal CDFs to each variable. In this case of the T the CDFs are well-known [34] and the key facts are summarized in Appendix A.

For analytical tractability much of the discussion will focus on the bivariate case. For this case this paper claims to have resolved the issued of creating a canonical picture completely for the special case of the bivariate Student-Normal distribution. There appear to be some genuine choices for the Student-Student case. For the first case, when one of the degrees of freedom tends to infinity, the choices coincide and we have a canonical representation for the density as well as methods for its simulation and copula. In particular we can write down simple formulae for such exotics as the correlated bivariate Cauchy-Normal distribution and Student-Normal with low integer degrees of freedom (on the first variable) in terms of elementary functions. The general Student-Normal case has a hypergeometric representation which we shall also exhibit. Such representations allow Monte-Carlo calculations of partially fat-tailed distributions to be done in parallel with analytical or semi-analytical calculations with a tractable density, thus allowing detailed checks to be made on numerical calculations, or in simple cases to be done entirely analytically.

1.3. *Elliptic Structures vs Independence*

The statistics literature is frequently concerned with distributions whose structure is *elliptical*, or, in more recent discussions, *meta-elliptical*. A good recent summary is given in [1]. In simple terms, such distributions arise naturally as it is often clear how to generalize the appearance of structures in a one-dimensional p.d.f. such as $f(\sigma^2 x^2)$, where σ is a standard deviation parameter, to a multi-dimensional structure where one writes down p.d.f.s such as

$$f(x^T A x) \tag{1.4}$$

where A is a matrix and x is a vector of random variables. We obtain zero correlation (what we mean by correlation is discussed presently) when A is diagonal. But we only obtain *independence* when f is exponential, i.e. the distribution is multivariate Gaussian. In our view *this is a fundamental drawback of the elliptical methodology*, particularly for financial modelling. This and other issues with elliptical methods have been discussed by others, for example [1]. Part of the motivation for this paper is to look at other options and to present structures for the Student T that get around this difficulty.

A good illustration of the issues about non-independence of non-normal elliptical distributions with zero covariance is given by the *local dependence function* of Holland and Wang [20]. An example of the local dependence function for the bivariate Cauchy distribution is given by Jones [22] - see Figure 2 of [22] and also Figure 2 of [23].

However, it should also be stressed that there are also certain advantages to elliptical structures in the context of financial risk management, and readers should see the thorough survey by Embrechts *et al* [10] for further discussion. Also, sometimes an elliptical model may be attractive[b] if one thinks that linear factors (producing correlations) and a systemic factor (producing tail dependence) co-exist. It is important that modellers can distinguish between the two and choose what they want. This paper provides modellers with some viable options.

1.4. *Reminder on the univariate case*

Many of our constructions will have a common root in the construction of the univariate Student T. We quickly remind the reader of how this works, using the terminology of Shaw [34]. Let $Z_0, Z_1, \dots Z_n$ be independent standard Normal random variables and set

$$\chi_n^2 = Z_1^2 + \cdots + Z_n^2 \tag{1.5}$$

The density function of χ_n^2 is easily worked out, using moment generating functions (see e.g. Sections 7.2 and 8.5 of Stirzaker [36]), and is given by

$$q_n(z) = \frac{1}{2\Gamma(\frac{n}{2})} e^{-z/2} \left(\frac{z}{2}\right)^{\frac{n}{2}-1} \tag{1.6}$$

χ_n^2 is a random variable with a mean of n and a variance of $2n$. We now define a Student T random variable by:

$$T = \frac{Z_0}{\sqrt{\chi_n^2/n}} \tag{1.7}$$

To obtain the density $f(t)$ of T we note that

$$f(t|\chi_n^2 = \nu) = \sqrt{\frac{\nu}{2\pi n}} e^{-\frac{t^2\nu}{2n}} \tag{1.8}$$

Then to get the *joint* density of T and χ_n^2 we need to multiply by $q_n(\nu)$. Finally, to extract the univariate density for T, which we shall call $f_n(t)$, we integrate out ν. The density $f_n(t)$ is then given by

$$\int_0^\infty f(t|\chi_n = \nu) q_n(\nu) d\nu \equiv \int_0^\infty \frac{d\nu}{2\Gamma(\frac{n}{2})} \sqrt{\frac{\nu}{2\pi n}} \left(\frac{\nu}{2}\right)^{(\frac{n}{2}-1)} e^{-(\frac{\nu}{2}+\frac{t^2\nu}{2n})} \tag{1.9}$$

and by the use of standard integral, we obtain the formula

$$f_n(t) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{1}{(1+t^2/n)^{\frac{n+1}{2}}} \tag{1.10}$$

The number n, often assumed to be an integer, is called the "degrees of freedom" of the distribution. It is evident that a sample from this distribution can easily be

[b]We are grateful to Dr Andreas Tsanakas for these observations

obtained by using $n+1$ samples from the standard Normal distribution, provided n is an integer. This is well known, as is the use of a Normal variate divided by the square root of a scaled sample from the χ^2 distribution, that itself being obtained by other methods.

More compactly and generally, we can think of a T variable as being given by the representation

$$T = \frac{Z}{\sqrt{C^2/a}} \tag{1.11}$$

where Z is normal and C^2 is independent of Z and has a χ^2 distribution with, in general a degrees of freedom, where a is real but not necessarily an integer. How C^2 is sampled is a matter of choice. If a is a low integer adding up the squares of normals is easy and fast. In general we might want to think of it as arising from a cunning simulation of a gamma variable, or formally as $F_{\chi^2}^{-1}(U)$, where F is the gamma CDF and U is uniform. As discussed by Shaw [34] it is actually quite straightforward to simulate the T directly in the univariate case, but to understand the multivariate case and associated copula, the "normal over root chi-squared" approach gives some obvious clues how to proceed.

1.5. *Concepts of Correlation*

The final initial task is to define the relevant correlation concepts. The usual *Pearson* or *product-moment* correlation will be denoted ρ and is given, for two random variables X_1, X_2 by

$$\rho = \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sqrt{Var(X_1)Var X_2)}} \tag{1.12}$$

The *Spearman rank* correlation ρ_S is given by applying the product-moment formula to the ranks of the X_i. To express it in terms of distributions we realize the ranks as the transformed variables $Y_i = F_i(X_i)$ where F_i is the cumulative distribution function. Since the Y_i are uniform the product-moment expression simplifies to

$$\rho_S = 12E[F(X_1)F(X_2)] - 3 \tag{1.13}$$

The third measure is Kendall's τ. While this is normally defined in terms of data its distributional form is given by the expression

$$\tau = 4E[F(X_1, X_2)] - 1 \tag{1.14}$$

where F is now the joint distribution function. For an elegant proof of both these expressions in the copula framework, see [31]. A substantial discussion of issues of correlation and dependence is given by Embrechts *et al* [10].

A complete knowledge of these parameters for complicated multivariate distributions is not available. However, in the multivariate Gaussian case we have

$$\rho_S = \frac{6}{\pi}\sin^{-1}(\frac{\rho}{2}) \tag{1.15}$$

$$\tau = \frac{2}{\pi} \sin^{-1}(\rho) \tag{1.16}$$

These results are not entirely obvious! The first formula linking the rank correlation was given by Kruskal [26] - see also [21]. The second formula, for Kendall's τ, actually applies for a large class of elliptical distributions - see e.g. [11]. We shall see that τ is closely related to the mixing parameter we shall employ throughout this paper.

2. Approaches to the multivariate Student T with equal marginals

Note the plural in the title of this Section! In the finance literature the Student T copula with equal marginals is commonly understood as being simulated by taking a linear combination of Gaussian variables defined by the Cholesky decomposition of the correlation structure, and dividing all of them by the same χ^2 variable. This is described in many papers - see for example the book by Cherubini *et al* [7]. We will describe the bivariate case to fix our notation for what follows, and also to explore an issue that is, so far as this author can tell, quite routinely ignored. Let W_i be independent Gaussians and let

$$Z_{01} = \alpha W_1 + \beta W_2, \quad Z_{02} = \gamma W_1 + \delta W_2 \tag{2.1}$$

subject to the transformed variables having unit variance:

$$\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2 \tag{2.2}$$

Then one constructs

$$T_1 = Z_{01}\sqrt{\frac{n}{C^2}}, \quad T_2 = Z_{02}\sqrt{\frac{n}{C^2}} \tag{2.3}$$

where C is a sample from the χ^2 distribution with parameter n.

To derive the density let us fix the notation further and introduce a "mixing angle" θ. The idea is that in the standard bivariate Normal case the linear correlation would turn out to be $\rho = \sin\theta$, and Kendall's τ is indeed just $2\theta/\pi$ in the Normal case. More generally, we set

$$Z_{01} = W_1, \quad Z_{02} = W_1 \sin\theta + W_2 \cos\theta \tag{2.4}$$

$$T_1 = \sqrt{\frac{n}{C^2}} W_1, \quad T_2 = \sqrt{\frac{n}{C^2}} (W_1 \sin\theta + W_2 \cos\theta) \tag{2.5}$$

Then we can invert this relationship as

$$W_1 = \sqrt{\frac{C^2}{n}} T_1, \quad W_2 = \sqrt{\frac{C^2}{n}} \frac{1}{\cos\theta} (T_2 - T_1 \sin\theta) \tag{2.6}$$

We know the standard normal density for the W_i in the form

$$\frac{1}{2\pi} \exp\{-\frac{1}{2}(w_1^2 + w_2^2)\} \tag{2.7}$$

and so the *conditional* density of the T_i *given* a fixed value of $C^2 = z$ is just

$$f(t_1, t_2|C^2 = z) = \frac{z}{2\pi n \cos\theta} \exp\left\{-\frac{z}{2n\cos^2\theta}\left(t_1^2 + t_2^2 - 2t_2t_2\sin\theta\right)\right\} \tag{2.8}$$

As in the univariate case we can now integrate out over the density of z, and obtain the result for the joint PDF as

$$f(t_1, t_2) = \frac{1}{2\pi\cos\theta} \frac{1}{(1 + \Delta/n)^{n/2+1}} \tag{2.9}$$

where

$$\Delta = \frac{t_1^2 + t_2^2 - 2t_2t_2\sin\theta}{\cos^2\theta} \tag{2.10}$$

This *joint* distribution has marginals which are each T-distributed with n degrees of freedom. However, in sharp contrast to the Normal case, when $\theta = 0$, we have

$$f(t_1, t_2) = \frac{1}{2\pi} \frac{1}{(1 + (t_1^2 + t_2^2)/n)^{n/2+1}} \tag{2.11}$$

which is *not* the product of the two marginal density functions unless $n \to \infty$ and we are back in the Normal case. So we have a manifest failure to secure the desired structure for independence.

2.1. *An alternative T copula with equal degrees of freedom*

The failure to achieve the desirable product structure suggests we look for an alternative. There is an obvious one, which is to use *independent* denominators rather than the same one. So with the same notation as before, we have

$$T_1 = \sqrt{\frac{n}{C_1^2}} W_1, \quad T_2 = \sqrt{\frac{n}{C_2^2}} (W_1 \sin\theta + W_2 \cos\theta) \tag{2.12}$$

Then we can invert this relationship as

$$W_1 = \sqrt{\frac{C_1^2}{n}} T_1, \quad W_2 = \frac{1}{\cos\theta}\left(\sqrt{\frac{C_2^2}{n}} T_2 - \sqrt{\frac{C_1^2}{n}} T_1 \sin\theta\right) \tag{2.13}$$

Proceeding as before, the conditional density given fixed values z_1, z_2 of C_1^2, C_2^2, is $f(t_1, t_2|C_1^2 = z_1; C_2^2 = z_2)$ and is given by

$$\frac{\sqrt{z_1 z_2}}{2\pi n \cos\theta} \exp\left\{-\frac{1}{2n\cos^2\theta}\left(z_1 t_1^2 + z_2 t_2^2 - 2t_2t_2\sin\theta\sqrt{z_1 z_2}\right)\right\} \tag{2.14}$$

Note that this reduces to the previous conditional density if $z_1 = z_2$. But now we integrate over a gamma distribution for each z_i, each with the "same" n - they are i.i.d. So we integrate the conditional density against the product density for the z_i, this being

$$\frac{1}{2^n \Gamma^2(n/2)} (z_1 z_2)^{(n/2-1)} e^{-\frac{1}{2}(z_1+z_2)} \tag{2.15}$$

The resulting formula can be given in terms of hypergeometric functions as follows. We introduce intermediate variables as follows:

$$\alpha_1 = 1 + \frac{t_1^2}{n\cos^2(\theta)}, \quad \alpha_2 = 1 + \frac{t_2^2}{n\cos^2(\theta)}, \quad \gamma = \frac{2t_1 t_2 \sin(\theta)}{n\cos^2(\theta)} \tag{2.16}$$

and the normalizing constant

$$C' = \frac{1}{\cos(\theta)\pi n \Gamma(n/2)^2} \tag{2.17}$$

and the density is then given by

$$\begin{aligned} C'(\alpha_1\alpha_2)^{-\frac{n}{2}-1}\Bigg[\Gamma\left(\frac{n+1}{2}\right)^2 {}_2F_1\left(\frac{n+1}{2}, \frac{n+1}{2}; \frac{1}{2}; \frac{\gamma^2}{4\alpha_1\alpha_2}\right)\sqrt{\alpha_1}\sqrt{\alpha_2} \\ -\gamma\Gamma\left(\frac{n}{2}+1\right)^2 {}_2F_1\left(\frac{n}{2}+1, \frac{n}{2}+1; \frac{3}{2}; \frac{\gamma^2}{4\alpha_1\alpha_2}\right)\Bigg] \end{aligned} \tag{2.18}$$

However, we can check very directly that as $\theta \to 0$ this expression simplifies (as it must, by its construction) to just

$$\frac{\Gamma[\frac{n+1}{2}]^2}{\Gamma[\frac{n}{2}]^2 a\pi}\left(\frac{1}{1+t_1^2/n}\right)^{(n+1)/2}\left(\frac{1}{1+t_2^2/n}\right)^{(n+1)/2} \tag{2.19}$$

So we see that we can recover a product structure in the independent case, albeit with a moderately complicated bivariate density function. However, the simulation remains trivial, as does the copula construction. We are not aware of this density appearing before in the literature.[c]

3. Approaches to the Student T with unequal marginals

We shall now turn to the multivariate case and look at the definitions of the multivariate T that correspond to the natural generalizations of the normal over chi approach. In order to keep the notation straightforward we write down the bivariate versions. In all cases we have degrees of freedom n_1 and n_2 which are neither necessarily equal nor integer.

3.1. *Grouped approaches*

I first describe an approach developed by Demarta and McNeil [9] and Daul *et al* [8], that I will refer to as the *tightly grouped* approach. In the bivariate case this simplifies to writing down two independent normal variables W_1 and W_2, and forming

$$Z_{01} = \alpha W_1 + \beta W_2, \quad Z_{02} = \gamma W_1 + \delta W_2 \tag{3.1}$$

[c]We have not been able to recognize it in any paper or book to which we have access, though correction on this matter is appreciated!

subject to

$$\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2 \tag{3.2}$$

Then one constructs

$$T_1 = Z_{01}\sqrt{\frac{n_1}{C_1^2}}, \quad T_2 = Z_{02}\sqrt{\frac{n_2}{C_2^2}} \tag{3.3}$$

where the C_i^2 are *dependent* χ^2 variables with degrees of freedom n_i. They are obtained by taking one sample U from a uniform distribution and setting

$$C_i^2 = G_{n_i}^{-1}(U) \tag{3.4}$$

where G_{n_i} is the CDF for the χ^2 distributed with parameter n_i. The *grouping* refers to the fact that in the general multivariate case we group together all the marginals with the same degrees of freedom and use the same χ^2 denominator to turn the normal variable to a T variable within each group. I refer to this as the "tight" case as the same U is used for all the groups. It has the nice property that because the same U is employed this approach coalesces into the ordinary T copula when all degrees of freedom are the same. However, there are at least two other routes one might take.

The first and rather obvious option would be to take the C_i to be independent - I would call this the *loosely grouped* approach. This approach is straightforward and transparent. However, it does have a feature that should be noted. As long as $n_2 \neq n_1$ the groups remain distinct, *and* as $n_2 \rightarrow n_1$ the groups remain distinct. We can now recognize the loosely grouped approach as being the natural generalization of our alternative equal d.o.f. method.

The second further option is a generalization of the method of Jones (see below).

So far we have just discussed the simulation part of the construction, which is very easy. The construction of the copula follows by the application of the marginal CDFs with degrees of freedom n_i to the T_i. The formulae to be used are given in Appendix A.

3.2. *A generalization of the approach of Jones*

Another elegant approach has already been given by Jones [23] and is also summarized in [25]. In general one sorts the degrees of freedom into increasing order and exploits the additive properties of the χ^2 distribution. This is an idea we believe was first introduced by Bulgren *et al* [6]. Suppose that $n_1 \leq n_2$. We let $a = n_1$ and $b = n_2 - n_1$. In Jones' original specification the dependence is only through the chi-squared variables. So in our notation his model is of the form

$$T_1 = W_1\sqrt{\frac{a}{C_1^2}}, \quad T_2 = W_2\sqrt{\frac{a+b}{C_1^2 + C_2^2}} \tag{3.5}$$

where C_1^2 has a χ^2 distribution with a degrees of freedom, and C_2 has a χ^2 distribution with b degrees of freedom. This gives a model with strictly zero correlation, despite the dependency in the denominators. However, it is straightforward to generalize this to include dependency in the numerator, as introduced in [27], by using the mixed variables:

$$T_1 = Z_{01}\sqrt{\frac{a}{C_1^2}}, \quad T_2 = Z_{02}\sqrt{\frac{a+b}{C_1^2 + C_2^2}} \tag{3.6}$$

The copula implied by this “generalized Jones” method is obtained by applying the marginal CDFs with degrees of freedom a and $a+b$ respectively to T_1 and T_2 respectively.

3.3. *The meta-elliptical distribution*

The meta-elliptical distributions were introduced by Fang *et al* [11]. Further very useful discussion is provided by Abdous *et al* [1] and the corrigendum [12] to [11]. It does not appear to be possible to give an elementary closed-form expression for the PDF in this case, even for the simpler bivariate case.

The simulation of these distributions fall outside (so far as we can establish) the family of methods described as ”normal over root chi-squared”. A simulation algorithm is given in Section 4.2 of [11].

3.4. *A canonical limit for the bivariate case?*

It is evident that while the tightly and loosely grouped and Jones methods are all different in general, they do have a common limit in the bivariate case when $n_2 \to +\infty$. One just has to note that for either grouped case, $C_2^2/n_2 \to 1$, and that in the Jones case the same thing happens as $b \to \infty$. So there is in a sense a natural or *canonical* common limit, where we take

$$T_1 = Z_{01}\sqrt{\frac{n_1}{C_1^2}}, \quad T_2 = Z_{02} \tag{3.7}$$

This gives the recipe for simulating a Student-Normal distribution with a correlation. The associated copula is again obvious. The commonality of this limit suggests that it has a role to play as the *canonical* Student-Normal distribution.

3.5. *What do we want to know?*

The approaches discussed above give a construction method. We want to know various associated things, including, but not necessarily limited to:

- What is the associated density function?
- How precisely do we feed in measured correlations?
- What are the mathematical and implementation drawbacks?

We will take a look at these issues first with the most straightforward case. As we have already observed, there is a candidate *canonical* construction, in the sense that the Jones and grouping approaches coincide, for the Student-Normal case, so this is considered in some detail first. The common nature of the limit that these have in common with the grouped case suggests that it is helpful to characterize them in more detail. In particular we shall be able to characterize the densities in rather more elementary terms that would allow much greater accessibility to the results for both applications and education.

4. The Canonical Student-Normal Distribution

In the general case the grouped and Jones copulas discussed previously represent different objects. However, it is clear than when $n_2 \to \infty$ the approaches coalesce into the same entity. We first reduce the problem by parametrizing the linkage between the two distributions by a rotation angle θ. With W_1, W_2 as before, we set

$$T_1 = \frac{W_1}{\sqrt{C_1^2/a}}, \quad T_2 = W_1 \sin\theta + W_2 \cos\theta \tag{4.1}$$

with no variance randomization in the latter. Inverting this definition, we have

$$W_1 = \sqrt{\frac{C_1^2}{a}} T_1 \ , \quad W_2 = \frac{1}{\cos\theta}(T_2 - \sqrt{\frac{C_1^2}{a}} T_1 \sin\theta) \tag{4.2}$$

We know the standard normal density for the W_i in the form

$$\frac{1}{2\pi} \exp\{-\frac{1}{2}(w_1^2 + w_2^2)\} \tag{4.3}$$

and so the *conditional* density of the T_i *given* a fixed value of $C_1^2 = z$ is just

$$f(t_1, t_2 | C_1^2 = z) = \frac{1}{2\pi\cos\theta}\sqrt{\frac{z}{a}} \exp\left\{-\frac{1}{2\cos^2\theta}(\frac{z}{a}t_1^2 + t_2^2 - 2t_2 t_2 \sin\theta\sqrt{\frac{z}{a}})\right\} \tag{4.4}$$

The unconditional joint density of the T_i is then given by integrating this conditional density against the chi-squared density function for z. That is,

$$f(t_1, t_2) = \int_0^\infty dz f(t_1, t_2 | \chi^2 = z) \frac{1}{\Gamma(a/2) 2^{a/2}} z^{a/2-1} e^{-z/2} \tag{4.5}$$

So that

$$f(t_1, t_2) = c\int_0^\infty dz\ z^{(a-1)/2} \exp\left\{-\frac{z}{2} - \frac{1}{2\cos^2\theta}(\frac{z}{a}t_1^2 + t_2^2 - 2t_2 t_2 \sin\theta\sqrt{\frac{z}{a}})\right\} \tag{4.6}$$

where $c = 1/(2\pi 2^{a/2}\Gamma(a/2)\sqrt{a}\cos\theta)$. Some simplification and the change of variables $z = q^2$ gives us

$$f(t_1, t_2) = 2ce^{-t_2^2/(2\cos^2\theta)} \int_0^\infty dq\ q^a \exp\left\{-\frac{q^2}{2}\left(1 + \frac{t_1^2}{a\cos^2\theta}\right) + \frac{t_1 t_2 q \sin\theta}{\sqrt{a}\cos^2\theta}\right\} \tag{4.7}$$

We shall now proceed to investigate this expression for some cases of interest. First we write down some moments.

4.1. *Simple Moments*

The basic moments can be calculated from the conditional distribution (since it is a correlated bivariate Gaussian) followed by integration over q. The results are

$$E[t_1] = 0, \forall a > 1 \tag{4.8}$$

$$E[t_1^2] = \frac{a}{a-2}, \forall a > 2 \tag{4.9}$$

$$E[t_2] = 0, \forall a > 0 \tag{4.10}$$

$$E[t_2^2] = 1, \forall a > 0 \tag{4.11}$$

$$E[t_1 t_2] = \frac{\sqrt{a}\sin(\theta)\Gamma\left(\frac{a-1}{2}\right)}{\sqrt{2}\Gamma\left(\frac{a}{2}\right)}, \forall a > 1 \tag{4.12}$$

From these results we may infer that the ordinary product-moment correlation ρ exists provided $a > 2$ and is given by

$$\rho = \sin(\theta)\frac{\Gamma\left(\frac{a-1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\sqrt{\frac{a}{2} - 1} \tag{4.13}$$

We note that $\rho \to \sin(\theta)$ as $a \to +\infty$ so that the usual bivariate Normal result is recovered, but for finite a we have $|\rho| < |\sin\theta|$ with $\rho/\sin(\theta)$ increasing monotonically with increasing a, from zero when $a = 2$ to unity as $a \to \infty$.

4.2. *The uncorrelated general case*

The integral is easy if we decouple the variables by setting $\sin\theta = 0, \cos\theta = 1$ and we recover the *product density* formula after some simplification:

$$f(t_1, t_2) = \frac{1}{\sqrt{a\pi}}\frac{\Gamma((a+1)/2)}{\Gamma(a/2)}\left(1 + t_1^2/a\right)^{(a+1)/2} \times \frac{1}{\sqrt{2\pi}}e^{-t_2^2/2} \tag{4.14}$$

But now we have an expression also valid when there is mixing between the two distributions, and we can look at some special cases of interest.

4.3. *The correlated general case*

In the general case we have the intermediate variables

$$\alpha = 1 + \frac{t_1^2}{a\cos^2\theta}\ , \quad \beta = \frac{t_1 t_2 \sin\theta}{\sqrt{a}\cos^2\theta}\ , \quad \gamma = \frac{\beta}{\sqrt{\alpha}} \tag{4.15}$$

and the density function is given by

$$\frac{2^{\frac{a-1}{2}-\frac{a}{2}} e^{\frac{\gamma^2}{2}-\frac{1}{2}\sec^2(\theta)t_2^2}\alpha^{-\frac{a}{2}-\frac{1}{2}}}{\sqrt{a}\pi\Gamma\left(\frac{a}{2}\right)\cos\theta} \times \tag{4.16}$$

$$\left[\sqrt{2}\gamma\Gamma\left(\frac{a}{2}+1\right)\,{}_1F_1\left(\frac{1}{2}-\frac{a}{2};\frac{3}{2};-\frac{\gamma^2}{2}\right)+\Gamma\left(\frac{a+1}{2}\right)\,{}_1F_1\left(-\frac{a}{2};\frac{1}{2};-\frac{\gamma^2}{2}\right)\right]$$

It is useful to see how this simplifies when a is a low integer, since then the hypergeometric function can be simplified in terms of the cumulative normal distribution.

4.4. *The correlated Cauchy-Normal Distribution*

If we allow θ to be general, i.e. we allow for a "correlation", and also set $a = 1$ to try and get one marginal to be a Cauchy the integral integrates to elementary functions. To keep the algebra tractable we introduce parameters

$$\alpha = 1 + \frac{t_1^2}{\cos^2\theta}\ ,\quad \beta = \frac{t_1 t_2 \sin\theta}{\cos^2\theta}\ ,\quad \gamma = \frac{\beta}{\sqrt{\alpha}} \tag{4.17}$$

and then

$$f(t_1,t_2) = \frac{1}{\cos\theta\sqrt{2\pi}}e^{-t_2^2/(2\cos^2\theta)}\frac{1}{\pi\alpha}\left[1+e^{\gamma^2}\gamma\sqrt{2\pi}\Phi(\gamma)\right] \tag{4.18}$$

gives the density of the joint Cauchy-Normal distribution, where $\Phi(x)$ as usual denotes the Normal CDF evaluated at x.

4.5. *The correlated T_2-Normal Distribution*

The intermediate variables are

$$\alpha = 1 + \frac{t_1^2}{2\cos^2\theta}\ ,\quad \beta = \frac{t_1 t_2 \sin\theta}{\sqrt{2}\cos^2\theta}\ ,\quad \gamma = \frac{\beta}{\sqrt{\alpha}} \tag{4.19}$$

and the density function is given by

$$f(t_1,t_2) = \frac{e^{-\frac{1}{2}\sec^2(\theta)t_2^2}\left(\gamma+e^{\frac{\gamma^2}{2}}\sqrt{2\pi}\left(\gamma^2+1\right)\Phi(\gamma)\right)}{\cos(\theta)2\sqrt{2}\pi\alpha^{3/2}} \tag{4.20}$$

4.6. *The correlated T_3-Normal Distribution*

The intermediate variables are

$$\alpha = 1 + \frac{t_1^2}{3\cos^2\theta}\ ,\quad \beta = \frac{t_1 t_2 \sin\theta}{\sqrt{3}\cos^2\theta}\ ,\quad \gamma = \frac{\beta}{\sqrt{\alpha}} \tag{4.21}$$

and the density function is given by

$$f(t_1,t_2) = \frac{e^{-\frac{1}{2}\sec^2(\theta)t_2^2}\left(\gamma^2+e^{\frac{\gamma^2}{2}}\sqrt{2\pi}\left(\gamma^2+3\right)\Phi(\gamma)\gamma+2\right)}{\cos(\theta)\sqrt{6}\pi^{3/2}\alpha^2} \tag{4.22}$$

4.7. *The correlated T_4-Normal Distribution*

The intermediate variables are

$$\alpha = 1 + \frac{t_1^2}{4\cos^2\theta}\ ,\quad \beta = \frac{t_1 t_2 \sin\theta}{2\cos^2\theta}\ ,\quad \gamma = \frac{\beta}{\sqrt{\alpha}} \tag{4.23}$$

and the density function is given by

$$f(t_1,t_2) = \frac{e^{-\frac{1}{2}\sec^2(\theta)t_2^2}\left(\gamma\left(\gamma^2+5\right) + e^{\frac{\gamma^2}{2}}\sqrt{2\pi}\left(\gamma^4+6\gamma^2+3\right)\Phi(\gamma)\right)}{8\cos(\theta)\pi\alpha^{5/2}} \tag{4.24}$$

4.8. *What do these densities look like?*

Contour plots of the density functions for $a = 1, 2, 4, 20$ are shown in Figure 1. Each row represents a value of a and the near-elliptical quality for $a = 20$ is geometrically evident. The low values of a are manifestly and highly non-elliptical in character. The three columns represent $\theta = 0, \pi/4, -\pi/4$.

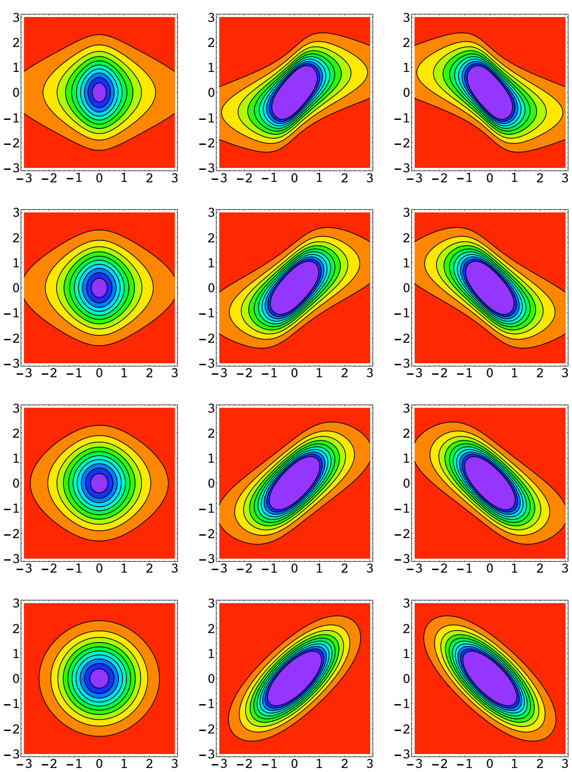

Fig. 1. Contour plots of densities with $a = 1$ (top), $a = 2, 4, 20$ (bottom) and $\theta = 0, \pi/4 - \pi/4$ (left to right)

5. Densities, Correlations for Student-Student case

5.1. *Jones approach and strongly dependent generalization*

The density function for the case $\theta = 0$ and $n_1 \leq n_2$ has been given by Jones [23]. I will reproduce his formula here as although it appeared correctly normalized in Jones original paper it has been given incorrectly normalized elsewhere [29,25]. With the (correct) normalization constant

$$C = \frac{\Gamma\left(\frac{1}{2}(n_1+1)\right)\Gamma\left(\frac{n_2}{2}+1\right)}{\pi\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{1}{2}(n_2+1)\right)\sqrt{n_1 n_2}} \tag{5.1}$$

the density is then given by

$$C\,{}_2F_1\left(\frac{n_2}{2}+1, \frac{1}{2}(n_2-n_1); \frac{1}{2}(n_2+1); \frac{t_1^2}{n_1}\left(1+\frac{t_1^2}{n_1}+\frac{t_2^2}{n_2}\right)^{-1}\right)\left(1+\frac{t_1^2}{n_1}+\frac{t_2^2}{n_2}\right)^{-\frac{n_2}{2}-1} \tag{5.2}$$

It is easy to see that when $n_1 = n_2$ the hypergeometric function evaluates to unity leaving the standard formula

$$f(t_1, t_2) = \frac{1}{2\pi}\frac{1}{(1+(t_1^2+t_2^2)/n_1)^{n_1/2+1}} \tag{5.3}$$

Jones established that the product-moment correlation is zero when it exists, but also exhibited the dependency features.

The case with $\theta \neq 0$ is significantly harder to characterize. Suppose we condition on the chi-squared variables being fixed at $C_1^2 = z$ and $C_2^2 = w$, with degrees of freedom $a = n_1$ and $b = n_2 - n_1$. We note first that the independent normal variables are given in terms of the dependent T variables by

$$W_1 = \sqrt{\frac{z}{a}}T_1, \quad W_2 = \frac{1}{\cos(\theta)}\left[\sqrt{\frac{z+w}{a+b}}T_2 - \sin(\theta)\sqrt{\frac{z}{a}}T_1\right] \tag{5.4}$$

Making the change of variables gives us

$$f(t_1, t_2 | C_1^2 = z; C_2^2 = w) =$$

$$\frac{\sqrt{z(z+w)}}{2\pi\cos(\theta)\sqrt{a(a+b)}}\exp\left\{\frac{-1}{2\cos^2(\theta)}\left[\frac{z}{a}t_1^2 + \frac{(z+w)}{(a+b)}t_2^2 - 2\sin(\theta)t_1 t_2\sqrt{\frac{z(z+w)}{a(a+b)}}\right]\right\} \tag{5.5}$$

By an ordinary Gaussian calculation we can see right away that the conditional product-moment expectation is given by

$$E[t_1 t_2 | C_1^2 = z; C_2^2 = w] = \sqrt{\frac{a(a+b)}{z(z+w)}}\sin(\theta) \tag{5.6}$$

So the unconditional product moment expectation is given by the integral of

$$\frac{\sqrt{a(a+b)}}{\sqrt{z(z+w)}}\sin(\theta) \tag{5.7}$$

against the joint density function

$$\frac{2^{-\frac{a}{2}-\frac{b}{2}}e^{-\frac{w}{2}-\frac{z}{2}}w^{\frac{b}{2}-1}z^{\frac{a}{2}-1}}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)} \tag{5.8}$$

over the range $0 \leq w, z < \infty$. Combining this with the expressions for the variances we see that the product-moment correlation is given for $a > 2, a+b > 2$ by

$$\rho = \sin(\theta)\frac{\sqrt{(a-2)(a+b-2)}}{2^{((a+b)/2)}\Gamma[a/2]\Gamma[b/2]}\int_0^\infty dz\int_0^\infty dw\frac{w^{(b/2-1)}z^{(a/2-1)}}{\sqrt{z(z+w)}}e^{-(z+w)/2} \tag{5.9}$$

This integral may easily be evaluated by setting $z = p^2,\ w = q^2$ and then changing to polar coordinates to deal with the denominator. After doing some standard integrals we finally obtain

$$\rho = \sin(\theta)\frac{\Gamma\left(\frac{a-1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\sqrt{\frac{a}{2}-1}\frac{\Gamma\left(\frac{a+b-2}{2}\right)}{\Gamma\left(\frac{a+b-1}{2}\right)}\sqrt{\frac{a+b}{2}-1} \tag{5.10}$$

and we remind the reader that $a = n_1,\ b = n_2 - n_1$. Note that this formula has the elegant properties that

$$\lim_{b\to 0}\rho = \sin(\theta), \quad \lim_{b\to\infty}\rho = \sin\theta\frac{\Gamma\left(\frac{n_1-1}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)}\sqrt{\frac{n_1}{2}-1} \tag{5.11}$$

The first result gives the standard (elliptical) result for equal degrees of freedom while the second is in agreement with what we established for the canonical Student-Normal limit.

We do not yet have a closed-form for the density for $\theta \neq 0$. The simulation is of course straightforward, but details on the rank correlation and Kendall's τ remain to be elucidated. A density function was given as a doubly-infinite sum for the similar problem considered by Bulgren *et al.* Our own integral representation for the density is given in Appendix B.

5.2. *The (loosely) grouped approach*

We can carry out the calculation of ordinary correlation and the density by following a similar route to that taken by our generalization of the Jones approach. There are minor modifications along the way arising from the fact that we now have $a = n_1, b = n_2$.

$$W_1 = \sqrt{\frac{z}{n_1}}T_1, \quad W_2 = \frac{1}{\cos(\theta)}\left[\sqrt{\frac{w}{n_2}}T_2 - \sin(\theta)\sqrt{\frac{z}{n_1}}T_1\right] \tag{5.12}$$

Making the change of variables gives us

$$f(t_1, t_2|C_1^2 = z; C_2^2 = w) =$$

$$\frac{\sqrt{zw}}{2\pi\cos(\theta)\sqrt{n_1n_2}}\exp\left\{\frac{-1}{2\cos^2(\theta)}\left[\frac{z}{n_1}t_1^2 + \frac{w}{n_2}t_2^2 - 2\sin(\theta)t_1t_2\sqrt{\frac{zw}{n_1n_2}}\right]\right\} \tag{5.13}$$

We deal first with the ordinary correlation. By an ordinary Gaussian calculation we can see right away that the conditional product-moment expectation is given by

$$E[t_1 t_2 | C_1^2 = z; C_2^2 = w] = \sqrt{\frac{n_1 n_2}{zw}} \sin(\theta) \tag{5.14}$$

So the unconditional product moment expectation is given the integral of

$$\frac{\sqrt{n_1 n_2}}{\sqrt{zw}} \sin(\theta) \tag{5.15}$$

against the joint density function

$$\frac{2^{-\frac{n_1}{2}-\frac{n_2}{2}} e^{-\frac{w}{2}-\frac{z}{2}} w^{\frac{n_2}{2}-1} z^{\frac{n_1}{2}-1}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \tag{5.16}$$

over the range $0 \leq w, z < \infty$. Combining this with the expressions for the variances we see that the product-moment correlation is given for $n_1 > 2, n_2 > 2$ by

$$\rho = \sin(\theta) \frac{\Gamma\left(\frac{n_1-1}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)} \sqrt{\frac{n_1}{2} - 1} \frac{\Gamma\left(\frac{n_2-1}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)} \sqrt{\frac{n_2}{2} - 1} \tag{5.17}$$

One can check that as $n_2 \to \infty$ we recover the formula of Section 4.1. A reasonably simple expression is available for the density in terms of hypergeometric functions of type ${}_2F_1$. We introduce intermediate variables

$$\alpha_1 = 1 + \frac{t_1^2}{n_1 \cos^2(\theta)}, \quad \alpha_2 = 1 + \frac{t_2^2}{n_2 \cos^2(\theta)}, \quad \gamma = \frac{2 t_1 t_2 \sin(\theta)}{\sqrt{n_1 n_2} \cos^2(\theta)} \tag{5.18}$$

and the normalizing constant

$$C' = \frac{1}{\cos(\theta) \pi \sqrt{n_1 n_2} \Gamma(n_1/2) \Gamma(n_2/2)} \tag{5.19}$$

and the density is then given by

$$\begin{aligned} C' \alpha_1^{-\frac{n_1}{2}-1} \alpha_2^{-\frac{n_2}{2}-1} \Bigg[& \Gamma\left(\frac{n_1+1}{2}\right) \Gamma\left(\frac{n_2+1}{2}\right) {}_2F_1\left(\frac{n_1+1}{2}, \frac{n_2+1}{2}; \frac{1}{2}; \frac{\gamma^2}{4\alpha_1\alpha_2}\right) \sqrt{\alpha_1 \alpha_2} \\ & -\gamma \Gamma\left(\frac{n_1}{2}+1\right) \Gamma\left(\frac{n_2}{2}+1\right) {}_2F_1\left(\frac{n_1}{2}+1, \frac{n_2}{2}+1; \frac{3}{2}; \frac{\gamma^2}{4\alpha_1\alpha_2}\right) \Bigg] \end{aligned} \tag{5.20}$$

It may now be seen very explicitly that when $\theta = 0$ this reduces to

$$\frac{\Gamma\left(\frac{n_1+1}{2}\right) \Gamma\left(\frac{n_2+1}{2}\right) \left(1 + \frac{t_1^2}{n_1}\right)^{-\frac{n_1}{2}-\frac{1}{2}} \left(1 + \frac{t_2^2}{n_2}\right)^{-\frac{n_2}{2}-\frac{1}{2}}}{\sqrt{n_1}\sqrt{n_2} \pi \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \tag{5.21}$$

which is the desired product of two independent T distributions with non-equal degrees of freedom. Although we have a formula for ρ, the rank correlation and Kendall's τ are not yet known. Given that the ordinary correlation vanishes as $n_1, n_2 \to 2_+$ we need further insight into this, but the situation is not that surprising given this is the limit at which the variance of either marginal distribution tends to infinity.

5.3. *The (tightly) grouped approach*

In this case no elementary density has been given. However, there is a very useful correlation approximation, obtained by Daul *et al* [8].

$$\tau \sim \frac{2}{\pi} \arcsin \tilde{\rho} \tag{5.22}$$

where $\tilde{\rho}$ is the correlation between the underlying Gaussian variables in the numerator. In our terminology this means that

$$\tau \sim \frac{2}{\pi}\theta \tag{5.23}$$

is approximately true. This powerful result makes calibration of the method tractable.

5.4. *The meta-elliptical approach*

In this case no closed-form density has been given. A formula for the density using the distribution functions implicitly has been given in the original paper [11]. Again, a powerful and this time exact relationship between Kendall's τ and the mixing parameter exists in the form

$$\tau = \frac{2}{\pi}\theta \tag{5.24}$$

where θ is the analogous parameter in the meta-elliptical framework.

6. Summary

In this paper we have surveyed the various options by which a "normal over root chi-squared" approach can be used to generate bivariate T copulas and distributions. We are able to proceed for the case of unequal marginals just as easily as for equal marginals.

We have demonstrated the existence of a natural alternative to the elliptical method that allows distributions satisfying the independence condition to be constructed, with or without equal degrees of freedom, and have exhibited the bivariate density for this case. We have also shown how to generalize Jones' method to the situation of strong dependence and non-zero correlation, though we have not been able to create a closed-form density for this case.

The simulation methods are easy, irrespective of the complexity of the density, and so provide a clear options further to the standard T and grouped T copulas already in use, but with the independence condition in place.

We have also made some progress on the correlation aspects, though the rank measures require further clarification. There are of other issues that are worth consideration. Are these distributions manifestly unimodal? What is an efficient method of extracting the degrees of freedom from a data set? Indeed - the estimation of the

n_i is part of a more general question of estimating a model based on the T with unknown mean, variance (which is coupled to n_i) and perhaps slope or other non-linear parameters. Such questions are best set in the context of estimating a regression model with T-distributed noise. Some progress has been made on this but will be reported elsewhere.

Appendix A. Student CDFs

In order to simulate the T copula with *any* of the approaches for simulation discussed in this paper it is necessary to apply the appropriate marginal distribution functions. We start with the formula

$$F_n(x) = \int_{-\infty}^{x} f_n(t)dt = \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \int_{-\infty}^{x} \frac{1}{(1+t^2/n)^{\frac{n+1}{2}}} dt \tag{A.1}$$

An obvious approach is to make a trigonometric substitution, $t = \sqrt{n}\tan\theta$. We can then obtain the integral as a collection of powers of trigonometric functions. The resulting expressions are given by expressions 26.7.3 and 26.7.4 of Abramowitz and Stegun [2] (on-line at [3]).

$F_n(x)$ can be written in "closed form", albeit in terms of hypergeometric functions, for general n. Integration in Mathematica [38] gives

$$F_n(x) = \frac{1}{2} + \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} x \; {}_2F_1\left(\frac{1}{2}, \frac{n+1}{2}; \frac{3}{2}; -\frac{x^2}{n})\right) \tag{A.2}$$

The CDF may also be thought of in terms of β-functions, for we can rewrite the hypergeometric function to obtain (see Section 26.7.1 of [2,3], bearing in mind the conversion from one- to two-sided results):

$$F_n(x) = \frac{1}{2}\left(1 + \mathrm{sgn}(x)(1 - I_{(\frac{n}{x^2+n})}\left(\frac{n}{2}, \frac{1}{2}\right)\right) \tag{A.3}$$

giving an expression in terms of regularized β-functions. As usual $\mathrm{sgn}(x)$ is $+1$ if $x > 0$ and -1 if $x < 0$. The regularized beta function $I_x(a,b)$ is given by

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} \tag{A.4}$$

where $B(a,b)$ is the ordinary β-function and $B_x(a,b)$ is the incomplete version

$$B_x(a,b) = \int_0^x t^{(a-1)}(1-t)^{(b-1)} dt \tag{A.5}$$

For our immediate purposes it will be useful to look at some cases of $F_n(x)$ for small n very explicitly. We tabulate the cases $n = 1$ to $n = 6$ explicitly in terms of rational and trigonometric functions.

$$
\begin{array}{ll}
n & F_n(x) \\
1 & \frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x) \\
2 & \frac{1}{2}+\frac{x}{2\sqrt{x^2+2}} \\
3 & \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{\sqrt{3}x}{\pi(x^2+3)} \\
4 & \frac{1}{2}+\frac{x\left(x^2+6\right)}{2(x^2+4)^{3/2}} \\
5 & \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+\frac{\sqrt{5}x\left(3x^2+25\right)}{3\pi(x^2+5)^2} \\
6 & \frac{1}{2}+\frac{x\left(2x^4+30x^2+135\right)}{4(x^2+6)^{5/2}}
\end{array}
\tag{A.6}
$$

This establishes the general pattern. We can see that odd n contains a mixture of algebraic and trigonometric functions, but the case of n even is always algebraic. The CDF for the case of *any* even n can be written in the form:

$$F_n(x)=\frac{1}{2}+x\left(\frac{x^2}{n}+1\right)^{\frac{1-n}{2}}\left(\sum_{k=0}^{\frac{n}{2}-1}x^{2k}a(k,n)\right) \tag{A.7}$$

where the coefficients are defined recursively by the relations

$$a(0,n)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \tag{A.8}$$

$$a(k,n)=\frac{(n-2k)a(k-1,n)}{(2k+1)n} \tag{A.9}$$

This may be proved by elementary differentiation and noting the the recurrence relation causes cancellations of all non-zero powers of x in the numerator of the resulting expression.

Appendix B. A representation of the density for the additive χ^2 model

The strongly dependent generalization of the Jones approach [23] (see also Bulgren *et al* [6]) involves a rather awkward density function. Although we cannot give it in closed form we have a simple representation as a finite integral as follows. We take the conditional distribution given by Equation (5.6) and the joint density function of Equation (5.9). As with the correlation calculation, we let $z=p^2$, $w=q^2$ and then make a further change of variables $p=r\cos(\theta),\ q=r\sin(\theta)$. The r integral can be done and we are left with the following representation as an integral over $u = \cos(\theta)$. Recall that $a = n_1$ and $b = n_2 - n_1$. We introduce intermediate variables

$$\alpha_1 = 1 + \frac{t_1^2}{a\cos^2(\theta)} + \frac{t_2^2}{(a+b)\cos^2(\theta)}, \quad \alpha_2 = 1 + \frac{t_2^2}{(a+b)\cos^2(\theta)} \tag{B.1}$$

$$\beta = \frac{2\sin(\theta)t_1t_2}{\sqrt{a(a+b)}\cos^2(\theta)} \tag{B.2}$$

We also have the normaling constant

$$D = \frac{2\Gamma\left(\frac{a+b+2}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})\pi\cos(\theta)\sqrt{a(a+b)}} \tag{B.3}$$

Then the density function is given by

$$f(t_1, t_2) = D\int_0^1 du\ u^a(1-u^2)^{(b-2)/2}[\alpha_2 + (\alpha_1 - \alpha_2)u^2 - \beta u]^{-(a+b+2)/2} \tag{B.4}$$

One can show that when $\beta = 0$, for example when $\theta = 0$, this evaluates to a Gauss hypergeometric function in precisely the form originally given by Jones and reproduced here in Equations (5.1) and (5.2). This is readily checked by using the integral identities and transformation identities for the Gauss hypergeometric functions given in [19]. This integral expression is valid for zero or non-zero θ and is a good basis for numerical computation provided $b = n_2 - n_1 > 0$. If $b = 0$ our constructions reduce to that given in the much simpler form by Equations (2.14) and (2.15) with $n_1 = n_2 = n$.

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