General Relativity with Torsion:
Extending Wald's Chapter on Curvature

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Abstract

Most applications of differential geometry, including general relativity, assume that the connection is "torsion free": that vectors do not rotate during parallel transport. Because some extensions of GR (such as string theory) do include torsion, it is useful to see how torsion appears in standard geometrical definitions and formulas in modern language. In this review article, I step through chapter 3, "Curvature", of Robert Wald's textbook *General Relativity* and show what changes when the torsion-free condition is relaxed.

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1 Introduction

Robert Wald's textbook *General Relativity* [1], like most work on differential geometry, includes the assumption that derivative operators ∇_a are "torsion free": for all smooth functions f, $\nabla_a \nabla_b f - \nabla_b \nabla_a f = 0$. This property corresponds intuitively to the condition that vectors not be rotated by parallel transport. Such a condition is natural to impose, and the theory of general relativity itself includes this assumption.

However, differential geometry is equally well defined with torsion as without, and some extensions of general relativity include torsion terms. The first of these was "Einstein-Cartan theory", as introduced by Cartan in 1922 [2] (translated to English with commentary as an appendix to [3]). One review of work in this area is [4]. Of greater current interest, the low energy limit of string theory includes a massless 2-form field whose field strength plays the role of torsion. While torsion can always be treated as an independent tensor field rather than as part of the geometry, the latter approach can be more efficient and may potentially give greater insight into the theory.

I have written this document primarily for my own reference, but I am happy to share it with others. It is written with the assumption that the reader has a copy of Wald's book close at hand; I have not attempted to make it stand on its own. (Readers with a decent knowledge of GR may be able to follow most of it unaided.) I have tried to provide appropriate generalizations of every numbered equation in chapter 3 ("Curvature") that changes in the presence of torsion; in the handful of cases where I have simply described the change in the text, the equation number is printed in **bold**. (I have also done this for appendix B.1 on differential forms.) Any equation not shown with a correction here remains unchanged in the presence of torsion. I have done my best to avoid mistakes in either presentation or results, but I will be grateful for any corrections or feedback on what follows.

2 Defining Torsion

As with most geometric concepts, there are several ways to define torsion. Following Wald's presentation, I will define it in terms of the commutator of derivative operators. (The formulas are provided here for reference; I will explain them in more detail as they arise over the course of the chapter.) As explained in the footnote on Wald's p. (31), the torsion of a connection is characterized by the torsion tensor ${T^c}_{ab}$, which is defined by

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = -{T^c}_{ab} \nabla_c f$$

for any smooth function f. It is also directly related to commutators of vector fields,

$$[v, w]^c = v^a \nabla_a w^c - w^a \nabla_a v^c - {T^c}_{ab} v^a w^b ,$$

and to an antisymmetric component in the Christoffel symbols,

$$2{\Gamma^c}_{[ab]} = {\Gamma^c}_{ab} - {\Gamma^c}_{ba} = {T^c}_{ab} .$$

In string theory, the field strength of the massless 2-form field B_{ab} is the negative of the torsion as defined above:

$$3\nabla_{[a} B_{bc]} = H_{abc} = -T_{abc} .$$

(In the language of differential forms, $\boldsymbol{H} = d\boldsymbol{B}$.) This form is considerably more constrained than general torsion (which need not even be antisymmetric on all three covariant indices), and this will not be our definition in most of what follows. A few further comments on this are given in appendix A.

3 Curvature (with Torsion)

The section and equation numbers here are set to match those in Wald's book. When those equations are not altered by the presence of torsion, they will not be given here, so the numbering may jump occasionally.

3.1 Derivative Operators and Parallel Transport

The first change to Wald's presentation must be to omit the torsion free condition (his condition 5) when defining a derivative operator. Problem 3.1.a in his book asks the reader to show the existence of the torsion tensor (and gives a hint on how to do so, by echoing the derivation of the "change of derivative" tensor C^c_{ab}). The proof is not difficult, so to avoid doing people's homework for them I will simply cite the result here:

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = -{T^c}_{ab} \nabla_c f .$$

This formula comes straight from the footnote to condition 5. The torsion is antisymmetric in its second and third indices, but in general no symmetry involving the first index is required at all.

As noted in Problem 3.1.b, the formula for the commutator of two vector fields changes in the presence of torsion.

$$\begin{aligned} [v, w](f) &= v^a \nabla_a (w^b \nabla_b f) - w^a \nabla_a (v^b \nabla_b f) \\ &= (v^a \nabla_a w^b - w^a \nabla_a v^b) \nabla_b f - v^a w^b T^c{}_{ab} \nabla_c f \ , \end{aligned} \tag{3.1.1}$$

where we have simply applied the Leibnitz rule. This leads to the modified expression

$$[v, w]^c = v^a \nabla_a w^c - w^a \nabla_a v^c - T^c{}_{ab} v^a w^b \ . \tag{3.1.2}$$

Wald's derivation of the tensor $C^c{}_{ab}$ relating two derivative operators is unchanged in the presence of torsion, but the symmetry of that tensor is lost. Taking the commutator of Eq. (3.1.8) under exchange of a and b yields:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = (\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a) f - (C^c{}_{ab} - C^c{}_{ba}) \nabla_c f$$

(where of course ∇_c and $\tilde{\nabla}_c$ are equal when applied to f in the final term). Substituting the definition of torsion in the first two terms shows that the commutator is

$$C^c{}_{ab} - C^c{}_{ba} = T^c{}_{ab} - \tilde{T}^c{}_{ab} \ (\equiv \Delta T^c{}_{ab}) \ . \tag{3.1.9}$$

If the two derivative operators have equal torsion, then these coefficients will have the usual symmetry.

The expressions showing how $C^c{}_{ab}$ relates derivative operators when applied to general tensors are unchanged by the presence of torsion. And with the torsion-free condition relaxed, any $C^c{}_{ab}$ will define a new derivative operator, regardless of its symmetry. In particular, the definition of the Christoffel symbol $\Gamma^c{}_{ab}$ will now also incorporate torsion.

The definition of parallel transport is not changed in the presence of torsion, and it still defines a connection on the manifold. But when seeking a derivative operator compatible with the metric g_{ab}, torsion remains unconstrained. Requiring that parallel transport leave inner products $g_{ab} v^a w^b$ invariant essentially means that the vectors' lengths and angles relative to one another must be unchanged from point to point. But this does not specify anything about "global" rotations of the tangent space during parallel transport: that is the physical meaning of torsion.

When we allow derivative operators with torsion, the statement of Theorem 3.1.1 must be modified as explained in Problem 3.1.c:

Theorem 3.1.1 *Let g_{ab} be a metric and $T^c{}_{ab}$ be a torsion. Then there exists a unique derivative operator ∇_a with this torsion satisfying $\nabla_a g_{bc} = 0$.*

Wald's proof holds as written up through Eq. (3.1.26). We have seen that torsion changes the symmetry rule in Eq. (3.1.9), so the next step in the proof becomes

$$2C_{cab} = \tilde{\nabla}_a g_{cb} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} - \Delta T_{abc} - \Delta T_{bac} + \Delta T_{cab} \ . \tag{3.1.27}$$

We have already seen that the difference in torsion is responsible for the antisymmetric part $C_{c[ab]}$, but this expression shows that it can contribute to the symmetric part $C_{c(ab)}$ as well. The torsion contribution to this symmetric part is zero if and only if the difference in torsion is totally antisymmetric in its three indices, $\Delta T_{cab} = \Delta T_{[cab]}$.

It is clear from this that Eq. (3.1.28) becomes

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{db} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} - \Delta T_{abd} - \Delta T_{bad} + \Delta T_{dab} \right) \ . \tag{3.1.28}$$

That in turn leads to modified expressions for the Christoffel symbols:

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab} - T_{abd} - T_{bad} + T_{dab} \right) \ , \tag{3.1.29}$$

or in components,

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} \sum_\sigma g^{\rho\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - T_{\mu\nu\sigma} - T_{\nu\mu\sigma} + T_{\sigma\mu\nu} \right) \ . \tag{3.1.30}$$

With this expression for the Christoffel symbols in hand, intuition for the effect of torsion can be developed by considering Wald's Eq. (3.1.19) on a manifold with a flat Cartesian metric but non-zero torsion for various choices of curve (with tangent vector t^a) and vector to be parallel transported v^a. In the particularly simple case where the torsion is totally antisymmetric, that equation becomes

$$\frac{dv^\nu}{dt} + \frac{1}{2} \sum_{\mu,\lambda} t^\mu T^\nu{}_{\mu\lambda} v^\lambda = 0 \qquad (\text{when } g_{ab} = \eta_{ab} \text{ and } T_{cab} = T_{[cab]}) \ .$$

So for instance, if $T^z{}_{xy} > 0$, parallel transport along the x direction will cause v to rotate about the x-axis in a left-handed manner.

In this special case, it is clear from this expression that a vector v^a tangent to the curve (i.e. parallel to t^a) will not be affected by torsion. This is not the case for more general choices of torsion (which can contribute to the symmetric part of $\Gamma^c{}_{ba}$). General torsion can lead to significant changes when we consider geodesics.

3.2 Curvature

Our definition of curvature must also be generalized in the presence of torsion. Wald's approach is still valid, but the first term on the right hand side of Eq. (3.2.1) no longer cancels in the next step. After subtracting $\nabla_b \nabla_a (f\omega_c)$ from Eq. (3.1.1), we find

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f\omega_c) = (-T^d{}_{ab} \nabla_d f)\omega_c + f(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c \; .$$

Adding an appropriate derivative of ω_c to both sides then gives

$$(\nabla_a \nabla_b - \nabla_b \nabla_a + T^d{}_{ab} \nabla_d)(f\omega_c) = f(\nabla_a \nabla_b - \nabla_b \nabla_a + T^d{}_{ab} \nabla_d)\omega_c \; . \quad (3.2.2)$$

Thus, by the same reasoning as in the torsion-free case, the expression in parentheses is a tensor:

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c + T^d{}_{ab} \nabla_d \omega_c = R_{abc}{}^d \omega_d \; . \quad (3.2.3)$$

This defines the Riemann curvature tensor in the presence of torsion.

This simple change in definition,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \to (\nabla_a \nabla_b - \nabla_b \nabla_a + T^d{}_{ab} \nabla_d) \; ,$$

is the only modification necessary for some time. This substitution arises in the middle line of **Eq. (3.2.8)** due to the change in Eq. (3.1.2) for the commutator of vector fields, but all of the results in that section are the same: the new Riemann tensor correctly measures the path dependence of parallel transport. And the generalized definition is used in **Eq. (3.2.10)**, **Eq. (3.2.11)**, and **Eq. (3.2.12)**, which show that it is still valid for arbitrary tensor fields. (There is also an extra term in the second line of **Eq. (3.2.10)**, but it is straightforward to find and has no broad significance.)

The generalizations of the "four key properties of the Riemann tensor" are more interesting. Property (1) ($R_{abc}{}^d = -R_{bac}{}^d$) still follows directly from the definition, because $T^c{}_{ab}$ is antisymmetric in a and b. Property (2) is much less attractive:

$$R_{[abc]}{}^d = -\nabla_{[a}T^d{}_{bc]} + T^e{}_{[ab}T^d{}_{c]e} \; . \qquad (3.2.14)$$

Property (3) ($R_{abcd} = -R_{abdc}$) still holds. And the Bianchi identity, property (4), is modified:

$$\nabla_{[a}R_{bc]d}{}^e = T^f{}_{[ab}R_{c]fd}{}^e \; . \qquad (3.2.16)$$

In the proof of property (2), the starting point is modified:

$$2\nabla_{[a}\nabla_b\omega_{c]} + T^d{}_{[ab|}\nabla_d\omega_{|c]} = -\nabla_{[a}T^d{}_{bc]}\omega_d + T^e{}_{[ab}T^d{}_{c]e}\omega_d \; . \qquad (3.2.17)$$

In deriving this equation, we have used Eq. (B.1.7) for $\nabla^2\boldsymbol{\omega}$. This provides the appropriate form for Eq. (3.2.18):

$$\begin{aligned} R_{[abc]}{}^d\omega_d &= \nabla_{[a}\nabla_b\omega_{c]} - \nabla_{[b}\nabla_a\omega_{c]} + T^d{}_{[ab|}\nabla_d\omega_{|c]} \\ &= 2\nabla_{[a}\nabla_b\omega_{c]} + T^d{}_{[ab|}\nabla_d\omega_{|c]} \\ &= \left(-\nabla_{[a}T^d{}_{bc]} + T^e{}_{[ab}T^d{}_{c]e}\right)\omega_d \; . \end{aligned} \qquad (3.2.18)$$

The proof of property (3) in **Eq. (3.2.19)** is essentially unchanged; the only change in the equation is the usual substitution of definitions. However, the change in property (2) means that the Riemann tensor with torsion is no longer symmetric under exchange of the first pair of indices with the second. In the absence of torsion, we were able to write:

$$2R_{cdab} = R_{cdab}\overbrace{-R_{dacb}}\overbrace{-R_{acdb}} = R_{dcba}\overbrace{\underbrace{-R_{abdc}}-R_{bdac}}\overbrace{\underbrace{-R_{bacd}}-R_{cbad}} = \underbrace{2R_{abcd}}$$

But when we use our modification of property (2), this becomes much more complicated. All four uses of Eq. (3.2.14) add different torsion factors:

$$\begin{aligned} 2R_{cdab} = 2R_{abcd} + 3\,\big(\, &\nabla_{[b|}T_{a|cd]} - \nabla_{[a|}T_{b|cd]} - \nabla_{[d|}T_{c|ab]} + \nabla_{[c|}T_{d|ab]} \\ &+ T_{ae[b}T^e{}_{cd]} - T_{be[a}T^e{}_{cd]} - T_{ce[d}T^e{}_{ab]} + T_{de[c}T^e{}_{ab]}\big) \; . \end{aligned} \qquad (3.2.20)$$

If the torsion is totally antisymmetric, $T_{cab} = T_{[cab]}$, this expression simplifies enormously. The torsion-squared terms cancel out completely, and the derivatives of torsion combine and simplify to

$$R_{abcd} + \nabla_{[a}T_{b]cd} = R_{cdab} + \nabla_{[c}T_{d]ab} \qquad (\text{when } T_{cab} = T_{[cab]}). \qquad (3.2.20a)$$

In the special case where $\boldsymbol{dT} = 4\partial_{[a}T_{bcd]} = 0$, an even more elegant relation holds:

$$R_{cdab} = R_{abcd}|_{T\to -T} \qquad (\text{when } T_{cab} = T_{[cab]} \text{ and } \partial_{[a}T_{bcd]} = 0). \tag{3.2.20b}$$

The proof of this relation is given in section 3.4a, once we have found an explicit expression for the Riemann tensor.

Finally, we come to the Bianchi identity, property (4). We will simply apply our usual change in definition to the basic formulas used by Wald:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a + T^e{}_{ab}\nabla_e)\nabla_c\omega_d = R_{abc}{}^e\nabla_e\omega_d + R_{abd}{}^e\nabla_c\omega_e \tag{3.2.21}$$

and

$$\nabla_a\left[(\nabla_b\nabla_c - \nabla_c\nabla_b + T^e{}_{bc}\nabla_e)\omega_d\right] = \nabla_a\left[R_{bcd}{}^e\omega_e\right] = \omega_e\nabla_a R_{bcd}{}^e + R_{bcd}{}^e\nabla_a\omega_e \ . \tag{3.2.22}$$

Without torsion, antisymmetrizing over a, b, and c makes the left hand sides of these equations equal. But the torsion term in the second equation adds considerable complication:

$$\begin{aligned} \nabla_{[a}\left(T^e{}_{bc]}\nabla_e\omega_d\right) &= (\nabla_{[a}T^e{}_{bc]})\nabla_e\omega_d + T^e{}_{[bc}\nabla_{a]}\nabla_e\omega_d \\ &= (\nabla_{[a}T^e{}_{bc]})\nabla_e\omega_d + T^e{}_{[bc}R_{a]ed}{}^f\omega_f \\ &\quad + T^e{}_{[bc|}\nabla_e\nabla_{|a]}\omega_d - T^e{}_{[bc}T^f{}_{a]e}\nabla_f\omega_d \ . \end{aligned}$$

The third term on the right hand side is finally of the proper form to match the torsion term in Eq. (3.2.21), so after antisymmetrization, the second equation is equal to the first *plus* the first, second, and fourth terms immediately above.

Meanwhile, the antisymmetrized first term on the right hand side of Eq. (3.2.21) is also non-trivial once torsion is included, as we must use Eq. (3.2.14):

$$R_{[abc]}{}^e\nabla_e\omega_d = (-\nabla_{[a}T^e{}_{bc]} + T^f{}_{[ab}T^e{}_{c]f})\nabla_e\omega_d \ .$$

These terms will cancel with the first and fourth extra terms from the previous correction. Showing terms from the second equation first:

$$\begin{aligned} \omega_e\nabla_{[a}R_{bc]d}{}^e + R_{[bc|d}{}^e\nabla_{a]}\omega_e &= R_{[abc]}{}^e\nabla_e\omega_d + R_{[ab|d}{}^e\nabla_{|c]}\omega_e + T^e{}_{[bc}R_{a]ed}{}^f\omega_f \\ &\quad + (\nabla_{[a}T^e{}_{bc]} - T^f{}_{[bc}T^e{}_{a]f})\nabla_e\omega_d \\ &= R_{[ab|d}{}^e\nabla_{|c]}\omega_e + T^e{}_{[bc}R_{a]ed}{}^f\omega_f \ . \end{aligned} \tag{3.2.23}$$

After a final cancellation of terms between the two sides, this leaves

$$\omega_e \nabla_{[a} R_{bc]d}{}^e = \omega_e T^f{}_{[bc} R_{a]fd}{}^e \ , \tag{3.2.24}$$

which leads to the Bianchi identity as stated above.

In the orthonormal tetrad method of section 3.4b, the Riemann tensor is treated as a differential 2-form in its first two indices, $R_{ab\mu}{}^{\nu} \equiv \boldsymbol{R}_{\mu}{}^{\nu}$. In the notation introduced in my appendix B.1, the left hand side of the Bianchi identity with torsion is written $\nabla \boldsymbol{R}_{\mu}{}^{\nu}$. As shown below Eq. (B.1.6), ∇ can be related to the usual derivative d by

$$(\nabla \boldsymbol{R}_{\mu}{}^{\nu})_{abc} = (d\boldsymbol{R}_{\mu}{}^{\nu})_{abc} + 3T^d{}_{[ab} R_{c]d\mu}{}^{\nu} \ \left(= (d\boldsymbol{R}_{\mu}{}^{\nu})_{abc} + T^{\sigma} \wedge \boldsymbol{R}_{(\mathbf{1})}{}_{\sigma\mu}{}^{\nu} \right) \ .$$

Comparing this with our result for the Bianchi identity yields the result $d\boldsymbol{R}_{\mu}{}^{\nu} = 0$, just as in the torsion-free case. (On the other hand, a direct form-based proof of that identity would make this an alternate derivation of our Bianchi identity.)

We can still define the Ricci tensor as the trace of the Riemann tensor, and its symmetry can be found by contracting Eq. (3.2.20) with g^{bd}:

$$R_{ac} = R_{ca} - 3\nabla_{[a} T^b{}_{bc]} + T^b{}_{be} T^e{}_{ac} \ . \tag{3.2.26}$$

If the torsion is totally antisymmetric, the final term vanishes and the equation reduces to $R_{ac} = R_{ca} + \nabla_b T^b{}_{ac}$; if $d\boldsymbol{T} = 0$, Eq. (3.2.20b) indicates that $R_{ac} = R_{ca}|_{T \to -T}$. The Ricci scalar's definition is entirely unchanged.

The Weyl tensor C_{abcd} can still be defined as the trace free part of the Riemann tensor. Wald's expression assumes that the Ricci tensor is symmetric, but only minor re-ordering is required:

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(R_{a[c} g_{d]b} - R_{b[c} g_{d]a}) - \frac{2}{(n-1)(n-2)} R g_{a[c} g_{d]b} \ . \tag{3.2.28}$$

The Weyl tensor still satisfies properties (1) and (3), and its analog of condition (2) can be computed from $C_{[abc]d} = R_{[abc]d} + 2/(n-2)\, R_{[ab} g_{c]d}$ using results above. (I have not checked whether $C_{abc}{}^d$ remains invariant under conformal transformations.)

The modified Bianchi identity remains complicated after contraction:

$$\nabla_a R_{cbd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 2T^e{}_{a[b} R_{c]ed}{}^a - T^e{}_{bc} R_{ed} \ . \tag{3.2.29}$$

Contracting again with g^{bd} then gives

$$2\nabla_a R_c{}^a - \nabla_c R = -T^{eab} R_{ceab} - 2T^{ea}{}_c R_{ea} \,. \tag{3.2.30}$$

Because no derivatives are explicit on the right hand side, there is no obvious generalization of the Einstein tensor for arbitrary torsion.

In the special case where $T_{cab} = T_{[cab]}$, the antisymmetry allows us to use identities related to property (2) to reduce the right hand side of this expression to $(T^{eab}\nabla_c T_{eab} - T^{eab}\nabla_e T_{cab} - 2T_{cab}\nabla_e T^{eab})/2$. This is not quite a total (covariant) derivative, so even in this case no obvious generalization of the Einstein tensor exists: there is no clear analog of **Eq. (3.2.31)** or **Eq. (3.2.32)**. In Einstein-Cartan theory, the stress-energy tensor is modified by terms related to spin and has similarly non-vanishing divergence.

3.3 Geodesics

The concept of a geodesic can be formulated in two ways. One is the definition given by Wald in Eq. (3.3.1): a "straightest possible line" whose tangent vector is parallel propagated along itself. The other is the source of the name (as I understand it): a "shortest possible path" between any two of its points (or more generally, an "extremal length path" between them). In the presence of torsion, these two concepts need no longer be equivalent. For the sake of consistency with Wald's definition, we will take the term "geodesic" to imply the first meaning but not necessarily the second.

Most of Wald's discussion in this section requires no modification at all. As mentioned at the end of section 3.1, the geodesic equation depends only on the symmetric part of the Christoffel symbols, so if $T_{cab} = T_{[cab]}$, torsion will not affect the equation at all.

The first statement that may be changed by the presence of torsion is the assertion that in Gaussian normal coordinates, geodesics remain orthogonal to the hypersurfaces S_t. Because the definition of the commutator of vector fields has changed, we find

$$\begin{aligned} n^b\nabla_b(n_a X^a) &= n_a n^b \nabla_b X^a \\ &= n_a X^b \nabla_b n^a + n_a T^a{}_{bc} n^b X^c \\ &= \frac{1}{2} X^b \nabla_b (n^a n_a) + T_{abc} n^a n^b X^c \\ &= T_{abc} n^a n^b X^c \,. \end{aligned} \tag{3.3.6}$$

Thus, orthogonality is maintained if and only if the torsion is totally antisymmetric.

The next change, as stated earlier, comes in the proof that the shortest path between two points is a geodesic. The essence of the change is that while the geodesic equation may depend on torsion, distances depend only on the metric (so even on a manifold with torsion, the "shortest paths" must correspond to torsion-free geodesics).

Formally, all of the mathematics leading up to Eq. (3.3.13) remain unchanged, but the result may no longer match our generalized expression for the Christoffel symbols in equation (3.1.30). The antisymmetric part of the Christoffel symbols will not contribute here in any case, but the torsion will change the symmetric part (and thus invalidate the conclusion that extremal length paths are geodesics) if and only if $T_{cab} \neq T_{[cab]}$. I do not believe that a geodesic equation including torsion can be obtained from a "point particle" Lagrangian as in Eq. (3.3.14), although that Lagrangian can still be used to find the part of the Christoffel symbols that depends only on the metric.

Finally, in the discussion of the geodesic deviation equation, the first change is directly related to that of Eq. (3.3.6):

$$T^b \nabla_b X^a = X^b \nabla_b T^a + T^a{}_{bc} T^b X^c \; ; \tag{3.3.16}$$

as in Eq. (3.3.6), $X^a T_a$ need no longer be constant along the geodesics.

The acceleration $a^a = T^c \nabla_c v^a$ is then given by:

$$\begin{aligned} a^a &= T^c \nabla_c (T^b \nabla_b X^a) \\ &= T^c \nabla_c (X^b \nabla_b T^a + T^a{}_{de} T^d X^e) \\ &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a + T^c \nabla_c (T^a{}_{de} T^d X^e) \\ &= (X^c \nabla_c T^b)(\nabla_b T^a) + (T^b{}_{cd} T^c X^d)(\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a \\ &\quad - X^b T^c T^d{}_{cb} \nabla_d T^a - R_{cbd}{}^a X^b T^c T^d + T^c \nabla_c (T^a{}_{de} T^d X^e) \\ &= X^c \nabla_c (T^b \nabla_b T^a) - R_{cbd}{}^a X^b T^c T^d + T^c \nabla_c (T^a{}_{de} T^d X^e) \\ &= R_{bcd}{}^a X^b T^c T^d + T^c \nabla_c (T^a{}_{de} T^d X^e) \; . \end{aligned} \tag{3.3.18}$$

This result may be more intuitive when written explicitly in terms of the relative velocity: $T^c \nabla_c (v^a - T^a{}_{de} T^d X^e) = R_{bcd}{}^a X^b T^c T^d$. The left hand side shows that the relative velocity of two geodesics naturally changes during parallel transport due to the effects of torsion. Only the deviations from that direct torsion contribution are due to the manifold's curvature.

3.4 Methods for Computing Curvature

3.4a Coordinate Component Method

The given expressions for derivatives of a dual vector field still hold for non-zero torsion. When constructing the Riemann tensor, the second line of Eq. (3.4.2) no longer vanishes after a and b are antisymmetrized, but its contribution precisely cancels out the explicit torsion term in our modified definition of the Riemann tensor. Thus, Wald's expression for the Riemann tensor is essentially correct, except that it assumes the symmetry of the Christoffel symbol in one place. The proper index order is

$$R_{abc}{}^{d}\omega_d = \left[-2\partial_{[a}\Gamma^{d}{}_{b]c} + 2\Gamma^{e}{}_{[a|c}\Gamma^{d}{}_{|b]e}\right]\omega_d \; . \tag{3.4.3}$$

In components, this yields the following expression for the Riemann tensor:

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^{\nu}}\Gamma^{\sigma}{}_{\mu\rho} - \frac{\partial}{\partial x^{\mu}}\Gamma^{\sigma}{}_{\nu\rho} + \sum_{\alpha}\left(\Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\alpha}\right) \; . \tag{3.4.4}$$

It may at times be useful to decompose the Christoffel symbols into a sum of metric- and torsion-derived parts:

$$\Gamma^{c}{}_{ab} \equiv \hat{\Gamma}^{c}{}_{ab} + \frac{1}{2}\left(T^{c}{}_{ab} - T_{ab}{}^{c} - T_{ba}{}^{c}\right) \; .$$

Here, $\hat{\Gamma}^{c}{}_{ab}$ is the part of the Christoffel symbol that is independent of torsion. (As usual, the final two torsion terms cancel if the torsion is totally antisymmetric.) We can use this to expand out the torsion contribution to the Riemann tensor, writing the torsion independent part as $\hat{R}_{abc}{}^{d}$:

$$\begin{aligned} R_{abc}{}^{d} &= \hat{R}_{abc}{}^{d} - \partial_{[a}T^{d}{}_{b]c} + \partial_{[a}T_{b]c}{}^{d} + \partial_{[a|}T_{c|b]}{}^{d} \\ &\quad + \hat{\Gamma}^{e}{}_{[a|c}\left(T^{d}{}_{|b]e} - T_{b]e}{}^{d} - T_{e|b]}{}^{d}\right) + \left(T^{e}{}_{[a|c} - T_{[a|c}{}^{e} - T_{c[a}{}^{e}\right)\hat{\Gamma}^{d}{}_{|b]e} \\ &\quad + \frac{1}{2}\left(T^{e}{}_{[a|c} - T_{[a|c}{}^{e} - T_{c[a}{}^{e}\right)\left(T^{d}{}_{|b]e} - T_{b]e}{}^{d} - T_{e|b]}{}^{d}\right) \; . \end{aligned}$$

For totally antisymmetric torsion, this simplifies substantially:

$$\begin{aligned} R_{abc}{}^{d} &= \hat{R}_{abc}{}^{d} - \partial_{[a}T^{d}{}_{b]c} + \hat{\Gamma}^{e}{}_{[a|c}T^{d}{}_{|b]e} + T^{e}{}_{[a|c}\hat{\Gamma}^{d}{}_{|b]e} + \frac{1}{2}T^{e}{}_{[a|c}T^{d}{}_{|b]e} \\ &= \hat{R}_{abc}{}^{d} - \nabla_{[a}T_{b]c}{}^{d} + \frac{1}{2}\left(T^{e}{}_{[a|c}T_{e|b]}{}^{d} - T^{e}{}_{ab}T_{ec}{}^{d}\right) \; . \end{aligned}$$

To compare this with Eq. (3.2.20a) for totally antisymmetric torsion, note that if the d index is lowered, $\hat{R}_{abcd}$ and both terms in parentheses on the second line are invariant under $\{a, b\} \leftrightarrow \{c, d\}$.

We can now derive the elegant expression $R_{cdab} = R_{abcd}|_{T\to -T}$ that holds in the special case $d\boldsymbol{T} = 4\partial_{[a}T_{bcd]} = 0$ (as stated in Eq. (3.2.20b) above). In the final line above, all of the terms are manifestly invariant under this transformation except $-\nabla_{[a}T_{b]c}{}^{d}$. Thus, our goal is to show that $\nabla_{[a}T_{b]cd} = \tilde{\nabla}_{[c}\tilde{T}_{d]ab}$, where $\tilde{\nabla}_a$ is a new derivative operator that differs from ∇_a only in its torsion: $\tilde{T}^c{}_{ab} = -T^c{}_{ab}$.

Using results from appendix B.1,

$$2\nabla_{[a}T_{bcd]} = 2\partial_{[a}T_{bcd]} - 3T^e{}_{[ab}T_{cd]e} = -3T^e{}_{[ab}T_{cd]e} \ .$$

Therefore,

$$\nabla_{[a}T_{b]cd} = 2\nabla_{[a}T_{bcd]} - \nabla_{[c}T_{d]ab} = -3T^e{}_{[ab}T_{cd]e} - \nabla_{[c}T_{d]ab} \ .$$

We must next convert to the new derivative operator $\tilde{\nabla}_a$, which is related to ∇_a by the tensor $C^c{}_{ab} = T^c{}_{ab}$. We find that

$$\begin{aligned} \nabla_{[c}T_{d]ab} &= \tilde{\nabla}_{[c}T_{d]ab} - T^e{}_{cd}T_{eab} - T^e{}_{[c|a}T_{|d]eb} - T^e{}_{[c|b}T_{|d]ae} \\ &= \tilde{\nabla}_{[c}T_{d]ab} - T^e{}_{cd}T_{eab} + 2T^e{}_{[c|a}T_{e|d]b} \ . \end{aligned}$$

Putting this all together, the result is

$$\nabla_{[a}T_{b]cd} = -3T^e{}_{[ab|}T_{e|cd]} - \tilde{\nabla}_{[c}T_{d]ab} + T^e{}_{cd}T_{eab} - 2T^e{}_{[c|a}T_{e|d]b} = \tilde{\nabla}_{[c}\tilde{T}_{d]ab} \ ,$$

as the explicit torsion squared terms cancel out when the antisymmetrizations are expanded. This proves the desired relation.

Getting back to the flow of Wald's presentation, the next step is to correct his expression for the Ricci tensor. Again, it is essentially correct apart from some index ordering:

$$R_{\mu\rho} = \sum_{\nu} \frac{\partial}{\partial x^\nu}\Gamma^\nu{}_{\mu\rho} - \frac{\partial}{\partial x^\mu}\sum_{\nu}\Gamma^\nu{}_{\nu\rho} + \sum_{\alpha,\nu}\left(\Gamma^\alpha{}_{\mu\rho}\Gamma^\nu{}_{\nu\alpha} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\nu{}_{\mu\alpha}\right) \ . \qquad (3.4.5)$$

The formula for the contracted Christoffel symbol requires the addition of a torsion piece,

$$\Gamma^a{}_{a\mu} = \sum_{\nu}\Gamma^\nu{}_{\nu\mu} = \frac{1}{2}\sum_{\nu,\alpha} g^{\nu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \sum_{\nu} T^\nu{}_{\nu\mu} \ . \qquad (3.4.7)$$

As far as I know, there is no further simplification to be found for this torsion contribution in general (for totally antisymmetric torsion, it vanishes entirely). Thus, the final simple formula for the contracted Christoffel symbol is

$$\Gamma^{a}{}_{a\mu} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^{\mu}} + \sum_{\nu} T^{\nu}{}_{\nu\mu} = \frac{\partial}{\partial x^{\mu}} \ln\sqrt{|g|} + \sum_{\nu} T^{\nu}{}_{\nu\mu} \; . \tag{3.4.9}$$

And following from this, the divergence of a vector field is

$$\nabla_a T^a = \partial_a T^a + \Gamma^{a}{}_{ab} T^b = \sum_{\mu} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\mu}} (\sqrt{|g|}\, T^{\mu}) + \sum_{\mu,\alpha} T^{\alpha}{}_{\alpha\mu} T^{\mu} \; . \tag{3.4.10}$$

3.4b Orthonormal Basis (Tetrad) Methods

Wald names the torsion free condition "ingredient (2)" in determining the curvature, so that term is a signal indicating that changes are required. Most of the basic definitions used in this approach remain unchanged, so the first change is simply to substitute the modified definition in the expression for the components of the Riemann tensor:

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= R_{abcd}(e_\rho)^a(e_\sigma)^b(e_\mu)^c(e_\nu)^d \\ &= (e_\rho)^a(e_\sigma)^b(e_\mu)^c(\nabla_a\nabla_b - \nabla_b\nabla_a + T^{d}{}_{ab}\nabla_d)(e_\nu)_c \; . \end{aligned} \tag{3.4.17}$$

The extra term can be expressed very simply using the definition of the connection 1-forms, leading to the result

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= (e_\rho)^a(e_\sigma)^b \left\{ \nabla_a\omega_{b\mu\nu} - \nabla_b\omega_{a\mu\nu} + T^{d}{}_{ab}\omega_{d\mu\nu} \right. \\ &\qquad \left. - \sum_{\alpha,\beta} \eta^{\alpha\beta} \left[\omega_{a\beta\mu}\omega_{b\alpha\nu} - \omega_{b\beta\mu}\omega_{a\alpha\nu}\right] \right\} \; . \end{aligned} \tag{3.4.20}$$

This leads directly to a corresponding change in the expression written in terms of the Ricci rotation coefficients:

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= (e_\rho)^a\nabla_a\omega_{\sigma\mu\nu} - (e_\sigma)^a\nabla_a\omega_{\rho\mu\nu} + \sum_{\lambda} T^{\lambda}{}_{\rho\sigma}\omega_{\lambda\mu\nu} \\ &\quad - \sum_{\alpha,\beta} \eta^{\alpha\beta} \left\{\omega_{\rho\beta\mu}\omega_{\sigma\alpha\nu} - \omega_{\sigma\beta\mu}\omega_{\rho\alpha\nu} + \omega_{\rho\beta\sigma}\omega_{\alpha\mu\nu} - \omega_{\sigma\beta\rho}\omega_{\alpha\mu\nu}\right\} \; . \end{aligned} \tag{3.4.21}$$

We next come to the the discussion of "ingredient (2)", and the first change is due to the corrected formula for the commutator of vector fields:

$$\begin{aligned} (e_\sigma)_a[e_\mu, e_\nu]^a &= (e_\sigma)_a \left\{(e_\mu)^b\nabla_b(e_\nu)^a - (e_\nu)^b\nabla_b(e_\mu)^a - T^{a}{}_{bc}(e_\mu)^b(e_\nu)^c\right\} \\ &= \omega_{\mu\sigma\nu} - \omega_{\nu\sigma\mu} - T_{\sigma\mu\nu} \; . \end{aligned} \tag{3.4.23}$$

Eq. (3.4.24) for the antisymmetrized derivative of the connection 1-forms still holds, and Eq. (B.1.6) shows precisely how to write it in terms of the ordinary derivative:

$$\partial_{[a}(e_\sigma)_{b]} = \nabla_{[a}(e_\sigma)_{b]} + \frac{1}{2}T^c{}_{ab}(e_\sigma)_c = \sum_{\mu,\nu} \eta^{\mu\nu}(e_\mu)_{[a}\omega_{b]\sigma\nu} + \frac{1}{2}T^c{}_{ab}(e_\sigma)_c \ . \quad (3.4.25)$$

Finally, we can translate these results into the language of differential forms. The torsion becomes a collection of 2-forms, $(\boldsymbol{T}^\sigma)_{ab} = (e^\sigma)_c T^c{}_{ab}$. Then Eq. (3.4.25) can be written

$$d\boldsymbol{e}_\sigma = \sum_\mu \boldsymbol{e}_\mu \wedge \boldsymbol{\omega}_\sigma{}^\mu + \boldsymbol{T}_\sigma \ , \quad (3.4.27)$$

or, using the notation introduced in appendix B.1 for a derivative with torsion, $\nabla \boldsymbol{e}_\sigma = \sum_\mu \boldsymbol{e}_\mu \wedge \boldsymbol{\omega}_\sigma{}^\mu$. Similarly, we can write Eq. (3.4.20) as

$$\boldsymbol{R}_\mu{}^\nu = d\boldsymbol{\omega}_\mu{}^\nu + \sum_\alpha \boldsymbol{\omega}_\mu{}^\alpha \wedge \boldsymbol{\omega}_\alpha{}^\nu \ . \quad (3.4.28)$$

This equation is identical to the torsion free result, but that fact is sufficiently surprising (and the equation is sufficiently important) that I have chosen to duplicate it here anyway. In terms of the derivative with torsion, this equation is less elegant $\boldsymbol{R}_\mu{}^\nu = \nabla\boldsymbol{\omega}_\mu{}^\nu + \sum_\alpha(\boldsymbol{\omega}_\mu{}^\alpha \wedge \boldsymbol{\omega}_\alpha{}^\nu + \boldsymbol{T}^\alpha \omega_{\alpha\mu}{}^\nu)$. In general, it seems that using the ordinary derivative d is the simplest approach: the torsion appears in the equations of structure exactly once, and in a very straightforward way.

A B-Fields and Non-symmetric Metrics

[*Note that this appendix has nothing to do with Wald's appendix A, which for its part has nothing to do with torsion.*]

In string theory (and several related theories), the metric g_{ab} is accompanied by a 2-form field B_{ab} whose field strength $H_{abc} = 3\partial_{[a}B_{cd]}$. The two sometimes appear in the combination $G_{ab} \equiv g_{ab} + B_{ab}$, which can in some ways be thought of as a generalized metric that is not necessarily symmetric (but remains nondegenerate).

However, this perspective should not be taken too seriously. Tensor indices are still raised and lowered with g^{ab} and g_{ab} alone (otherwise, the map between the tangent space and its dual would be more difficult to define, as $G_{ab}v^b \neq G_{ba}v^b$). And the derivative operator is still chosen so as to satisfy $\nabla_a g_{bc} = 0$. (If the derivative operator instead obeyed $\nabla_a G_{bc} = 0$, this would imply both $\nabla_a g_{bc} = 0$ and $\nabla_a B_{bc} = 0$. The latter leads to a constraint on the torsion $T^d{}_{[ab}B_{c]d} = -\partial_{[a}B_{bc]}$. An appropriate torsion could probably be chosen to satisfy this constraint, which would make Eq. (3.1.27) hold as written above, but this is not the approach that is relevant here.)

The actual effect of the B-field in string theory is to introduce a torsion directly. Eq. (3.1.29) becomes

$$\begin{aligned}\Gamma^c{}_{ab} &= \frac{1}{2}g^{cd}\left(\partial_a G_{db} + \partial_b G_{ad} - \partial_d G_{ab}\right) \\ &= \frac{1}{2}g^{cd}\left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab} - 3\partial_{[a}B_{bd]}\right) \ .\end{aligned}$$

This corresponds to a torsion $T_{abc} = -H_{abc}$. (The relative sign here is purely conventional: if we had reversed the order of indices on the three G_{ab} terms, the sign of H would have been positive.) In components, this reads

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial B_{\nu\sigma}}{\partial x^\mu} - \frac{\partial B_{\sigma\mu}}{\partial x^\nu} - \frac{\partial B_{\mu\nu}}{\partial x^\sigma}\right) \ .$$

B Differential Forms, et cetera

B.1 Differential Forms

Perhaps the greatest value of differential forms as they are usually presented is that their properties are independent of the choice of derivative operator. In the presence of torsion that independence is partly broken, although they are still independent of the metric.

Naturally, the results change only when derivatives are involved. This first appears when relating the antisymmetrized derivatives for different derivative operators:

$$\nabla_{[b}\omega_{a_1\cdots a_p]} - \tilde{\nabla}_{[b}\omega_{a_1\cdots a_p]} = -\sum_{j=1}^{p} C^d{}_{[ba_j}\omega_{a_1\cdots|d|\cdots a_p]} = (-1)^p\,\frac{p}{2}\,\Delta T^d{}_{[ba_1}\omega_{a_2\cdots a_p]d} \ . \tag{B.1.6}$$

(Note that this corrects a sign error in Wald's equation, which had no effect when torsion was zero.) Thus, this map is only unique when the manifold's torsion is specified, but it still does not require a preferred metric. I will denote this map by ∇, and I will continue to use d to refer to the torsion free case. (This notation is almost certainly not standard, but it seems sensible.) In particular,

$$(\nabla\boldsymbol{\omega})_{ba_1\cdots a_p} = (d\boldsymbol{\omega})_{ba_1\cdots a_p} + (-1)^p \, \frac{p(p+1)}{2} \, T^d{}_{[ba_1}\omega_{a_2\cdots a_p]d} \, .$$

The final term is clearly a differential form as well, but it is not clear (to me, at least) how to express it in pure form language. It is tempting to write it as $(-1)^p \boldsymbol{T}^d \wedge (\boldsymbol{\omega}_{(p-1)})_d$, simply treating the contracted indices as labels on a set of forms much like μ in the tetrad $(e_\mu)^a$. In fact, I am fairly confident that this approach would at least work in the orthonormal tetrad context: there, the torsion is treated as a set of 2-forms T^σ, and I see no danger in decomposing a p-form $\boldsymbol{\omega}$ into a set of $(p-1)$-forms $(\boldsymbol{\omega}_{(p-1)})_\sigma$.

It is clear that for general torsion, the Poincaré lemma will not hold: $\nabla^2 = \nabla \circ \nabla \neq 0$. In particular, for a scalar field f this is simply the definition of torsion: $(\nabla^2 f)_{ab} = -T^c{}_{ab}\nabla_c f$. For more general forms, the formula becomes

$$\begin{aligned}
\frac{2(\nabla^2\boldsymbol{\omega})_{bca_1\cdots a_p}}{(p+2)(p+1)} &= 2\nabla_{[b}\nabla_c\omega_{a_1\cdots a_p]} \\
&= \sum_{j=1}^{p} R_{[bca_j}{}^d\,\omega_{a_1\cdots|d|\cdots a_p]} - T^d{}_{[bc|}\nabla_d\omega_{|a_1\cdots a_p]} \\
&= -p(-1)^p R_{[bca_1}{}^d\,\omega_{a_2\cdots a_p]d} - T^d{}_{[bc|}\nabla_d\omega_{|a_1\cdots a_p]} \qquad \text{(B.1.7)} \\
&= p(-1)^p(\nabla_{[b}T^d{}_{ca_1|} - T^e{}_{[bc}T^d{}_{a_1|e})\omega_{|a_2\cdots a_p]d} - T^d{}_{[bc|}\nabla_d\omega_{|a_1\cdots a_p]}
\end{aligned}$$

In the final line, we have substituted for the antisymmetrized Riemann tensor using Eq. (3.2.14). Because this expression is much less elegant (and much less useful) than $d^2 = 0$, most equations involving differential forms will still be best expressed in terms of the torsion-free derivative.

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