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1 Languages and structures

Language: alphabet, terms, formulas. The **alphabet of a language** L consists of, by definition, the following symbols:

(i) relation symbols P_i, $(i \in I)$, function symbols f_j, $(j \in J)$, and constant symbols c_k, $(k \in K)$ with some index sets I, J, K. Further, to each $i \in I$ and $j \in J$ is assigned a positive integer ρ_i, μ_j, respectively, called the **arity** of the relation symbol P_i or the function symbol f_j.
The symbols in (i) are called **non-logical** symbols and their choice determines L. In addition any language has the following symbols:
(ii) $\doteq$ - the equality symbol;
(iii) $v_1, \dots, v_n, \dots$ - the variables;
(iv) $\wedge$, $\neg$ - the connectives;
(v) $\exists$ - the existential quantifier;
(vi) (,), , - parentheses and comma.

Words of the alphabet of L constructed in a specific way are called L-terms and L-formulas:

L**-terms** are given by recursive definition as follows:
(i) v_i is an L-term (any $i \geq 1$);
(ii) c is an L-term (any constant symbol c of L);
(iii) if f is a function symbol of L of arity μ, and $\tau_1, \dots \tau_\mu$ are L-terms, then $f(\tau_1, \dots, \tau_\mu)$ is an L-term;
(iv) nothing else is an L-term.

We define the **complexity** of a term τ to be just the length of τ as a word in the alphabet of L. It is obvious from the definition that any term of complexity $l > 1$ is obtained by an application of (iii) to terms of lower complexity.

We sometimes refer to a term τ as $\tau(v_{i_1}, \dots, v_{i_n})$ to mark the fact that the variables occurring in τ are among $v_{i_1}, \dots, v_{i_n}$. It may happen that no variables occur in τ, such terms are called **closed.**

Atomic L**-formulas** are the words of the form
(i) $\tau_1 \doteq \tau_2$ for any L-terms τ_1 and τ_2

or
(ii) $P(\tau_1, \ldots, \tau_\rho)$ for any relational L-symbol P of arity ρ and L-terms $\tau_1, \ldots, \tau_\rho$.

Notice, that (i) can be seen as a special case of (ii) if we view $\doteq$ as a relational symbol of arity 2.

We sometimes refer to an atomic formula φ of the form $P(\tau_1, \ldots, \tau_\rho)$ as $\varphi(v_{i_1}, \ldots, v_{i_n})$ to mark the fact that all the variables occurring in $\tau_1, \ldots, \tau_\rho$ are among $v_{i_1}, \ldots, v_{i_n}$.

An **L-formula** is defined by the following recursive definition:
(i) any atomic L-formula is an L-formula;
(ii) if φ is an L-formula, so is $\neg\varphi$;
(iii) if φ, ψ are L-formulas, so is $(\varphi \wedge \psi)$;
(iv) if φ is an L-formula, so is $\exists v\varphi$ for any variable v;
(v) nothing else is an L-formula.

We define the **complexity of an L-formula** φ to be just the number of occurrences of $\wedge$, $\neg$ and $\exists$ in φ. It is obvious from the definition that an atomic formula is of complexity 0 and that any formula of complexity $l > 0$ is obtained by an application of (ii),(iii) or (iv) to formulas of lower complexity.

For an atomic formula $\varphi(v_{i_1}, \ldots, v_{i_n})$ the distinguished variables are said to be **free in** φ. The variables which are free in φ and ψ in (ii) and (iii) are, by definition, also free in $\neg\varphi$ and $(\varphi \wedge \psi)$. The variable v in (iv) is called **bounded** in $\exists v\varphi$ and the list of free variables for this formula is given by the free variables of φ except v.
An L-formula with no free variables is called also an **L-sentence**.

We define a language L to be the set of all L-formulas. Thus $|L|$ is the cardinality of the set.

Exercise 1.1 *Show that*

$$|L| = \max\{\aleph_0, \text{card }(I), \text{card }(J), \text{card }(K)\}.$$

To give a meaning or **interpretation** of symbols of a language L we introduce a notion of an L-**structure.** An L-structure $\mathcal{A}$ consists of

(i) a non-empty set A, called a **domain** of the L-structure;
(ii) an assignment of an r-ary relation (subset) $P^{\mathcal{A}} \subseteq A^r$ to any relation symbol P of L of arity r;
(iii) an assignment of an m-ary function $f^{\mathcal{A}} : A^m \to A$ to any function symbol f of L of arity m;
(iv) an assignment of an element $c^{\mathcal{A}} \in A$ to any constant symbol c of L.

Thus an L-structure is an object of the form

$$\mathcal{A} = \left\langle A; \{P_i^{\mathcal{A}}\}_{i\in I}; \{f_j^{\mathcal{A}}\}_{j\in J}; \{c_k^{\mathcal{A}}\}_{k\in K} \right\rangle .$$

$\{P_i^{\mathcal{A}}\}_{i\in I}$, $\{f_j^{\mathcal{A}}\}_{j\in J}$ and $\{c_k^{\mathcal{A}}\}_{k\in K}$ are called **the interpretations** of the predicate, function and constant symbols correspondingly.
We write $A = \text{dom }(\mathcal{A})$.

Example Groups can be considered L-structures where L is having one constant symbol e, one binary and one unary operation symbols $\cdot$ and $^{-1}$ and no relation symbols.

If $\mathcal{A}$ and $\mathcal{B}$ are both L-structures we say that $\mathcal{A}$ is **isomorphic** to $\mathcal{B}$, written $\mathcal{A} \cong \mathcal{B}$, if there is a bijection $\pi : \text{dom }(\mathcal{A}) \to \text{dom }(\mathcal{B})$ which **preserves** corresponding relation, function and constant symbols, i.e. for any $i \in I$, $j \in J$ and $k \in K$:
(i) $\bar{a} \in P_i^{\mathcal{A}}$ iff $\pi(\bar{a}) \in P_i^{\mathcal{B}}$;
(ii) $\pi(f_j^{\mathcal{A}}(\bar{a})) = f_j^{\mathcal{B}}(\pi(\bar{a}))$;
(iii) $\pi(c_k^{\mathcal{A}}) = c_k^{\mathcal{B}}$.
The map π is then called an **isomorphism**. If π is only assumed being injective but still satisfies (i)-(iii), then it is called an **embedding** and can be written as $\pi : \mathcal{A} \to \mathcal{B}$ or $\mathcal{A} \subseteq_\pi \mathcal{B}$.
An isomorphism $\pi : \mathcal{A} \to \mathcal{A}$ of the structure onto itself is called an **automorphism**.

Interpretation of terms in a structure.

Given an L-structure $\mathcal{A}$, we assign to each L-term $\tau(v_1,\ldots,v_n)$ a function

$$\tau^{\mathcal{A}} : A^n \to A$$

by the following natural rule:
(i) if $\tau(v_1,\ldots,v_n)$ is just a variable v_j then $\tau^{\mathcal{A}}$ is the corresponding coordinate function $\langle a_1,\ldots a_n\rangle \mapsto a_j$;
(ii) if $\tau(v_1,\ldots,v_n)$ is a constant symbol c then $\tau^{\mathcal{A}}(a_1,\ldots,a_n) = c^{\mathcal{A}}$;
(iii) if $\tau(v_1,\ldots,v_n)$ is $f(\tau_1(v_1,\ldots,v_n),\ldots,\tau_m(v_1,\ldots,v_n))$ then $\tau^{\mathcal{A}}(a_1,\ldots,a_n) = f^{\mathcal{A}}(\tau_1^{\mathcal{A}}(a_1,\ldots,a_n),\ldots,\tau_m^{\mathcal{A}}(a_1,\ldots,a_n))$.

Exercise 1.2 *Prove that if $\pi : \mathcal{A} \to \mathcal{B}$ is an embedding then π preserves L-terms, that is for any term $\tau(\bar{v})$*

$$\pi(\tau^{\mathcal{A}}(\bar{a})) = \tau^{\mathcal{B}}(\pi(\bar{a})).$$

Assigning truth values to L-formulas in an L-structure.
Suppose $\mathcal{A}$ is an L-structure with domain A, $\varphi(v_1,\ldots,v_n)$ an L-formula with free variables $v_1,\ldots,v_n$ and $\bar{a} = \langle a_1,\ldots,a_n\rangle \in A^n$. Given these data we assign a truth value **true**, written $\mathcal{A} \vDash \varphi(\bar{a})$, or **false**, $\mathcal{A} \nvDash \varphi(\bar{a})$, by the following rules:
(i) $\mathcal{A} \vDash \tau_1(\bar{a}) \doteq \tau_2(\bar{a})$ iff $\tau_1^{\mathcal{A}}(\bar{a}) = \tau_2^{\mathcal{A}}(\bar{a})$;
(ii) $\mathcal{A} \vDash P(\tau_1(\bar{a}),\ldots,\tau_r(\bar{a}))$ iff $\langle \tau_1^{\mathcal{A}}(\bar{a}),\ldots,\tau_r^{\mathcal{A}}(\bar{a})\rangle \in P_i^{\mathcal{A}}$;
(iii) $\mathcal{A} \vDash \varphi_1(\bar{a}) \wedge \varphi_2(\bar{a})$ iff $\mathcal{A} \vDash \varphi_1(\bar{a})$ and $\mathcal{A} \vDash \varphi_2(\bar{a})$;
(iv) $\mathcal{A} \vDash \neg\varphi(\bar{a})$ iff $\mathcal{A} \nvDash \varphi(\bar{a})$;
(v) $\mathcal{A} \vDash \exists v_n \varphi(a_1,\ldots,a_{n-1},v_n)$ iff there is an $a_n \in A$ such that $\mathcal{A} \vDash \varphi(a_1,\ldots,a_n)$.

Given an L-structure $\mathcal{A}$ and an L-formula $\varphi(v_1,\ldots,v_n)$ we can define the set

$$\varphi(\mathcal{A}) = \{\bar{a} \in A^n : \mathcal{A} \vDash \varphi(\bar{a})\}.$$

Sets of this form are called **definable**.

Since any subset of A^n can be viewed as an n-ary relation, $\varphi(\bar{v})$ determines also an **L-definable relation.** If $\varphi(\mathcal{A})$ coincides with a graph of a function $f : A^{n-1} \to A$, we say then that f is an **L-definable function.**

Lemma 1.1 *An embedding $\pi : \mathcal{A} \to \mathcal{B}$ of L-structures preserves atomic L-formulas, i.e. for any atomic $\varphi(v_1, \ldots, v_n)$ for any $\bar{a} \in A^n$*

$$(*) \qquad \mathcal{A} \models \varphi(\bar{a}) \ \text{ iff } \ \mathcal{B} \models \varphi(\pi(\bar{a})).$$

Proof By exercise 1.2 π preserves terms.
Let φ be an atomic formula of the form $P(\tau_1(\bar{v}), \ldots, \tau_r(\bar{v}),\ P$ a relation symbol. Denote

$$\alpha_i = \tau_i^{\mathcal{A}}(\bar{a}), \quad \beta_i = \tau_i^{\mathcal{B}}(\pi(\bar{a})) \quad i = 1, \ldots, r.$$

Since terms are preserved, $\pi(\alpha_i) = \beta_i$. By the definition of an isomorphism

$$\langle \alpha_1, \ldots, \alpha_r \rangle \in P^{\mathcal{A}} \ \text{ iff } \ \langle \beta_1, \ldots, \beta_r \rangle \in P^{\mathcal{B}}.$$

This means that $(*)$ holds for our formula.
Since $\doteq$ can be treated as a binary relation symbol, the above proves $(*)$ for all atomic formulas. $\square$

Proposition 1 *An isomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between L-structures preserves any L-formula $\varphi(v_1, \ldots, v_n)$ ($n \geq 0$), i.e. for any $\bar{a} \in A^n$*

$$(*) \qquad \mathcal{A} \models \varphi(\bar{a}) \ \text{ iff } \ \mathcal{B} \models \varphi(\pi(\bar{a})).$$

Proof By induction on the complexity of φ.
For atomic formulas we have Lemma 1.1.
Suppose now that the complexity of φ is $l > 0$, and for all formulas of complexity less than l (*) holds. Then φ is obtained by applying either (ii), (iii) or (iv) of the definition of formula to formulas of lower complexities.
Consider e.g. case (iv). Then $\varphi(\bar{v}) = \exists v_{n+1} \psi(\bar{v}, v_{n+1})$ and

$\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{A} \models \exists v_{n+1} \psi(\bar{a}, v_{n+1})$ iff there is an a_{n+1} such that $\mathcal{A} \models \psi(\bar{a}, a_{n+1})$.

The latter by the induction hypothesis is equivalent to $\mathcal{B} \models \psi(\bar{b}, b_{n+1})$ where $\bar{b} = \pi(\bar{a})$ and $b_{n+1} = \pi(a_{n+1})$. Continuing in the reverse order we come to the equivalent statement $\mathcal{B} \models \varphi(\bar{b})$, which proves $(*)$ in this case. Similarly $(*)$ holds in cases (ii) and (iii) and this completes the proof.$\square$

Corollary 1 *For definable subsets (relations)*

$$\pi(\varphi(\mathcal{A})) = \varphi(\mathcal{B}),$$

in particular, when $\pi : \mathcal{A} \to \mathcal{A}$ is an automorphism (an isomorphism onto itself),

$$\pi(\varphi(\mathcal{A})) = \varphi(\mathcal{A}).$$

The latter is very useful in checking non-definability of some subsets or relations.

Exercise 1.3 *The multiplication is not definable in $\langle \mathbb{R}, + \rangle$.*

Agreement about notations. The proposition above about the properties of isomorphic structures says that there is no harm in identifying elements of $\mathcal{A}$ with its images under an isomorphism. Correspondingly, when speaking about embedding $\pi : \mathcal{A} \to \mathcal{B}$ we identify $A = \mathrm{dom}\, \mathcal{A}$ with its image $\pi(A) \subseteq B = \mathrm{dom}\, \mathcal{B}$ element-wise. And so, by default, $\mathcal{A} \subseteq \mathcal{B}$ assumes $A \subseteq B$.

Given two L-structures $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$ is **elementarily equivalent to** $\mathcal{B}$, written $\mathcal{A} \equiv \mathcal{B}$, if for any L-sentence φ

$$\mathcal{A} \vDash \varphi \text{ iff } \mathcal{B} \vDash \varphi.$$

Two typical model-theoretic problems:
I. Given $\mathcal{A}$, what are the definable subsets of A, or A^n?
II. Given $\mathcal{A}$, what are the $\mathcal{B}$ such that

$$\mathcal{A} \equiv \mathcal{B}?$$

Some abbreviations
Let ϕ and ψ be L-formulas.
$(\phi \vee \psi)$ is an abbreviation for the formula $\neg(\neg\phi \wedge \neg\psi)$;
$(\phi \to \psi)$ is an abbreviation for the formula $\neg(\phi \wedge \neg\psi)$;
$(\phi \leftrightarrow \psi)$ is an abbreviation for the formula $((\phi \to \psi) \wedge (\psi \to \phi))$;
$\forall v \psi$ is an abbreviation for the formula $\neg\exists v \neg\psi$.

2 The Compactness Theorem

Let Σ be a set of L-sentences. We write $\mathcal{A} \vDash \Sigma$ if, for any $\sigma \in \Sigma$, $\mathcal{A} \vDash \sigma$.
An L-sentence σ is said to be **a logical consequence of** a finite Σ, written $\Sigma \vDash \sigma$, if $\mathcal{A} \vDash \Sigma$ implies $\mathcal{A} \vDash \sigma$ for every L-structure $\mathcal{A}$. For Σ infinite, $\Sigma \vDash \sigma$ means that there is a finite $\Sigma^0 \subset \Sigma$ such that $\Sigma^0 \vDash \sigma$.

σ is called **logically valid**, written $\vDash \sigma$, if $\mathcal{A} \vDash \sigma$ for every L-structure $\mathcal{A}$.

A set Σ of L-sentences is said to be **satisfiable** if there is an L-structure $\mathcal{A}$ such that $\mathcal{A} \vDash \Sigma$. $\mathcal{A}$ is then called **a model** of Σ.
Σ is said to be **finitely satisfiable (f.s.)** if any finite subset of Σ is satisfiable.

Σ is said to be **complete** if, for any L-sentence σ, $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Exercise 2.1 *Let $\alpha, \alpha_1, \ldots, \alpha_n, \beta, \beta_1, \ldots, \beta_n, \gamma$ be closed L-terms, P, f L-symbols for n-ary predicate and n-ary function, correspondingly, and $\psi(v_0, v_1, \ldots, v_n)$ an L-formula with free variables $v_0, v_1, \ldots, v_n$. Prove that*

1. $\alpha \doteq \beta \vDash \beta \doteq \alpha$;
2. $\alpha \doteq \beta, \beta \doteq \gamma \vDash \alpha \doteq \gamma$;
3. $\vDash \alpha \doteq \alpha$;
4. $\alpha_1 \doteq \beta_1, \ldots, \alpha_n \doteq \beta_n, P(\alpha_1, \ldots, \alpha_n) \models P(\beta_1, \ldots, \beta_n)$;
5. $\alpha \doteq \beta, \alpha_1 \doteq \beta_1, \ldots, \alpha_n \doteq \beta_n, f(\alpha_1, \ldots, \alpha_n) \doteq \alpha \models f(\beta_1, \ldots, \beta_n) \doteq \beta$;
6. $\psi(\beta, \alpha_1, \ldots, \alpha_n) \models \exists v_0 \psi(v_0, \alpha_1, \ldots, \alpha_n)$.

A set of L-sentences Σ is said to be **deductively closed** if

$$\Sigma \vDash \sigma \text{ implies } \sigma \in \Sigma.$$

Exercise 2.2 *(i) If $\Sigma' \subseteq \Sigma$ and $\Sigma' \vDash \sigma$ then $\Sigma \vDash \sigma$;*
(ii) A complete f.s. Σ is deductively closed.

Proposition 2 (Lindenbaum's Theorem) *For any f.s. set of L-sentences Σ there is a complete f.s. set of L-sentences $\Sigma^{\#}$ such that $\Sigma \subseteq \Sigma^{\#}$.*

Proof Let
$$\mathcal{S} = \{\Sigma' : \Sigma \subseteq \Sigma' \text{ a f.s. set of } L\text{-sentences } \}.$$

Clearly $\mathcal{S}$ satisfies the hypothesis of Zorn's Lemma, so it contains a maximal element $\Sigma^{\#}$ say. This is complete for otherwise, say $\sigma \notin \Sigma^{\#}$ and $\neg\sigma \notin \Sigma^{\#}$. By maximality neither $\{\sigma\} \cup \Sigma^{\#}$ nor $\{\neg\sigma\} \cup \Sigma^{\#}$ is f.s.. Hence there exist finite $S_1 \subseteq \Sigma^{\#}$ and $S_2 \subseteq \Sigma^{\#}$ such that neither $\{\sigma\} \cup S_1$ nor $\{\neg\sigma\} \cup S_2$ is satisfiable. However, $S_1 \cup S_2 \subseteq \Sigma^{\#}$, finite, so has a model, $\mathcal{A}$ say. But either $\mathcal{A} \vDash \sigma$, so $\mathcal{A} \vDash \{\sigma\} \cup S_1$, or $\mathcal{A} \vDash \neg\sigma$, so $\mathcal{A} \vDash \{\neg\sigma\} \cup S_2$, a contradiction.$\square$

A set Σ of L-sentences is said to be **full** if for any sentence in Σ of the form $\exists v \varphi(v)$ there is a closed L-term λ such that $\varphi(\lambda) \in \Sigma$.

Exercise 2.3 *If there exists a complete f.s. full set of L-sentences then there exists a closed L-term.*
Hint: Consider the L-sentence $\exists v\ v \doteq v$.

An L-structure $\mathcal{A}$ is called **canonical** if for every $a \in A$ there is a closed L-term λ such that $\lambda^{\mathcal{A}} = a$.

Proposition 3 *For any complete, full, f.s. set Σ of L-sentences there is a canonical model $\mathcal{A}$ of Σ.*

Proof Let Λ be the set of closed terms of L. This is nonempty by 2.3. For $\alpha, \beta \in \Lambda$ define $\alpha \backsim \beta$ iff $\alpha \doteq \beta \in \Sigma$.
This is an equivalence relation by 2.1.1- 2.1.3. and 2.2.
For $\alpha \in \Lambda$, let $\tilde{\alpha}$ denote the $\backsim$-equivalence class containing α. Let

$$A = \{\tilde{\alpha} : \ \alpha \in \Lambda\}.$$

This will be the domain of our model $\mathcal{A}$. We want to define relations, functions and constants of L on A.
Let P be an n-ary relation symbol of L and $\alpha_1, \ldots, \alpha_n \in \Lambda$. Define

$$\langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \rangle \in P^{\mathcal{A}} \text{ iff } P(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

By 2.1.4 the definition does not depend on the choice of representatives in the $\backsim$-classes.
For a unary function symbol f of L of arity m and $\alpha_1, \ldots, \alpha_m \in \Lambda$ define

$$f^{\mathcal{A}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) = \tilde{\tau}, \text{ where } \tau = f(\alpha_1, \ldots, \alpha_m).$$

By 2.1.5 this is well-defined.
Finally, for a constant symbol, $c^{\mathcal{A}}$ is just $\tilde{c}$.
We now prove by induction on complexity of an L-formula $\varphi(v_1, \ldots, v_n)$ that

$$(*) \quad \mathcal{A} \vDash \varphi(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \text{ iff } \varphi(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

For atomic formulas we have this by definition.
If $\varphi = (\varphi_1 \wedge \varphi_2)$ then
$\mathcal{A} \vDash (\varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \wedge \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n))$ iff $\mathcal{A} \vDash \varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ and $\mathcal{A} \vDash \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff (by induction hypothesis) $\varphi_1(\alpha_1, \ldots, \alpha_n), \varphi_2(\alpha_1, \ldots, \alpha_n) \in \Sigma$ iff (by 2.2) $(\varphi_1(\alpha_1, \ldots, \alpha_n) \wedge \varphi_2(\alpha_1, \ldots, \alpha_n)) \in \Sigma$. Which proves (*) in this case.
The case $\varphi = \neg\psi$ is proved similarly.
In case $\varphi = \exists v \psi$
$\mathcal{A} \vDash \exists v \psi(v, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff there is $\beta \in \Lambda$ such that $\mathcal{A} \vDash \psi(\tilde{\beta}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff there is $\beta \in \Lambda$ such that $\psi(\beta, \alpha_1, \ldots, \alpha_n) \in \Sigma$. The latter implies, by 2.1.6 and 2.2, that $\exists v \psi(v, \alpha_1, \ldots, \alpha_n) \in \Sigma$, and the converse holds because Σ is full. This proves (*) for the formula and finishes the proof of (*) for all formulas.
Finally notice that (*) implies that $\mathcal{A} \vDash \Sigma$.$\square$

We sometimes need to expand or reduce our language.
Let L be a language with non-logical symbols $\{P_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$ and $L' \subseteq L$ with non-logical symbols $\{P_i\}_{i \in I'} \cup \{f_j\}_{j \in J'} \cup \{c_k\}_{k \in K'}$ ($I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$). Let

$$\mathcal{A} = \langle A; \{P_i^{\mathcal{A}}\}_{i \in I}; \{f_j^{\mathcal{A}}\}_{j \in J}; \{c_k^{\mathcal{A}}\}_{k \in K} \rangle$$

and

$$\mathcal{A}' = \langle A; \{P_i^{\mathcal{A}}\}_{i \in I'}; \{f_j^{\mathcal{A}}\}_{j \in J'}; \{c_k^{\mathcal{A}}\}_{k \in K'} \rangle.$$

Under these conditions we call $\mathcal{A}'$ **the L'-reduct of $\mathcal{A}$** and, correspondingly, $\mathcal{A}$ is **an L-expansion of $\mathcal{A}'$**.

Remark Obviously, under the notations above for an L'-formula $\varphi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in A$

$$\mathcal{A}' \vDash \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{A} \vDash \varphi(a_1, \ldots, a_n).$$

Exercise 2.4 *Let, for each $i \in \mathbb{N}$, Σ_i denote a set of L sentences. Suppose*

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \Sigma_i \ldots$$

and each Σ_i is finitely satisfiable.
Then the union of the chain, $\bigcup_{i \in \mathbb{N}} \Sigma_i$, is finitely satisfiable.

Theorem 1 (Compactness Theorem) *Any finitely satisfiable set of L-sentences Σ is satisfiable. Moreover, Σ has a model of cardinality less or equal to $|L|$, the cardinality of the language.*

Proof We introduce new languages L_i and complete set of L_i-sentences Σ_i $(i = 0, 1, \ldots)$. Let $L_0 = L$. By Lindenbaum's Theorem there exists $\Sigma_0 \supseteq \Sigma$, a complete set of L_0-sentences.
Given f.s. Σ_i in language L_i, introduce the new language

$$L_{i+1} = L_i \cup \{c_\phi : \ \phi \ \text{ a one variable } L_i\text{-formula}\}$$

and the new set of L_{i+1} sentences

$$\Sigma_i^* = \Sigma_i \cup \{(\exists v \phi(v) \rightarrow \phi(c_\phi)) \ : \ \phi \text{ a one variable } L_i\text{-formula}\}.$$

Claim. Σ_i^* is f.s. Indeed, for any finite $S \subseteq \Sigma_i^*$ let $S_1 = S \cap \Sigma_i$ and take a model $\mathcal{A}$ of S_1 with domain A, which we assume well-ordered. Assign constants to symbols c_ϕ as follows:

$$c_\phi = \begin{cases} \text{the first element in } \phi(\mathcal{A}) & \text{if } \phi(\mathcal{A}) \neq \emptyset \\ \text{the first element in } A & \text{if } \phi(\mathcal{A}) = \emptyset \end{cases}.$$

Denote the expanded structure $\mathcal{A}^*$. By the definition, for all $\phi(v)$,
$\mathcal{A}^* \vDash \exists v \phi(v) \rightarrow \phi(c_\phi)$. So $\mathcal{A}^* \vDash S$. This proves the claim.

Let Σ_{i+1} be a complete f.s. set of L_{i+1}-sentences containing Σ_i^*.
Take $\Sigma^* = \bigcup_{i \in \mathbb{N}} \Sigma_i$. This is finitely satisfiable by 2.4. By construction one sees immediately that Σ^* is also full and complete set of sentences in the

language $\bigcup L_i = L + \{$ new constants$\}$. Proposition 3 gives us the canonical model, $\mathcal{A}^*$, of Σ^*. The reduct of $\mathcal{A}^*$ to language L is a model of Σ.
The cardinality of the model we constructed is less or equal to $|L|$ (see also Exercise 1.1). $\square$

3 Method of diagrams and elementary embeddings

An embedding of L structures $\pi : \mathcal{A} \to \mathcal{B}$ is called **elementary** if π preserves any L-formula $\varphi(v_1, \ldots, v_n)$, i.e. for any $a_1, \ldots, a_n \in \mathrm{dom}\ \mathcal{A}$

$$\mathcal{A} \vDash \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \vDash \varphi(\pi(a_1), \ldots, \pi(a_n)).$$

We write the fact of elementary embedding as

$$\mathcal{A} \preccurlyeq \mathcal{B}.$$

We usually identify $A = \mathrm{dom}\ \mathcal{A}$ with the subset $\pi(A)$ of $B = \mathrm{dom}\ \mathcal{B}$. Then $\pi(a) = a$ for all $a \in A$ and so $\mathcal{A} \preccurlyeq \mathcal{B}$ usually mean

$$\mathcal{A} \vDash \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \vDash \varphi(a_1, \ldots, a_n).$$

Example Let $\mathcal{Z} = \langle \mathbb{Z}; +, 0 \rangle$ be the additive group of integers. Then, given an integer $m > 1$, the embedding

$$[m] : \mathcal{Z} \to \mathcal{Z},$$

defined as $[m](z) = m \cdot z$, is not elementary.

For an L-structure $\mathcal{A}$ let $L_A = L \cup \{c_a :\ a \in A\}$ be the expansion of the language, $\mathcal{A}^+$ the natural expansion of $\mathcal{A}$ to L_A assigning to c_a the element a, and

$$\mathrm{Diag}(\mathcal{A}) = \{\sigma :\ \sigma \text{ an atomic } L_A\text{-sentence or negation of an atomic } L_A\text{-sentence, such that } \mathcal{A}^+ \vDash \sigma\}.$$

$$\mathrm{CDiag}(\mathcal{A}) = \{\sigma :\ \sigma\ L_A\text{-sentence such that } \mathcal{A}^+ \models \sigma\}.$$

Theorem 2 (Method of Diagrams) *For an L structure $\mathcal{B}$,*
(i) there is an expansion $\mathcal{B}^+$ to the language L_A such that $\mathcal{B}^+ \models \mathrm{Diag}(\mathcal{A})$ *iff $\mathcal{A} \subseteq \mathcal{B}$.*
(ii) there is an expansion $\mathcal{B}^+$ to the language L_A such that $\mathcal{B}^+ \models \mathrm{CDiag}(\mathcal{A})$ *iff $\mathcal{A} \preccurlyeq \mathcal{B}$.*

Proof Indeed, by definitions and Lemma 1.1, $a \to c_a^{\mathcal{B}^+}$ is an embedding iff $\mathcal{B}^+ \models \mathrm{Diag}(\mathcal{A})$.
The elementary embedding case is straightforward by definition.$\square$

Corollary 2 *Given an L-structure $\mathcal{A}$ and a set of L-sentences T, (i) the set $T \cup \mathrm{Diag}(\mathcal{A})$ is f.s. iff there is a model $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$.*
(ii) the set $T \cup \mathrm{CDiag}(\mathcal{A})$ is f.s. iff there is a model $\mathcal{B}$ of T such that $\mathcal{A} \preccurlyeq \mathcal{B}$.

Theorem 3 (Upward Lowenheim-Skolem Theorem) *For any infinite L-structure $\mathcal{A}$ and a cardinal $\kappa \geq \max\{|L|, ||\mathcal{A}||\}$ there is an L-structure $\mathcal{B}$ of cardinality κ such that $\mathcal{A} \preccurlyeq \mathcal{B}$.*

Proof Let M be a set of cardinality κ. Consider an extension $L_{A,M}$ of language L obtained by adding to L_A constant symbols c_i for each $i \in M$. Consider now the set of $L_{A,M}$-sentences

$$\Sigma = \mathrm{CDiag}(\mathcal{A}) \cup \{\neg c_i \doteq c_j : \ i \neq j \in M\}.$$

We claim that Σ is f.s. Indeed, consider a finite subset $S \subseteq \Sigma$. Obviously

$$S \subseteq S_0 \cup \{\neg c_i \doteq c_j : \ i \neq j \in M_0\}$$

for some $S_0 \subseteq \mathrm{CDiag}(\mathcal{A})$ and $M_0 \subset M$, both finite. By definition $\mathcal{A}^+ \models S_0$. Now, since A is infinite, we can expand $\mathcal{A}^+$ to the model of S by assigning to c_i $(i \in M_0)$ distinct elements of A. This proves the claim.
It follows from the compactness theorem that Σ has a model of cardinality $|L_{A,M}|$, which is equal to κ. Let $\mathcal{B}^*$ be such a model. The L-reduct $\mathcal{B}$ of $\mathcal{B}^*$, by the method of diagrams, satisfies the requirement of the theorem. $\square$

Lemma 3.1 (Tarski-Vaught test) *Suppose $\mathcal{A} \subseteq \mathcal{B}$ are L-structures with domains $A \subseteq B$. Then $\mathcal{A} \preccurlyeq \mathcal{B}$ iff the following condition holds:*
for all L-formulas $\varphi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_{n-1} \in A$, $b \in B$ such that $\mathcal{B} \models \varphi(a_1, \ldots, a_{n-1}, b)$ there is $a \in A$ with $\mathcal{B} \models \varphi(a_1, \ldots, a_{n-1}, a)$

Proof Obviously, given $\bar{a} = \langle a_1, \ldots, a_{n-1}\rangle$ the existence of $b \in B$ as above is equivalent to $\mathcal{B} \models \exists v\varphi(\bar{a}, v)$.

Suppose $\mathcal{A} \preccurlyeq \mathcal{B}$. Then $\mathcal{B} \models \exists v\varphi(\bar{a}, v)$ is equivalent to $\mathcal{A} \models \exists v\varphi(\bar{a}, v)$ which is equivalent to the existence of an $a \in A$ with $\mathcal{A} \models \varphi(\bar{a}, a)$. The latter by $\mathcal{A} \preccurlyeq \mathcal{B}$ implies $\mathcal{B} \models \varphi(\bar{a}, a)$.
For the converse, we assume that for all φ

$$(*)\quad \mathcal{B} \models \exists v\varphi(\bar{a}, v) \text{ implies that for some } a \in A \quad \mathcal{B} \models \varphi(\bar{a}, a)$$

and want to prove that

$$(**)\quad \mathcal{A} \models \psi(\bar{a}) \text{ iff } \mathcal{B} \models \psi(\bar{a})$$

for all L-formulas $\psi(\bar{v})$.
Induction on the complexity of ψ. For ψ atomic (**) is given by Lemma 1.1 and the definition of the embedding $\mathcal{A} \subseteq \mathcal{B}$. The cases of $\psi = \psi_1 \wedge \psi_2$ and $\psi = \neg\psi_1$ are easy. In the case $\psi = \exists v\varphi$ the $\Rightarrow$ side of (**) follows immediately from the induction hypothesis and the meaning of $\exists$.
Proof of $\Leftarrow$:
$\mathcal{B} \models \exists v\varphi(\bar{a}, v)$ implies $\mathcal{B} \models \varphi(\bar{a}, b)$, some $b \in B$, implies $\mathcal{B} \models \varphi(\bar{a}, a)$, some $a \in A$, implies, by the induction hypothesis, $\mathcal{A} \models \varphi(\bar{a}, a)$, implies $\mathcal{A} \models \exists v\varphi(\bar{a}, v)$.
$\square$

Theorem 4 (Downward Lowenheim-Skolem Theorem) *Let $\mathcal{B}$ be an L-structure, S a subset of $B = \operatorname{dom}(\mathcal{B})$. Then there exists $\mathcal{A} \preccurlyeq \mathcal{B}$ such that $S \subseteq A = \operatorname{dom}(\mathcal{A})$ and $||\mathcal{A}|| \leq \max\{\operatorname{card}(S), |L|\}$. In particular, given $\mathcal{B}$ and a cardinal $||\mathcal{B}|| \geq \kappa \geq |L|$ we can have $\mathcal{A} \preccurlyeq \mathcal{B}$ of cardinality κ.*

Proof Fix some $b_0 \in B$. For each L-formula $\phi(v_1, \ldots, v_n)$ define a function $g_\phi : B^{n-1} \to B$ by

$$g_\phi(b_1, \ldots, b_{n-1}) = \begin{cases} \text{an element } b \in B : \mathcal{B} \models \phi(b_1, \ldots, b_{n-1}, b) \\ \qquad\qquad\qquad \text{if such one exists} \\ b_0 \quad \text{if not} \end{cases}$$

(g_ϕ are called *Skolem functions*).

Notice that for ϕ of the form $\tau(v_1, \ldots, v_{n-1}) \doteq v_n$, where τ is an L-term, g_ϕ coincides with the function $\tau^{\mathcal{B}}$.
Let A be the closure of S under all the g_ϕ, i.e.

$$A = \bigcup_{i \in \mathbb{N}} S_i : \quad S_0 = S \ \text{ and}$$

$$S_{i+1} = \{g_\phi(b_1, \ldots, b_{n-1}) : b_1, \ldots, b_{n-1} \in S_i,\ \phi(v_1, \ldots, v_n)\ L-\text{formulas}\}.$$

Notice that card $A \leq$ card $S + |L|$.
Define an L-structure $\mathcal{A}$ on the domain A interpreting the relation, function and constant symbols of L on A as induced from $\mathcal{B}$:
(i) for an n-ary relation symbol P or the equality symbol,
$P^{\mathcal{A}} = P^{\mathcal{B}} \cap A^n$;
(ii) for an m-ary function symbol f and $\bar{a} \in A^m$, $a \in A$,
$f^{\mathcal{A}}(\bar{a}) = a$ iff $f^{\mathcal{B}}(\bar{a}) = a$;
(iii) for a constant symbol c, $\quad c^{\mathcal{A}} = c^{\mathcal{B}}$.
(ii) and (iii) are possible since A is closed under L-terms.

Clearly then $\mathcal{A} \subseteq \mathcal{B}$ and the condition of Tarski Lemma is satisfied, for if $\mathcal{B} \vDash \exists v \phi(\bar{a}, v)$ then $\mathcal{B} \vDash \phi(\bar{a}, g_\phi(\bar{a}))$. Thus the lemma finishes the proof. $\Box$

Corollary 3 *Let Σ be a set of L-sentences which has an infinite model. Then for any cardinal $\kappa \geq |L|$ there is a model of Σ of cardinality κ.*

Example Let $\mathcal{M}$ be a model of a set theory in the language with one binary predicate symbol $\in$. Then there is a countable elementary submodel

$$\mathcal{M}_0 \preccurlyeq \mathcal{M}.$$

4 Axiomatizable classes and preservation theorems

A formula of the form $\exists v_1 \dots \exists v_m \theta$, where θ is a quantifier-free formula, is called **an existential formula** (or an E-formula). The negation of an existential formula is called **a universal** (A-formula) formula.

Exercise 4.1 *Let $\phi_1, \dots, \phi_n$ be existential formulas. Prove that*
(i) $(\phi_1 \vee \dots \vee \phi_n)$ and $(\phi_1 \wedge \dots \wedge \phi_n)$ are logically equivalent to existential formulas;
(ii) $(\neg\phi_1 \wedge \dots \wedge \neg\phi_n)$ and $(\neg\phi_1 \vee \dots \vee \neg\phi_n)$ are logically equivalent to universal formulas.

Given a set of sentences Σ denote $\Sigma_\exists$ its subset consisting of all existential formulas in Σ. Correspondingly, $\Sigma_\forall$ are the universal formulas of Σ.
Thus $\mathrm{Th}_\exists(\mathcal{A})$ is the set of all existential L-sentences which hold in $\mathcal{A}$.

Lemma 4.1 *Suppose $\mathcal{A} \subseteq \mathcal{B}$ and $a_1, \dots, a_n \in A$.*
(i) If $\mathcal{A} \models \varphi(a_1, \dots, a_n)$, for an existential formula $\varphi(v_1, \dots, v_n)$, then $\mathcal{B} \models \varphi(a_1, \dots, a_n)$.
(ii) If $\mathcal{B} \models \psi(a_1, \dots, a_n)$, for an universal formula $\psi(v_1, \dots, v_n)$, then $\mathcal{A} \models \psi(a_1, \dots, a_n)$.

Proof (i) Let $\varphi(v_1, \dots, v_n)$ be $\exists v_{n+1}, \dots, v_m \theta(v_1, \dots, v_n, v_{n+1}, \dots, v_n)$ and θ quantifier-free. Under this notation $\mathcal{A} \models \varphi(a_1, \dots, a_n)$ means that there are $a_{n+1}, \dots, a_m \in A$ such that $\mathcal{A} \models \theta(a_1, \dots, a_m)$. To prove the statement of the lemma it is enough to show that for quantifier-free θ

$$\mathcal{A} \models \theta(a_1, \dots, a_m) \Leftrightarrow \mathcal{B} \models \theta(a_1, \dots, a_m).$$

For θ atomic it is proved in Lemma 1.1. If the equivalence holds for θ_1 and θ_2, it holds by definitions for $\neg\theta_1$ and $(\theta_1 \wedge \theta_2)$. The statement (i) follows by induction.
(ii) Follows immediately from (i). $\square$

Lemma 4.2 $\Sigma \cup \mathrm{Diag}(\mathcal{A})$ *is satisfiable iff* $\Sigma \cup \mathrm{Th}_{\exists}(\mathcal{A})$ *is satisfiable.*

Proof Any model of a finite part of $\Sigma \cup \mathrm{Diag}(\mathcal{A})$ can be reduced to a model of a corresponding finite part of $\Sigma \cup \mathrm{Th}_{\exists}(\mathcal{A})$ and vice-versa, since for θ quantifier-free, by definitions,
$\exists v_1, \dots v_n \theta(v_1, \dots, v_n) \in \mathrm{Th}_{\exists}(\mathcal{A})$ iff $\mathcal{A} \models \exists v_1, \dots v_n \theta(v_1, \dots, v_n)$ iff $\mathcal{A} \models \theta(a_1, \dots, a_n)$ for some $a_1, \dots a_n$ iff $\mathrm{Diag}(\mathcal{A}) \vDash \theta(c_{a_1}, \dots c_{a_n})$.$\square$

A class C of L-structures is called **axiomatizable** if there is a set Σ of L-sentences such that

$$\mathcal{A} \in C \text{ iff } \mathcal{A} \vDash \Sigma.$$

We also write equivalently

$$C = \mathrm{Mod}\ (\Sigma).$$

Σ is then called **a set of axioms for** C.
C is called **finitely axiomatizable** iff there is a finite set Σ of axioms for C.
An axiomatizable class C is said to be $\exists$**-axiomatizable** ($\forall$-axiomatizable) if Σ can be chosen to consists of existential (universal) sentences only.

Conversely, we call **the theory of class** C the set

$$\mathrm{Th}(C) = \{\sigma : L\text{-sentence}, \mathcal{A} \vDash \sigma \text{ for all } \mathcal{A} \in C\}.$$

If C consists of a one structure $\mathcal{A}$ then we denote $\mathrm{Th}(\mathcal{A})$ the theory of this class and call it **the theory of structure** $\mathcal{A}$.

Exercise 4.2 *Show that*
$\mathrm{Th}(C)$ *is deductively closed, for every class* C;
$\mathrm{Th}(\mathcal{A})$ *is complete, for every structure* $\mathcal{A}$.

Exercise 4.3 *Show that*
if $\mathrm{Th}_{\forall}(C) \vDash \sigma$, *for an* $\forall$-L*-sentence* σ, *then* $\sigma \in \mathrm{Th}_{\forall}(C)$;
if $\mathrm{Th}_{\exists}(C) \vDash \sigma$, *for an* $\exists$-L*-sentence* σ, *then* $\sigma \in \mathrm{Th}_{\exists}(C)$.
That is, the universal and the existential parts of $\mathrm{Th}(C)$ *are deductively closed in the corresponding classes of formulas.*

Examples-exercises
1. The class of groups in the language with one binary function symbol $\cdot$, one unary function symbol $^{-1}$ (taking the inverse) and one constant symbol e is $\forall$-axiomatizable.
2. The class of finite groups is not axiomatizable.
3. The class of fields of characteristic zero is axiomatizable but not finitely axiomatizable.

Theorem 5 *Let C be an axiomatizable class. Then the following conditions are equivalent:*
(i) C is $\forall$-axiomatizable;
(ii) If $\mathcal{B} \in C$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \in C$.

Proof (i) implies (ii) by Lemma 4.1(ii).
To prove the converse consider $\mathrm{Th}(C)$, the theory of class C, and $\mathrm{Th}_\forall(C)$, its universal part. Let $\mathcal{A} \models \mathrm{Th}_\forall(C)$. We need to show that $\mathcal{A} \in C$ which would yield Mod $(\mathrm{Th}_\forall(C)) =$ Mod $(\mathrm{Th}(C)) = C$, as required.

Claim. $\mathrm{Th}(C) \cup \mathrm{Th}_\exists(\mathcal{A})$ is finitely satisfiable.
Indeed, otherwise, $\mathrm{Th}(C) \models \neg\sigma_1 \vee \ldots \vee \neg\sigma_n$, for some $\sigma_1, \ldots, \sigma_n \in \mathrm{Th}_\exists(\mathcal{A})$. Also $\neg\sigma_1 \vee \ldots \vee \neg\sigma_n \equiv \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ and $\mathcal{A} \models \sigma_1 \wedge \ldots \wedge \sigma_n$. On the other hand $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ is equivalent to an $\forall$-formula, and is a logical consequence of $\mathrm{Th}(C)$. So $\mathcal{A} \models \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$, the contradiction. Claim proved.

It follows from the claim and Lemma 4.2 that $\mathrm{Th}(C) \cup \mathrm{Diag}(\mathcal{A})$ is satisfiable. Let $\mathcal{B}^+$ be a model of $\mathrm{Th}(C) \cup \mathrm{Diag}(\mathcal{A})$ and $\mathcal{B}$ its reduct to the initial language. In particular, $\mathcal{B} \in C$ and, by Theorem 2, $\mathcal{A} \subseteq \mathcal{B}$. It follows by assumptions that $\mathcal{A} \in C$.$\Box$

Exercise 4.4 *Let C be an axiomatizable class. Then the following conditions are equivalent:*
(i) C is $\exists$-axiomatizable;
(ii) If $\mathcal{A} \in C$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{B} \in C$.

Definition Let
$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_i \subseteq \ldots \tag{1}$$
be a sequence of L-structures, $i \in \mathbb{N}$, forming a chain with respect to embeddings.
Denote $\mathcal{A}^* = \bigcup_n \mathcal{A}_n$ the L-structure with:
the domain $A^* = \bigcup_n A_n$,
predicates $P^{\mathcal{A}^*} = \bigcup_n P^{\mathcal{A}_n}$, for each predicate symbol P of L,
operations $f^{\mathcal{A}^*} : (A^*)^m \to A^*$ sending $\bar{a}$ to b iff $\bar{a}$ is in A_n for some n and $f^{\mathcal{A}_n}(\bar{a}) = b$, for each function symbol f of L,
and $c^{\mathcal{A}^*} = c^{\mathcal{A}_0}$, for each constant symbol from L.

By definition $\mathcal{A}_n \subseteq \mathcal{A}^*$, for each n.
A formula equivalent to one of the form $\forall v_1 \ldots \forall v_m \exists v_{m+1} \ldots \exists v_{k+m}\theta$, where θ is a quantifier-free formula, is called an AE-formula.
The negation of an AE-formula is called an EA-formula.

Exercise 4.5 *Given a chain of the form (1) and an AE-sentence σ assume that $\mathcal{A}_n \vDash \sigma$ for every $n \in \mathbb{N}$. Prove that*
$$\mathcal{A}^* \vDash \sigma.$$

Exercise 4.6 *If, for each n,*
$$\mathcal{A}_n \preccurlyeq \mathcal{A}_{n+1}$$
that is, the chain is elementary, *then $\mathcal{A}_n \preccurlyeq \mathcal{A}^*$, for each n.*

We state without proof

Theorem 6 *Let C be an axiomatizable class. Then the following conditions are equivalent:*
(i) C is AE-axiomatizable;
(ii) For any chain of the form (1) with $\mathcal{A}_n \in C$ for all $n \in \mathbb{N}$, the union $\mathcal{A}^*$ is in C.

5 Categoricity in powers

We continue the study of axiomatizable classes, but now our interest is mainly in those which are axiomatised by a complete set of axioms.
From now on **an L-theory** will stand for a satisfiable deductively closed set T of L-sentences.
It follows from definitions that the theory of a non-empty class of structures is deductively closed, so it is a theory in the above sense.
A theory T is said to be **categorical in power** κ (κ-categorical) if there is a model $\mathcal{A}$ of T of cardinality κ and any model of T of this cardinality is isomorphic to $\mathcal{A}$.

Theorem 7 (R.Vaught) *Let $\kappa \geq |L|$ and T be a κ-categorical L-theory without finite models. Then T is complete.*

Proof Let σ be an L-sentence and $\mathcal{A}$ the unique, up to isomorphism, model of T of cardinality κ. The either σ or $\neg\sigma$ holds in $\mathcal{A}$, let it be σ. Then $T \cup \{\neg\sigma\}$ does not have a model of cardinality κ, which by the Lowenheim-Skolem theorems means $T \cup \{\neg\sigma\}$ does not have an infinite model, which by our assumption means it is not satisfiable. It follows that $T \vDash \sigma$. $\square$

Example 0 The trivial theory $T_=$ axiomatised by the axiom $\forall v \; v \doteq v$ in the language $L_=$ with no non-logical symbols is categorical in every power. Indeed, any set A determines a model $\mathcal{A} = \langle A \rangle$ of $T_=$ and any other model of $T_=$ of the same cardinality is isomorphic to $\mathcal{A}$ by a bijection. So $T_=$ is categorical in every power.
Note that $T_=$ is not complete.

Example 1 Let K be a field and L_K be the language with alphabet $\{+, \lambda_k, 0\}_{k \in K}$ where $+$ is a symbol of a binary function and λ_k symbols of unary functions, 0 constant symbol. Define V_K to be the theory of vector spaces over K, i.e. V_K is axiomatised by:
$\forall v_1 \forall v_2 \forall v_3 \;\; (v_1 + v_2) + v_3 \doteq v_1 + (v_2 + v_3);$
$\forall v_1 \forall v_2 \;\; v_1 + v_2 \doteq v_2 + v_1;$
$\forall v \;\; v + 0 \doteq v;$

$\forall v_1 \exists v_2 \ \ v_1 + v_2 \doteq 0;$
$\forall v_1 \forall v_2 \ \ \lambda_k(v_1 + v_2) \doteq \lambda_k(v_1) + \lambda_k(v_2)$ an axiom for each $k \in K$;
$\forall v \ \ \lambda_1(v) \doteq v;$
$\forall v \ \ \lambda_0(v) \doteq 0;$
$\forall v \ \ \lambda_{k_1}(\lambda_{k_2}(v)) \doteq \lambda_{k_1 \cdot k_2}(v)$ an axiom for each $k_1, k_2 \in K$;
$\forall v \ \ \lambda_{k_1}(v) + \lambda_{k_2}(v) \doteq \lambda_{k_1 + k_2}(v)$ an axiom for each $k_1, k_2 \in K$.

Mod V_K is exactly the class of vector spaces over K.
To discuss the theory further let us recall the basic facts and definitions of the theory of vector spaces.
A **basis of a vector space** $\mathcal{A}$ is a maximal linearly independent subset of $\mathcal{A}$. By Zorn's Lemma any independent subset can be extended to a basis, so a basis exists in any vector space (and in general can be infinite).
If B_1 and B_2 are bases of the same vector space, then card $B_1 =$ card B_2. This allows to define **the dimension of a vector space** to be the cardinality of a basis of the vector space.
If B_1 is a basis of $\mathcal{A}_1$ and B_2 a basis of $\mathcal{A}_2$, vector spaces over K, and $\pi : B_1 \to B_2$ a bijection, then π can be extended in a unique way (linearly) to an isomorphism between the vector spaces. In other words the isomorphism type of a vector space over a given field is determined by its dimension.
Let $\mathcal{A}$ be a model of V_K of cardinality $\kappa > |L_K| = \max\{\aleph_0, \text{card } K\}$. Then $||\mathcal{A}|| = \dim \mathcal{A}$, the dimension of the vector space (check it). It follows that, if $\mathcal{B}$ is another model of V_K of the same cardinality, $\mathcal{A} \cong \mathcal{B}$. Thus we have checked the validity of the following statement.

Theorem 8 *V_K is categorical in any infinite power $\kappa >$* card K.

Example 2 Let L be the language with one binary symbol $<$ and DLO be the theory of dense linear order with no end elements:
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \to \neg\, v_2 < v_1);$
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \vee v_1 \doteq v_2 \vee v_2 < v_1)$
$\forall v_1 \forall v_2 \forall v_3 \ (v_1 < v_2 \wedge v_2 < v_3) \to v_1 < v_3;$
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \to \exists v_3 \ (v_1 < v_3 \wedge v_3 < v_2));$
$\forall v_1 \exists v_2 \exists v_3 \ \ v_1 < v_2 \wedge v_3 < v_1.$

Cantor's Theorem *Any two countable models of* DLO *are isomorphic. In other words* DLO *is $\aleph_0$-categorical.*

To prove that any two countable models of DLO are isomorphic we enumerate the two ordered sets and then apply the famous *back-and-forth construction* of a bijection preserving the orders.

Exercise 5.1 *Show that* DLO *is not* κ*-categorical*
(i) for $\kappa = 2^{\aleph_0}$;
(ii) for any $\kappa > \aleph_0$.

Example 3 ACFA_0, the theory of *algebraically closed fields of characteristic zero* is given by the following axioms in the language of fields L_{fields} with binary operations $+, \cdot$ and constant symbols 0 and 1 :

Axioms of fields:
$\forall v_1 \forall v_2 \forall v_3$

$$
\begin{aligned}
&(v_1 + v_2) + v_3 \doteq v_1 + (v_2 + v_3)\\
&(v_1 \cdot v_2) \cdot v_3 \doteq v_1 \cdot (v_2 \cdot v_3)\\
&v_1 + v_2 \doteq v_2 + v_1\\
&v_1 \cdot v_2 \doteq v_2 \cdot v_1\\
&(v_1 + v_2) \cdot v_3 \doteq v_1 \cdot v_3 + v_2 \cdot v_3\\
&v_1 + 0 \doteq v_1\\
&v_1 \cdot 1 \doteq v_1.
\end{aligned}
$$

$\forall v_1 \exists v_2 \;\; v_1 + v_2 \doteq 0$
$\forall v_1 (\neg v_1 \doteq 0 \to \exists v_2 \; v_1 \cdot v_2 \doteq 1).$

Axioms stating that the field is of characteristic zero, one for each positive integer n :

$$\neg \underbrace{1 + \ldots + 1}_{n} \doteq 0,$$

Solvability of polynomial equations axioms, one for each positive integer n :

$$\forall v_1 \ldots \forall v_n \exists v \;\; v^n + v_1 \cdot v^{n-1} + \ldots + v_i \cdot v^i + \ldots + v_n \doteq 0.$$

Basic facts and definitions of dimension theory in algebraically closed fields is similar to the dimension theory in vector spaces. We give below a loose survey of it.

Any field F of characteristic zero contains a copy of rational numbers $\mathbb{Q}$. Indeed,

$$\underbrace{1^F + \ldots + 1^F}_{n} \in F,$$

is an element representing integer n, denote it n^F. Then the additive inverse of n^F represents $-n$, and correspondingly we can represent n^{-1} and in general any rational number m/n by a unique element of F. So we may just assume $\mathbb{Q} \subseteq F$.

A finite subset $\{x_1, \ldots, x_n\}$ of a field F is said to be **algebraically independent** if, for any nonzero polynomial in n variables $P(v_1, \ldots, v_n)$ with integer coefficients,

$$P(x_1, \ldots, x_n) \neq 0.$$

A **transcendence basis of a field** F is a maximal algebraically independent subset of F.
By Zorn's Lemma any independent subset can be extended to a basis, so a basis exists in any field.
If B_1 and B_2 are bases of the same field, then card $B_1 =$ card B_2.
This allows to define **the transcendence degree of a field** F to be the cardinality of a basis of the field, denoted tr.d.F.

Steinitz Theorem *If B_1 is a basis of F_1 and B_2 a basis of F_2, algebraically closed fields of same characteristic, and $\pi : B_1 \to B_2$ a bijection, then π can be extended to an isomorphism between the fields.*

In other words the isomorphism type of an algebraically closed field of a given characteristic is determined by its transcendence degree. Also, the transcendence degree of a field F is equal to the cardinality of the field modulo $\aleph_0$. In other words, for uncountable fields tr.d.$F =$ card F.
It follows that, if F_1 and F_2 are two models of ACFA$_0$ of an uncountable cardinality κ, then $F_1 \cong F_2$. Thus ACFA$_0$ is categorical in any such power κ.

It is also useful to consider the following simle example.
Example The theory of successor, T_S.
The language contains a unary function symbol s and a constant symbol 0.
The axioms are:

(a) $\forall v_1 \forall v_2\ (s(v_1) \doteq s(v_2) \rightarrow v_1 \doteq v_2)$;
(b) $\forall v_1 \exists v_2\ (\neg v_1 \doteq 0 \rightarrow v_1 \doteq s(v_2))$;
(c)$_n$ $\forall v\ \neg\, s^n(v) \doteq v$ for any positive integer n, where $s^n(v) = s(...(s(v))...)$, n times;
(d) $\forall v\ \neg s(v) \doteq 0$.

Exercise Prove that the theory T_S is categorical in all uncountable cardinalities.

6 $\aleph_0$-categoricity

Fix a *countable* language L. Henceforce T denotes a complete L-theory having an infinite model, say $\mathcal{A}$. By the Lowenheim-Skolem downward Theorem we may assume $\mathcal{A}$ is countable. Also, by the definition, $T = \mathrm{Th}(\mathcal{A})$.

Denote F_n the set of all L-formulas with free variables $v_1 \dots v_n$ (abbreviated $\bar{v}$). Denote $E_n(T)$ the binary relation on F_n defined by

$$\varphi(\bar{v})\ E_n\ \psi(\bar{v}) \quad \text{iff} \quad T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\varphi)).$$

Equivalently, since T is complete, $\varphi(\mathcal{A}) = \psi(\mathcal{A})$.

Thus, E_n is an equivalence relation respecting the Boolean operations $\wedge$, $\vee$ and $\neg$.

Given a theory T and a number n, $F_n/E_n(T)$ is called the n**th Lindenbaum algebra of** T. As was mentioned above, its elements are in a one-to-one correspondence with definable subsets of $\mathcal{A}$ and $\wedge$, $\vee$ and $\neg$ correspond to the usual Boolean operations $\cap$, $\cup$ and the complement, on the sets.

Theorem 9 (Ryll-Nardzewski) *T is $\aleph_0$-categorical iff $F_n/E_n(T)$ is finite for all $n \in N$.*

Proof of the theorem will follow from intermediate statements.

Lemma 6.1 *Assume that F_n/E_n is finite. Then for any $\bar{a} \in \mathcal{A}^n$ there is $\varphi(\bar{v})$ such that*
(i) $\mathcal{A} \models \varphi(\bar{a})$
and
(ii) whenever $\psi(\bar{v})$ is such that $\mathcal{A} \models \psi(\bar{a})$,

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v})).$$

Proof By the finiteness assumption we can find the minimal definable subset of A^n containing $\bar{a}$, say $\varphi(\mathcal{A})$. Then for any $\psi(\mathcal{A})$ containig $\bar{a}$ we have necessarily $\varphi(\mathcal{A}) \subseteq \psi(\mathcal{A})$. $\square$

Call φ as above **principal** [for $\bar{a}$].

Lemma 6.2 *[$\Leftarrow$ of the Theorem] The finiteness of all $F_n/E_n(T)$ implies $\aleph_0$-categoricity of T.*

Proof Suppose $\mathcal{B}$ is another countable model of T. Enumerate

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}.$$

We will construct new enumerations $\{a'_1, a'_2, \ldots\}$ and $\{b'_1, b'_2, \ldots\}$ of the sets so that the enumerations establish a correspondence between the sets preserving L-formulas, by the **back-and-forth method:**
Suppose $a'_1, \ldots, a'_{n-1} \in A$ and $b'_1, \ldots, b'_{n-1} \in B$ satisfy for all $\psi \in F_{n-1}$

$$\mathcal{A} \models \psi(a'_1, \ldots, a'_{n-1}) \text{ iff } \mathcal{B} \models \psi(b'_1, \ldots, b'_{n-1}). \tag{2}$$

Notice that (2) is true for $n = 1$ since $\mathcal{A} \equiv \mathcal{B}$. Let n be odd and a'_n be the first member in $A = \{a_1, a_2, \ldots\}$ not occurring among $a'_1, \ldots a'_{n-1}$. Let φ be a principal formula for $a'_1, \ldots, a'_n$. Then $\mathcal{A} \models \varphi(a'_1, \ldots, a'_n)$ and so, $\mathcal{A} \models \exists v \varphi(a'_1, \ldots, a'_{n-1}, v)$. By (2) $\mathcal{B} \models \exists v \varphi(b'_1, \ldots, b'_{n-1}, v)$. Hence we may choose $b'_n \in B$ such that $\mathcal{B} \models \varphi(b'_1, \ldots, b'_n)$.
Now suppose $\psi \in F_n$ and $\mathcal{A} \models \psi(a'_1, \ldots, a'_n)$. Since φ is principal

$$T \models \forall \bar{v}(\varphi(\bar{v}) \rightarrow \psi(\bar{v})).$$

Hence $\mathcal{B} \models \psi(b'_1, \ldots, b'_n)$.
Thus (2) is satisfied for $a'_1, \ldots a'_n$ and $b'_1, \ldots b'_n$, too.
Similarly, when $n + 1$ is even, b'_{n+1} is the first element in $B = \{b_1, b_2, \ldots\}$ not occurring among $b'_1, \ldots b'_n$. Then we can find $a'_{n+1} \in A$ such that (2) is satisfied for $a'_1, \ldots a'_{n+1}$ and $b'_1, \ldots b'_{n+1}$.
Hence we may inductively construct in this way $A = \{a'_1, a'_2, \ldots a'_n \ldots\}$, $B = \{b'_1, b'_2, \ldots b'_n \ldots\}$ satisfying (2) for all n. Our construction guarantees that we get all of A and all of B. Now it follows from (2) that $a'_i \rightarrow b'_i$ is an isomorphism. $\Box$

A subset $p \subset F_n$ is called an n**-type** (over T) if
(i) for all $\varphi \in p \;\; T \models \exists \bar{v} \varphi(\bar{v})$;
(ii) if $\varphi, \psi \in p$ then $(\varphi \wedge \psi) \in p$.

Type p is called **complete** if also the following is satisfied:
(iii) for any $\varphi \in F_n$ either $\varphi \in p$ or $\neg\varphi \in p$.

Suppose $\bar{a} \in A^n$. Then we define **the L-type of $\bar{a}$ in $\mathcal{A}$**.

$$\mathrm{tp}_{\mathcal{A}}(\bar{a}) = \{\varphi \in F_n : \; \mathcal{A} \models \varphi(\bar{a})\}.$$

Clearly, $\mathrm{tp}_{\mathcal{A}}(\bar{a})$ is a complete n-type.

When $\mathcal{A} \subseteq \mathcal{B}$ then $\mathrm{tp}_{\mathcal{A}}(a)$ and $\mathrm{tp}_{\mathcal{B}}(a)$ may be different. But it follows immediately from definitions that

$$\mathcal{A} \preccurlyeq \mathcal{B} \;\text{ implies }\; \mathrm{tp}_{\mathcal{A}}(a) = \mathrm{tp}_{\mathcal{B}}(a).$$

We say that an n-type p is **realised** in $\mathcal{A}$ if there is $\bar{a} \in A^n$ such that $p \subseteq \mathrm{tp}_{\mathcal{A}}(\bar{a})$.
If there is no such $\bar{a}$ in $\mathcal{A}$ we say that p is **omitted** in $\mathcal{A}$.

Lemma 6.3 *Given a set $P = \{p^\alpha : \alpha < \kappa\}$ of n-types p, a model $\mathcal{A}$ of T and a cardinal $\kappa \geq |\mathcal{A}|$, there is $\mathcal{B} \succcurlyeq \mathcal{A}$ of cardinality κ such that all types from P are realised in $\mathcal{B}$. In particular, given a type p there is a countable model $\mathcal{B}$ of T which realises p.*

Proof Consider the expansion L^+ of $L_{\mathcal{A}}$ by new constants

$$\{c_1^\alpha, \ldots, c_n^\alpha : \alpha < \kappa\},$$

and the theory

$$T^+ = \mathrm{CDiag}(\mathcal{A}) \cup \{\varphi(c_1^\alpha, \ldots, c_n^\alpha) : \; \varphi \in p^\alpha, \; \alpha < \kappa\}$$

We claim that T^+ is f.s. in $\mathcal{A}$. Indeed, any finite subset S of T^+ contains only finitely many formulas φ from the types. Since types are closed under conjunction, we may assume that there is at most one formula of the form

$\varphi(c_1^\alpha, \ldots, c_n^\alpha)$ in S for a type p^α. Since $\exists \bar{v} \varphi(\bar{v})$ holds in $\mathcal{A}$, we can find in $\mathcal{A}$ for $\varphi(c_1^\alpha, \ldots, c_n^\alpha)$ an interpretation of $c_1^\alpha, \ldots, c_n^\alpha$ which makes each such formula true in the corresponding expansion of $\mathcal{A}$.
By the Compactness Theorem there is a model $\mathcal{B}^+ \models T^+$ of cardinality κ. Since $\mathcal{B}^+ \models \mathrm{CDiag}(\mathcal{A})$ the L-reduct $\mathcal{B}$ of $\mathcal{B}^+$ is an elementary extension of $\mathcal{A}$. Let, for each α, $a_1^\alpha, \ldots, a_n^\alpha$ be the elements assigned to $c_1^\alpha, \ldots, c_n^\alpha$ in $\mathcal{B}^+$. By the construction $\langle a_1^\alpha, \ldots, a_n^\alpha \rangle$ realize p^α in $\mathcal{B}$.
If we start with a countable model $\mathcal{A}$ of T and $\kappa \leq \aleph_0$, then $\mathcal{B}$ can be chosen countable. $\square$

Corollary 4 *For any n-type there is $p' \supseteq p$ which is a complete n-type.*

Indeed, put $p' = \mathrm{tp}_{\mathcal{B}}(\bar{a})$ for $\bar{a}$ in $\mathcal{B}$ realising p.

Remark If $\pi : \mathcal{A} \to \mathcal{B}$ is an isomorphism, $\bar{a} \in A^n$, $\bar{b} \in B^n$, and $\pi : \bar{a} \to \bar{b}$ then $\mathrm{tp}_{\mathcal{A}}(\bar{a}) = \mathrm{tp}_{\mathcal{B}}(\bar{b})$.

Example There is a countable elementary extension of the group of integers $\mathbb{Z} = (\mathbb{Z}; +; 0)$ which is not isomorphic to $\mathbb{Z}$.
Given $n > 0$ denote $n|v$ the formula $\exists w(v = w + \ldots + w)$ (n summands).
Let

$$p = \{1|v \& \ldots \& n|v : \; n \in \mathbb{N}\}.$$

p clearly is a type, thus it is realised in some countable elementary extension. But p is obviously omitted in $\mathbb{Z}$.

A type p is called **principal** if there is $\varphi \in F_n$ such that $T \models \exists \bar{v} \varphi(\bar{v})$ and for any $\psi \in p$ $\quad T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$.
φ is called then a **principal formula for type** p.
A type which is not principal is called **non-principal**.

Exercise 6.1 *(i) A principal type p is realised in any model $\mathcal{A}$ of T.*
(ii) If p is a complete type then a principal formula for p is a principal formula.

Lemma 6.4 *If $F_n/E_n(T)$ is infinite then there exists a non-principal complete n-type in T.*

Proof Take $p = \{\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k \in F_n : \varphi_i$ principal formulae $\}$. We claim that p is an n-type.
Suppose not. Then

$$T \models \forall\bar{v}(\varphi_1(\bar{v}) \vee \ldots \vee \varphi_k(\bar{v}))$$

for some principal formulas $\varphi_1, \ldots, \varphi_k \in F_n$.
Define for $\psi \in F_n$

$$W_\psi = \{i \in \{1, \ldots, k\} : \; T \models \exists\bar{v}(\varphi_i(\bar{v}) \wedge \psi(\bar{v})\}$$

Notice that since φ_i's are principal formulas
$T \models \exists\bar{v}(\varphi_i(\bar{v}) \wedge \psi(\bar{v})\}$ iff $T \models \forall\bar{v}(\varphi_i(\bar{v}) \rightarrow \psi(\bar{v})\}$.
It follows that for any $\psi, \chi \in F_n$ $\psi E_n \chi$ iff $W_\psi = W_\chi$. Thus card $F_n/E_n(T) = 2^k$. This contradicts the assumtions and proves the claim.
Take now a complete n-type extending p. It is non-principal by the construction.$\square$

Theorem 10 (Omitting Type Theorem) *Let p be a non-principal n-type in a complete theory T of a countable language L. Then there is a countable model of T which omits p.*

Proof Let $L' = L \cup C$, C a set of countably many new constant symbols. Let $\bar{c}_1, \ldots, \bar{c}_k, \ldots$ be an enumeration of all n-tuples of constant symbols of L' and $\phi_1, \ldots, \phi_l, \ldots$ an enumeration of all sentences in L'.
We construct a chain of finite sets of L'-sentences

$$S_0 \subseteq \ldots S_m \subseteq \ldots$$

by induction on $m \geq 1$ so that
(i) $T \cup S_m$ are satisfiable,
(ii) for $m \geq 1$ either ϕ_m or $\neg\phi_m$ is in S_m, ,
(iii) if ϕ_m is in S_m and has the form $\exists v \; \varphi(v)$, for some 1-variable L'-formula $\varphi(v)$, then $\varphi(c) \in S_m$ for some $c \in C$
(iv) for $m \geq 1$ there is a formula $\psi \in p$ such that $\neg\psi(\bar{c}_m) \in S_m$.

Start with $S_0 = \emptyset$.
Suppose $S_0 \subseteq \ldots S_{m-1}$ are constructed.
If $T \cup S_{m-1} \cup \{\phi_m\}$ is satisfiable then put $S'_m = S_{m-1} \cup \{\phi_m\}$. Otherwise $S'_m = S_{m-1} \cup \{\neg\phi_m\}$. It is easy to see that $T \cup S'_m$ is satisfiable.

Claim. There exists $\psi \in p$ such that $T \cup S'_m \cup \{\neg\psi(\bar{c}_m)\}$ is satisfiable.
Proof of Claim. Suppose for all $\psi \in p$ the converse holds. Let $\Phi = \bigwedge S'_m$. We can represent Φ as $\varphi(c_{m,1}, \ldots, c_{m,n}, d_1, \ldots, d_p)$, where $\varphi(v_1, \ldots v_n, u_1, \ldots, u_p)$ is an L-formula with free variables $v_1, \ldots v_n, u_1, \ldots, u_p$ and $\langle c_{m,1} \ldots, c_{m,n}\rangle = \bar{c}_m$, $d_1, \ldots, d_p$ constant symbols not in L and different from $c_{m,i}$'s. We write corresponding formulas in the short form $\varphi(\bar{c}_m, \bar{d})$ and $\varphi(\bar{v}, \bar{u})$.
Then, by our assumption, for any $\psi \in p$

$$T \models (\varphi(\bar{c}_m, \bar{d}) \rightarrow \psi(\bar{c}_m)).$$

Since no component of $\bar{c}_m$ and $\bar{d}$ do occur in T, it follows

$$T \models \forall\bar{v}\forall\bar{u}(\varphi(\bar{v}, \bar{u}) \rightarrow \psi(\bar{v})).$$

The formula can be equivalently rewritten as $\forall\bar{v}(\exists\bar{u}\varphi(\bar{v}, \bar{u}) \rightarrow \psi(\bar{v}))$, so

$$T \models \forall\bar{v}(\exists\bar{u}\varphi(\bar{v}, \bar{u}) \rightarrow \psi(\bar{v}))$$

for every $\psi \in p$. This means that $\exists\bar{u}\varphi(\bar{v}, \bar{u})$ is a principal formula for p. The contradiction, which proves the claim.

Now take $S''_m = S'_m \cup \{\neg\psi(\bar{c}_m)\}$.
Suppose ϕ_m is in S''_m and has the form $\exists v\ \varphi(v)$. Choose $c \in C$ which does not occur in S''_m. Then $T \cup S''_m \cup \{\varphi(c)\}$ has a model: any model $\mathcal{A}$ of $T \cup S''_m$ in the language $L \cup \{$ constants of $S''_m\}$ can be expanded by assigning to c the values of v for $\exists v\varphi(v)$.
Denote $S_m = S''_m \cup \{\varphi(c)\}$. If ϕ_m does not have this form then put $S_m = S''_m$.
This S_m satisfies (i)-(iv) by the construction.
To finish the proof of the theorem consider now

$$T^* = T \cup \bigcup_{m \in \mathbb{N}} S_m.$$

By the properties (i)-(iii) T^* is satisfiable, complete and full set of sentences. By Theorem 3 T^* has a canonical model $\mathcal{A}$. Notice that by (iii) for any closed term λ T^* says $\lambda = c$ for some $c \in C$. Thus all elements of the canonical model $\mathcal{A}$ are named by symbols from C. Consequently, (iv) says that no n-tuple in $\mathcal{A}$ realises type p.$\square$

End of the proof of the Ryll-Nardzewski Theorem: *If $F_n/E_n(T)$ is infinite then T is not $\aleph_0$-categorical.*
Indeed, from Lemma 6.4 it follows, under the assumption, that there is a non-principal n-type in T. By the Omitting Type Theorem there is a countable model $\mathcal{A}$ that omits p. On the other hand, by Lemma 6.3, there is a countable model $\mathcal{B}$ which realises p. It follows $\mathcal{A}$ is non-isomorphic to $\mathcal{B}$ and thus T is not $\aleph_0$ categorical. $\square$

Remark Slight changes in the proof of the Omitting Type Theorem yield

Theorem 11 (Omitting Types Theorem of R.Vaught) *Let P be a countable set of non-principal n-types in a complete theory T of a countable language L. Then there is a countable model of T which omits every type in P.*

7 Spaces of types. Theories with few types

Let T be a complete theory of a countable language L.
We denote $\mathrm{S}_n(T)$ the set of all complete n-types in T, **the (Stone) space of n-types of T.**

The Stone spaces are closely connected with the Lindenbaum algebras of T.
For T as before, T is called **0-stable** if card $\mathrm{S}_n(T) \leq \aleph_0$ for all $n \in \mathbb{N}$.

A structure $\mathcal{A}$ is called **atomic** if for all $n \in \mathbb{N}$, every complete n-type realised in $\mathcal{A}$ is principal.

Remark We can equivalenly say in the definition: *every n-tuple in $\mathcal{A}$ satisfies a principal formula.*

Warning 'Atomic' here is connected with the notion of atoms of the Boolean algebra $F_n/E_n(T)$. Nothing to do with atomic formulas.

A model $\mathcal{A}$ of T is called **prime** if for any model $\mathcal{B}$ of T there is an elementary embedding $\pi : \mathcal{A} \to \mathcal{B}$.

Theorem 12 *(i) Any countable atomic model of a complete theory T is prime.*
(ii) Any two countable atomic models of T are isomorphic.
(iii) Assume T is 0-stable. Then T has a countable atomic model.

Proof (i) and (ii) are left to the reader (Problem sheet 6). Use an inductive construction similar to the one in the proof of Lemma 6.2.
(iii) Since T is 0-stable, there are only countably many non-principal types in $\bigcup_n \mathrm{S}_n(T)$. By the Omitting Types Theorem of Vaught there is a countable model $\mathcal{A}$ of T which omits all the non-principal types. This $\mathcal{A}$ is atomic by definition. $\square$

A structure $\mathcal{A}$ is called $\aleph_0$-**saturated** if, for any expansion $\mathcal{A}_{c_1,\ldots,c_m}$ of $\mathcal{A}$ by finitely many contant symbols $c_1, \ldots, c_m$, every 1-type in $\mathrm{Th}(\mathcal{A}_{c_1,\ldots,c_m})$ is realised in $\mathcal{A}_{c_1,\ldots,c_m}$.

A model $\mathcal{A}$ of T is called **$\aleph_0$-universal** if, for any countable model $\mathcal{B}$ of T, there is an elementary embedding $\pi : \mathcal{B} \to \mathcal{A}$.

Theorem 13 *(i) Any countable $\aleph_0$-saturated model of a complete theory T is $\aleph_0$-universal.*
(ii) Any two countable $\aleph_0$-saturated models of T are isomorphic.
(iii) Assume T is 0-stable. Then T has a countable $\aleph_0$-saturated model.

Proof (i) and (ii) are exercises (Problem sheet 7). Use an inductive construction similar to the one in the proof of Lemma 6.2.

Proof of (iii). We start with

Lemma 7.1 *Let $T' = T(c_1, \ldots, c_m)$ be a complete theory extending T in the language $L(c_1, \ldots, c_m)$, the extension of L by finitely many extra constants symbols $c_1, \ldots, c_m$, and suppose T is 0-stable. Then T' is 0-stable too.*

Proof Fix n. For each $p \in \mathrm{S}_n(T')$ define

$$p^* = \{\phi(v_1, \ldots, v_{n+m}) \in F_{n+m} : \ \phi(v_1, \ldots, v_n, c_1, \ldots, c_m) \in p\}.$$

It follows from the definition that $p^* \in \mathrm{S}_{n+m}(T)$, and if $p_1 \neq p_2$ then $p_1^* \neq p_2^*$. Hence we have mapping $\mathrm{S}_n(T') \to \mathrm{S}_{m+n}(T)$, which is injective. Since card $\mathrm{S}_{m+n}(T) \leq \aleph_0$, by the hypothesis, we have $\mathrm{S}_n(T') \leq \aleph_0$. $\square$

End of the proof of (iii). Let $\mathcal{A}$ be a countable model of T. Enumerate $\{a_1, \ldots, a_n, \ldots\}$ elements of $\mathcal{A}$. Let $C = \{c_1, \ldots, c_n, \ldots\}$ be a set of new constant symbols, $\mathcal{A}_C$ the structure in the language L_C obtained by assigning a_i to c_i, T_C the theory of the structure, and $T_{\{c_1,\ldots,c_m\}}$ the fragment of the theory containing formulas with at most the first m constants symbols of C. By Lemma above the set of 1-types $\bigcup_m \mathrm{S}_1(T_{\{c_1,\ldots,c_m\}})$ is countable. By Lemma 6.3 we can construct a countable $\mathcal{B}_C \succ \mathcal{A}_C$ which realises all the types of $\bigcup_m \mathrm{S}_1(T_{\{c_1,\ldots,c_m\}})$. Clearly $\mathcal{B}$ has the property that any 1-type of an expanded theory $\mathrm{Th}(\mathcal{A}_{\{c_1,\ldots,c_m\}})$ is realised in $\mathcal{B}_C$.

Repeating this construction we get an elementary chain

$$\mathcal{A}^{(0)} \preccurlyeq \mathcal{A}^{(1)} \preccurlyeq \ldots \preccurlyeq \mathcal{A}^{(n)} \ldots$$

of countable models of T with $\mathcal{A}^{(0)} = \mathcal{A}$ and the property that any 1-type in $\mathrm{Th}(\mathcal{A}^{(n)}_{\{c_1,\ldots,c_m\}})$ is realised in $\mathcal{A}^{(n+1)}_{c_1,\ldots,c_m}$ for any assignment of constant symbols $c_1, \ldots, c_m$, any m.
Then the union $\mathcal{A}^* = \bigcup_n \mathcal{A}^{(n)}$ of the elementary chain, by Exercise 4.6, is an elemenary extension of $\mathcal{A}$ and indeed of each $\mathcal{A}^{(n)}$. It follows that $\mathcal{A}^*$ is a countable saturated model of T. This proves (iii). $\square$

8 Theories with many types

Theorem 14 *Suppose* card $\mathrm{S}_n(T) = \kappa > \aleph_0$. *Then T has at least κ non-isomorphic countable models.*

Proof For any n-type there is a countable model that realises the type, and in a countable model at most countably many complete types can be realized.$\square$

Theorem 15 *Suppose* $\mathrm{S}_n(T)$ *is uncountable. Then* card $\mathrm{S}_n(T) = 2^{\aleph_0}$.

We start the proof of the theorem by introducing a new notion and proving an intermediate lemma.
A formula $\varphi \in F_n$ is called **fat** (in T) if

$$U_\varphi = \{p \in S_n(T) : \ \varphi \in p\}$$

is uncountable.

Lemma 8.1 *For any fat φ there are fat φ_0 and φ_1 such that $\varphi \equiv \varphi_0 \vee \varphi_1$ and there is no n-type containing both of the formulas, that is $T \vDash \neg\exists\bar{v}(\varphi_0 \wedge \varphi_1)$.*

Proof Suppose not. Define

$$q_\varphi = \{\psi \in F_n : (\psi \wedge \varphi) \text{ is fat }\}.$$

This is a complete type. Indeed, (i) of the definition of type follows from the fact that every ψ in q_φ belongs to a type, since ψ is fat.
(ii) follows from the assumption that φ can not be divided into two fat parts: $\psi_1 \wedge \psi_2 \wedge \varphi$ is fat, if $\psi_1, \psi_2 \in q$.
(iii) is immediate from the same assumption.
Now notice that

$$U_\varphi = \{q_\varphi\} \cup \bigcup \{U_{\neg\psi\wedge\varphi} : \psi \in q_\varphi\}.$$

By assumptions $U_{\neg\psi\wedge\varphi}$ is at most countable, for every $\psi \in q_\varphi$, contradicting the fact that φ is fat. $\square$

Proof of the theorem. Notice first that the number of n-types is not greater than $2^{\aleph_0}$ since each type is just a subset of the countable set F^n. So we want to show that the number is not less than $2^{\aleph_0}$.
Let $\mathcal{M} = \{\mu : \mathbb{N} \to \{0,1\}\}$ be the set of all $\{0,1\}$-sequences. For each μ and $n \in \mathbb{N}$ define $\mu_{|n}$, the initial n-cut of μ, to be the reduction of μ to $\{1,\ldots,n\}$.
Define a fat formula $\varphi_{\mu,n}$ by induction on n :
For $n = 0$ let it be the formula $v_1 = v_1$.
If $\varphi_{\mu,n}$ is defined then $\varphi_{\mu,n+1}$ is either one of the two fat formulas that divide $\varphi_{\mu,n+1}$, as given by the lemma above, depending on whether $\mu(n+1)$ is 0 or 1. So if $\mu_{|n} = \nu_{|n}$ and $\mu_{|n+1} \neq \nu_{|n+1}$, then $\varphi_{\mu,n} = \varphi_{\nu,n}$, and $\varphi_{\mu,n+1}$ but $\varphi_{\nu,n+1}$ can not belong to a common type. Also $T \vDash \forall \bar{v}(\varphi_{\mu,n+1} \to \varphi_{\mu,n})$.
Let now for each μ

$$q_\mu = \{\varphi_{\mu,i_1} \wedge \ldots \wedge \varphi_{\mu,i_n} : i_1,\ldots,i_n \in \mathbb{N}\}.$$

This, by definition, is a type. So, there is an extension $p_\mu \supseteq q_\mu$ which is a complete type. If $\mu \neq \nu$, say n is the first number such that $\mu(n) \neq \nu(n)$, then $\varphi_{\mu,n} \in p_\mu$, $\varphi_{\nu,n} \in p_\nu$ are the two mutually inconsistent formulas dividing $\varphi_{\mu,n}$, and so $p_\mu \neq p_\nu$.
Thus the number of complete types is not less then the number of infinite $\{0,1\}$-sequences, which is $2^{\aleph_0}$. $\square$

Remark In fact, the theorem is a special case of the classical topological fact: *An uncountable compact Hausdorff space with countable basis is of cardinality continuum.*
Our U_ϕ's form a basis of such a topology on $\mathrm{S}_n(T)$.

Applying Theorem 14 and taking into account that, given a countable language L, there is at most $2^{\aleph_0}$ countable L-structures, we have:

Corollary 5 *Suppose for some* n, $\mathrm{S}_n(T)$ *is uncountable. Then* T *has exactly* $2^{\aleph_0}$ *non-isomorphic countable models.*

Glossary

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assignement section 1
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bounded variable section 1
canonical L-structure section 2
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