5. Faraday's Law—Electromagnetic Induction

5.1 Electromotive Force

Faraday's Law

Michael Faraday's law of electromagnetic induction can be stated as follows:

- When the magnetic flux $\Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ through a circuit is changing, an electromotive force is induced in the circuit.
- The magnitude of the e.m.f. is proportional to the rate of change of magnetic flux.

Faraday deduced his law in 1831-2, after performing a number of experiments that showed:

- If the current in a coil changes, a current is induced in a neighbouring coil, because the amount of the first coil's magnetic flux that **links** (i.e., crosses) the second coil changes.
- If a coil moves *relative* to a source of flux (of whatever type), such that the flux linked changes, then a current is induced in the coil.
- If part of a conducting circuit moves, and therefore cuts magnetic flux, then a current is induced in the circuit.
- The current induced is proportional to the conductance of the wire, so the change in flux gives a definite **electromotive force** (voltage) rather than a current. This e.m.f. is also called the **electromotance**.
- Faraday did not initially note that the e.m.f. was proportional to the rate of change of flux, only that one increased with the other. Neumann was the first to assume this (in 1845) and Faraday proved it in 1851-2.
- In 1832, Joseph Henry was the first to discover self-inductance, whereby a change in current in a coil induces an e.m.f. in its own circuit.

We will show that these observations lead to the law as stated at the start, namely that the e.m.f. is equal to the rate of change of magnetic flux, whether it be the magnetic field that is changing, or the circuit that is moving, or a combination of the two.

Lenz's Law

Faraday was vague about the direction of the induced e.m.f. in each case, and it was Emil Lenz in 1834 who first made a clear statement.

- The induced e.m.f. is always in such a direction as to promote a current flow that creates a magnetic field that opposes the change in flux.

5.2 Explanation of Faraday's Law

- Consider the force on a charge $+q$ in a wire moving at velocity $\boldsymbol{v}$ through a field $\boldsymbol{B}$

$$\boldsymbol{F} = q\,(\boldsymbol{v}\times\boldsymbol{B}) \tag{5.1}$$

- In the rest frame of the charge, this force appears to be due to an electric field

$$\boldsymbol{E} = \boldsymbol{F}/q = \boldsymbol{v}\times\boldsymbol{B} \tag{5.2}$$

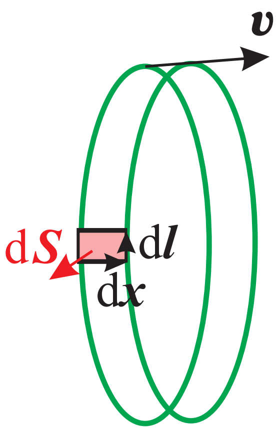
- The contribution to the e.m.f., for an element of length $\mathrm{d}\boldsymbol{l}$, is

$$\mathrm{d}\mathcal{E} \equiv \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = (\boldsymbol{v}\times\boldsymbol{B})\,.\mathrm{d}\boldsymbol{l} \tag{5.3}$$

- Now consider a loop of wire moving through a (constant) magnetic field $\boldsymbol{B}$:

  The total e.m.f. around the circuit

$$\mathcal{E} = \oint \mathrm{d}\mathcal{E} = -\oint (\boldsymbol{B}\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{l} = -\oint \boldsymbol{B}\cdot(\boldsymbol{v}\times\mathrm{d}\boldsymbol{l}). \tag{5.4}$$



- Let the loop move a distance $\mathrm{d}x$ in time $\mathrm{d}t$.
- Then

$$\boldsymbol{v}\times\mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\times\mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} \tag{5.5}$$

and so $\oint \boldsymbol{B}\cdot(\boldsymbol{v}\times\mathrm{d}\boldsymbol{l}) = \int_{\text{strip}} \boldsymbol{B}\cdot\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}$

$$= \frac{1}{\mathrm{d}t}\int_{\text{strip}} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{loop}} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \tag{5.6}$$

(The new area after time $\mathrm{d}t$ is the old area plus that of the strip around the edge.)

$$\Rightarrow \mathcal{E} = -\oint \boldsymbol{B}\cdot(\boldsymbol{v}\times\mathrm{d}\boldsymbol{l}) = -\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{5.7}$$

- The formula $\mathcal{E} = -\mathrm{d}\Phi/\mathrm{d}t$ works for both the following cases:
  1. Moving a loop, cutting stationary $\boldsymbol{B}$ field lines (as here);
  2. A stationary loop with a changing $\boldsymbol{B}$ field through it.

This formula for Faraday's law was derived above for the case of a rigid circuit moving in a static magnetic field. However, Faraday showed that it was only the relative motion of the circuit and the source of magnetic field that mattered. Thus the law also applies to the case of a stationary circuit where the flux linked changes with time because the source is moving.

However, we cannot tell how a magnetic field was produced just by measuring it at a point. Thus Faraday's law should hold when the flux varies for whatever reason, such as a varying current in another coil, and not just when the source is moving.

In some cases, the circuit may move while the field is changing. The law still holds.

5.3 Faraday's Law and Maxwell's Equations

- Faraday's law states that

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}. \tag{5.8}$$

- We can write Faraday's law in integral or differential form.
- To get the Maxwell equation we consider a fixed loop so that the only time variation comes from the time dependence of $\boldsymbol{B}$ at each position. Thus we change the full derivative to a partial derivative:

$$\text{Integral form: } \oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = -\int \mathrm{d}\boldsymbol{S}\cdot\frac{\partial \boldsymbol{B}}{\partial t}. \tag{5.9}$$

- Apply Stokes' theorem to the left-hand side to get the differential form:

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}. \tag{5.10}$$

- These are equal for all loops, so

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{(Maxwell 2)} \tag{5.11}$$

- This reduces, as expected, to $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$ for electrostatics.

5.4 Self-Inductance

Faraday's law requires us to know the flux linked by a circuit. We will now calculate the flux for a number of important cases. The inductance of a circuit element is the flux that links it per unit current, flowing either in the same circuit element (for the **self-inductance**) or in another circuit element (for the **mutual inductance**). We consider first the self-inductance.

- A circuit carrying current I is linked by the magnetic field lines produced by its own current.

- Let $\Phi =$ the total self-linked magnetic flux.

  Linearity $\Rightarrow \Phi \propto I$.

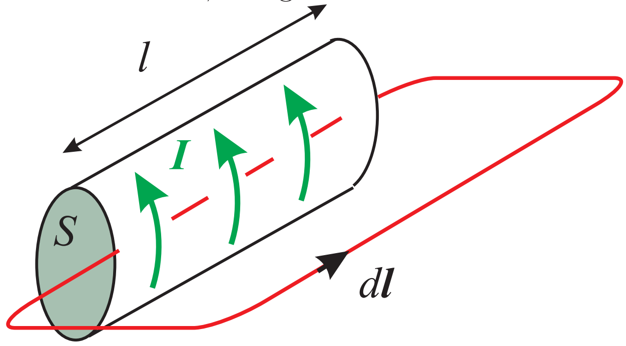
- Define the self-inductance:
$$L \equiv \Phi / I \tag{5.12}$$

- The unit of inductance (in SI) is the Henry:
$$1 \text{ H} \equiv 1 \text{ Wb A}^{-1}, \text{ where } 1 \text{ Wb} \equiv 1 \text{ T m}^2. \tag{5.13}$$

- L is determined by the geometry of the circuit: current loops have a larger L than straight wires, as we will show below.

5.5 Self-Inductance of a Long Solenoid

As a first example, we calculate the self-inductance of a long solenoid, of length l and cross-sectional area S; we ignore end effects.



Let there be n turns per unit length.

$B = 0$ outside the solenoid.

Inside the solenoid, $B \equiv B_{\mathrm{i}}$.

$$\text{Ampère:} \quad \oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = H_{\mathrm{i}} l = nlI \Rightarrow B_{\mathrm{i}} = \mu_0 n I \tag{5.14}$$

since nl is the total number of loops, each of which contributes to the field.

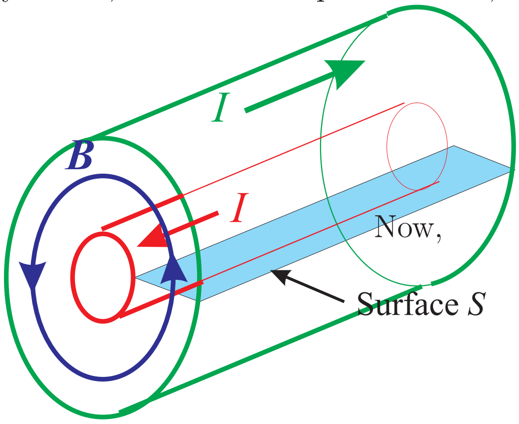
The total flux linked is
$$\Phi = nlB_{\mathrm{i}}S = n^2 l I \mu_0 S \tag{5.15}$$

$\Rightarrow$ The self-inductance $L \equiv \Phi/I = \mu_0 n^2 S l$.

$$\Rightarrow \quad \text{Self-inductance } \textit{per unit length} = \mu_0 n^2 S. \tag{5.16}$$

5.6 Self-Inductance of Coaxial Cylinders

The next example is a coaxial cable, which consists of a pair of coaxial cylinders, of length l, and inner and outer radii a and b. There is no magnetic field outside the outer wire, since no net current threads a loop of constant radius outside the outer cylinder. In between the cylinders, consider a loop of radius r, and apply Ampère's law:



$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = I \tag{5.17}$$

$$\Rightarrow B(r) = \frac{\mu_0 I}{2\pi r}. \tag{5.18}$$

Now,

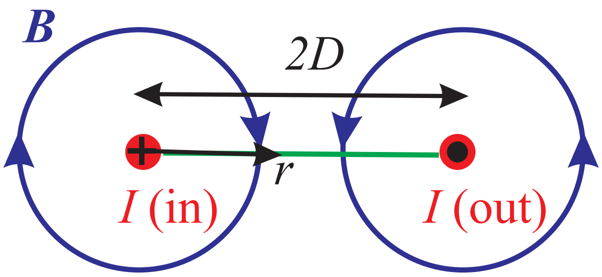
$$\Phi = l\int_a^b dr\ B(r) \tag{5.19}$$

$$\Rightarrow LI = \Phi = \frac{\mu_0 I l}{2\pi}\log\left(\frac{b}{a}\right) \tag{5.20}$$

$$\Rightarrow \text{The self-inductance per unit length} = \frac{\mu_0}{2\pi}\log\left(\frac{b}{a}\right). \tag{5.21}$$

5.7 Self-Inductance of a Pair of Wires

The third example is a pair of parallel wires (such as twin-pair cables, and transmission lines), of length l, and radius $a \ll D$, where $2D$ is the distance between the centres of the wires, as shown.



Assume the currents are uniformly distributed.
⇒ The magnetic field due to one wire

$$B(r) = \frac{\mu_0 I}{2\pi r}. \tag{5.22}$$

⇒ The flux due to one wire

$$\Phi = \frac{\mu_0 I l}{2\pi}\int_a^{2D-a}\frac{dr}{r} \approx \frac{\mu_0 I l}{2\pi}\log\left(\frac{2D}{a}\right) \quad (a \ll D) \tag{5.23}$$

The flux due to both wires is twice this.

$$\Rightarrow \text{The self-inductance per unit length} = \frac{\mu_0}{\pi}\log\left(\frac{2D}{a}\right). \tag{5.24}$$

Note that the integral does not include the field *inside* each wire. This is negligible at high frequencies, because of the "skin effect", whereby high-frequency current flows only in the surface of conductors. At low frequencies, this self-inductance of each wire can also be shown to be small—0.05μH m^{-1} (see e.g., Duffin pp225–6).

5.8 Energy Stored in Inductance

- If L is constant (rigid circuits) and I increases with time,

$$\text{Faraday} \Rightarrow \mathcal{E} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}(LI) = -L\frac{\partial I}{\partial t} \tag{5.25}$$

  since $\Phi = LI$.

- Consider an LR circuit + a voltage source:

$$V = RI \quad + \quad \underset{\text{“back e.m.f”}}{\underset{\uparrow}{L\frac{\partial I}{\partial t}}} \tag{5.26}$$

- The rate of energy loss in the voltage source $= VI$.

$$VI = \underset{\text{Dissipation in resistor}}{\underset{\uparrow}{I^2R}} + LI\frac{\partial I}{\partial t} = I^2R + \underset{\text{Rate of gain of magnetic energy in inductance}}{\underset{\uparrow}{\frac{\partial}{\partial t}\left(\tfrac{1}{2}LI^2\right)}} \tag{5.27}$$

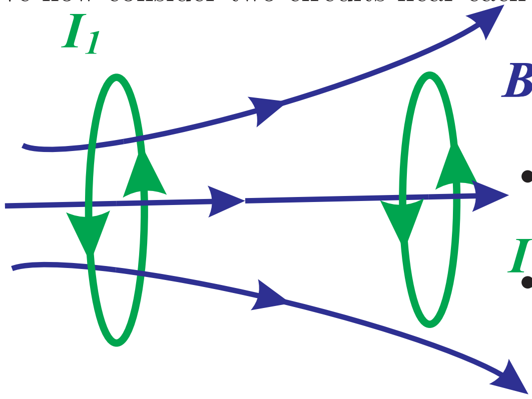
  $\Rightarrow$ Energy stored in LR circuit:

$$\boxed{U_{\mathrm{L}} = \tfrac{1}{2}LI^2} \tag{5.28}$$

  (c.f. capacitance $U_{\mathrm{C}} = \frac{1}{2}CV^2$).

5.9 Mutual Inductance

- We now consider two circuits near each other:



- The current I_1 produces a flux linkage Φ_2 in circuit #2.
- Linearity $\Rightarrow$ $\Phi_2 \propto I_1$.

- Define the **mutual inductance**:

$$M_{12} \equiv \frac{\Phi_2}{I_1}. \tag{5.29}$$

This is similar to $L_1 = \Phi_1 / I_1$ for the self-inductance.

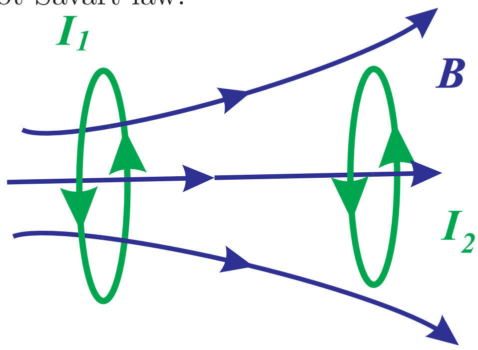
- Similarly, we can define $M_{21} \equiv \Phi_1 / I_2$.
- We will show that the mutual inductance is symmetric:

$$M_{21} = M_{12}. \tag{5.30}$$

- This is an example of a very general *reciprocity theorem*, which is applicable whenever there is a quadratic stored energy (see the Dynamics course).

5.10 Symmetry of Mutual Inductance

We will prove that the mutual inductance is symmetric in two ways, firstly by imagining how the currents are set up, and secondly by finding the flux enclosed directly from the Biot-Savart law.



- Suppose $I_1 = I_2 = 0$ and I_1 is gradually turned on.

  There is a back e.m.f.

$$\frac{\partial \Phi_1}{\partial t} = L_1 \frac{\partial I_1}{\partial t}. \tag{5.31}$$

I_1 does work against the e.m.f. at a rate

$$I_1 L_1 \frac{\partial I_1}{\partial t}. \tag{5.32}$$

⇒ The energy stored in the field

$$U = \tfrac{1}{2}L_1 I_1^2. \tag{5.33}$$

- Now suppose I_2 is turned on, keeping I_1 fixed.

  The extra energy due to the self-inductance is (as above) $\frac{1}{2}L_2 I_2^2$.

  The back e.m.f. in the first circuit due to the mutual inductance is

$$\frac{\partial \Phi_1}{\partial t} = M_{21}\frac{\partial I_2}{\partial t}. \tag{5.34}$$

  The battery in circuit #1 does work to keep I_1 constant at a rate

$$I_1 M_{21}\frac{\partial I_2}{\partial t}. \tag{5.35}$$

  (The battery in circuit #2 does no work against the mutual inductance since I_1 is a constant.)

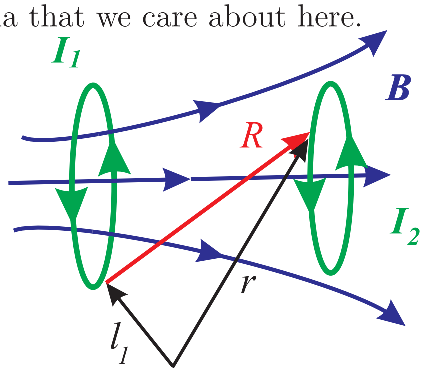
$$\Rightarrow \text{The total energy} \quad U = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2. \tag{5.36}$$

- This stored energy must be the same if, instead, I_2 is switched on before I_1

$$\Rightarrow \boxed{M_{21} = M_{12}} \tag{5.37}$$

5.11 Mutual Inductance From the Biot-Savart Law

We now use the Biot-Savart law to write down the mutual inductance directly, by integration. The formula itself for M may sometimes be useful, but it is the symmetry of the formula that we care about here.



- This is a high-tech direct proof. It is not for examination
- $R = \boldsymbol{r} - \boldsymbol{l}_1$
  ($\boldsymbol{r} = \boldsymbol{l}_2$ on loop #2)

- Remember that the vector potential $\boldsymbol{A}$ is defined through $\boldsymbol{B} = \boldsymbol{\nabla}\times \boldsymbol{A}$. In section 3.3 we showed that the Biot-Savart law can be written

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\times\left(\frac{\mathrm{d}\boldsymbol{l}}{R}\right), \tag{5.38}$$

  so we can choose

$$\mathrm{d}\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{l}}{R}. \tag{5.39}$$

- The flux linkage through circuit #2 (using Stokes's theorem)

$$\begin{aligned}
\Phi_2 = \int \mathrm{d}\boldsymbol{S}_2 \cdot \boldsymbol{B}_1 &= \int \mathrm{d}\boldsymbol{S}_2 \cdot \boldsymbol{\nabla} \times \boldsymbol{A}_1 \\
&= \oint_{\text{loop 2}} \mathrm{d}\boldsymbol{l}_2 \cdot \boldsymbol{A}_1 \\
&= \oint_{\text{loop 2}} \mathrm{d}\boldsymbol{l}_2 \cdot \oint_{\text{loop 1}} \mathrm{d}\boldsymbol{A}_1 \\
&= \frac{\mu_0 I_1}{4\pi} \oint_{\text{loop 2}} \mathrm{d}\boldsymbol{l}_2 \cdot \oint_{\text{loop 1}} \frac{\mathrm{d}\boldsymbol{l}_1}{|\boldsymbol{l}_2 - \boldsymbol{l}_1|}
\end{aligned} \tag{5.40}$$

$$\Rightarrow M_{12} = \frac{\mu_0}{4\pi} \oint_{\text{loop 2}} \oint_{\text{loop 1}} \frac{\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2}{|\boldsymbol{l}_2 - \boldsymbol{l}_1|} = M_{21}. \tag{5.41}$$

This is obviously symmetric.

Note that the mutual inductance is not symmetric if there is a non-linear magnetic material (a ferromagnet) present. μ is different for the primary and secondary loops if the number of turns is different, so that the field, and hence μ, is different.

5.12 Combination of L and M

- The mutual inductance is

$$M_{12} = M_{21} \equiv M. \tag{5.42}$$

- The self-inductance is L.

- Sign convention: M is positive if the currents are in the same sense

$$\Rightarrow \Phi_2 = L_2 I_2 + M I_1. \tag{5.43}$$

- The mutual inductance produces an e.m.f. $\mathcal{E}_1$ in circuit #1 if the current in circuit #2 changes. This adds to the e.m.f. due to any change in I_1 itself.

$$\begin{aligned}
\mathcal{E}_1 &= -L_1 \frac{\partial I_1}{\partial t} - M \frac{\partial I_2}{\partial t}, \\
\mathcal{E}_2 &= -L_2 \frac{\partial I_2}{\partial t} - M \frac{\partial I_1}{\partial t}.
\end{aligned} \tag{5.44}$$

5.13 Energy in Coupled Circuits

A voltage source connected to an inductor can drive only as much current as the back e.m.f. allows, i.e., the applied voltage and the back e.m.f. (the negative of the e.m.f.) balance each other.

$$\begin{aligned} V_1 &= L_1 \frac{\partial I_1}{\partial t} + M \frac{\partial I_2}{\partial t} \\ V_2 &= L_2 \frac{\partial I_2}{\partial t} + M \frac{\partial I_1}{\partial t} \end{aligned} \tag{5.45}$$

- The energy loss from the voltage sources

$$= V_1 I_1 + V_2 I_2. \tag{5.46}$$

$$= I_1 L_1 \frac{\partial I_1}{\partial t} + I_1 M \frac{\partial I_2}{\partial t} + I_2 L_2 \frac{\partial I_2}{\partial t} + I_2 M \frac{\partial I_1}{\partial t} \tag{5.47}$$

$$= \underset{\substack{\uparrow \\ \text{energy gain} \\ \text{in } L_1}}{\frac{\partial}{\partial t}\left(\tfrac{1}{2} L_1 I_1^2\right)} + \underset{\substack{\uparrow \\ \text{energy gain} \\ \text{in } L_2}}{\frac{\partial}{\partial t}\left(\tfrac{1}{2} L_2 I_2^2\right)} + \underset{\substack{\uparrow \\ \text{energy gain in} \\ \text{mutual field}}}{\frac{\partial}{\partial t}\left(M I_1 I_2\right)} \tag{5.48}$$

⇒ The total energy is $U_{\mathrm{M}} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$.

- The first two terms are > 0, but the third can take either sign.

5.14 The Link Between L and M

- $U_{\rm M} = \frac{1}{2}(L_1 I_1^2 + 2MI_1I_2 + L_2I_2^2)$

$$= \frac{1}{2}L_1\left(I_1 + \frac{M}{L_1}I_2\right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right)I_2^2. \tag{5.49}$$

- The total $U_{\rm M}$ must be positive so there is a restriction on the value of M. $U_{\rm M}$ must be positive even when the first bracket is zero.

$$\Rightarrow L_2 - \frac{M^2}{L_1} > 0$$

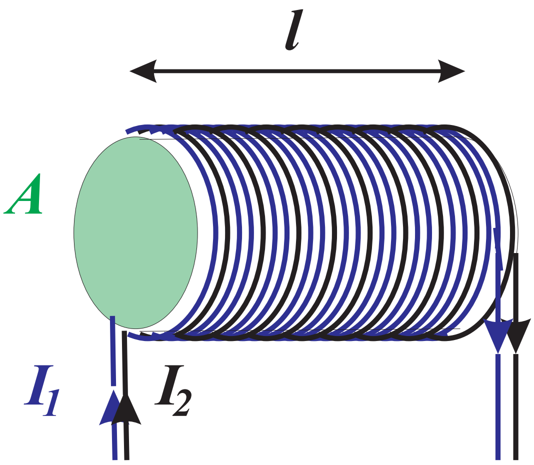
$$\Rightarrow M^2 \leq L_1L_2. \tag{5.50}$$

Define

$$M = k(L_1L_2)^{1/2} \quad (0 \leq k \leq 1). \tag{5.51}$$

k is the coefficient of coupling; $k = 1 \Rightarrow$ perfect coupling.

- Example: two long solenoids wound over each other.



The number of turns per unit length is n_1 and n_2, respectively (A and l are the same).

$B_1 = \mu_0 n_1 I_1$ as usual.

The flux linking #1 $= B_1An_1l$

$$\Rightarrow L_1 = \mu_0 n_1^2 Al. \tag{5.52}$$

The flux linking #2 $= B_1An_2l$

$$\Rightarrow M = \mu_0 n_1 n_2 Al. \tag{5.53}$$

Similarly

$$L_2 = \mu_0 n_2^2 Al. \tag{5.54}$$

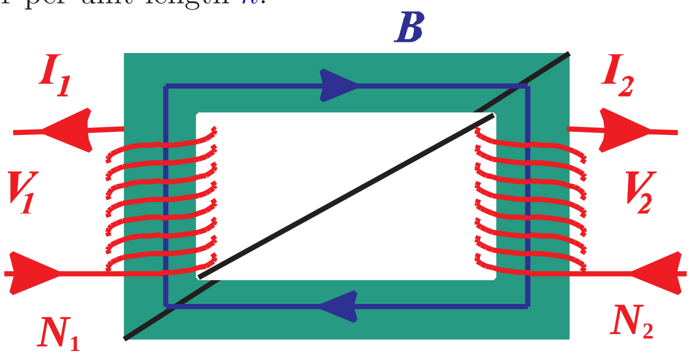
$$\Rightarrow M = (L_1L_2)^{1/2} \tag{5.55}$$

for this case, since the flux enclosed is the same for each coil.

5.15 The Ideal Transformer

We now consider the case of an ideal transformer, where two coils are connected by a loop of ferromagnet, as shown. We assume that all the flux is confined to the ferromagnet, i.e., that there is perfect coupling between the coils.

Here it is more convenient to use the total number of turns in a coil N, rather than the number per unit length n.



No flux losses

$$\Rightarrow k = 1. \tag{5.56}$$

The flux in the core is Φ, generated by either coil.

- We assume that there are no energy losses in the wires and no hysteresis. So the coil 'former' (or core) is, say, soft iron, with high μ; the field is not high enough to saturate the iron.
- The flux of each turn of coil #1 links each turn of coil #2.

$$\begin{aligned} \Phi_1 &= N_1\Phi \ ; \quad \Phi_2 = N_2\Phi \\ \Phi &= \text{flux linkage per turn} \\ V_1 &= -\frac{\partial \Phi_1}{\partial t} = -N_1\frac{\partial \Phi}{\partial t} \\ V_2 &= -\frac{\partial \Phi_2}{\partial t} = -N_2\frac{\partial \Phi}{\partial t} \end{aligned} \tag{5.57}$$

- The two coils behave as a transformer:

$$\boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}} \tag{5.58}$$

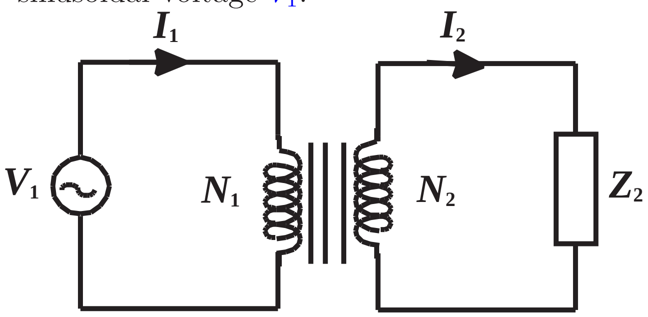
- The self-inductance $L = \mu_0(N/l)^2 Sl$ for a coil of length l and cross-sectional area S with N turns in total.

$$\Rightarrow \frac{L_1}{L_2} = \left(\frac{N_1}{N_2}\right)^2 \frac{l_2}{l_1}. \tag{5.59}$$

- This analysis assumes that the voltages generate a changing flux. Thus transformers are most useful for oscillating voltages, though they can also be used to give a large voltage spike when a circuit carrying a current is broken suddenly, e.g., to cause a spark in a car's sparkplugs.
- For non-ideal transformers, losses may arise from eddy currents in the core, resistive heating in the coils, hysteresis losses, and flux leakage.

5.16 Transformer — Example

- Consider a perfect transformer with coil #1 driven by a voltage generator giving a sinusoidal voltage V_1.



The coils are wound in the same direction, but the current I_1 goes into coil #1 at the same end that I_2 comes out of coil #2, so I_2 gives a flux through coil #1 in the opposite direction to the flux from I_1

$$\Rightarrow \Phi_1 = LI_1 - MI_2. \tag{5.60}$$

- Coil #2 is connected to a load impedance Z_2.

$$\begin{aligned} &\text{(primary)} & V_1 &= L_1\frac{\partial I_1}{\partial t} - M\frac{\partial I_2}{\partial t} \\ &\text{(secondary)} & V_2 = 0 &= L_2\frac{\partial I_2}{\partial t} - M\frac{\partial I_1}{\partial t} + I_2Z_2 \end{aligned} \tag{5.61}$$

- Look for a time dependence like $e^{j\omega t}$:

$$\Rightarrow \frac{\partial V_1}{\partial t} = j\omega V_1 \ ; \ \frac{\partial I_1}{\partial t} = j\omega I_1 \ ; \ \text{etc.} \tag{5.62}$$

- Rearranging:

$$V_2 = 0 \Rightarrow \quad (j\omega L_2 + Z_2)I_2 = j\omega M I_1 \tag{5.63}$$

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 = j\omega L_1 I_1 - \frac{(j\omega M)^2 I_1}{Z_2 + j\omega L_2} \tag{5.64}$$

$$\Rightarrow \frac{V_1}{I_1} = j\omega L_1 - \frac{(j\omega M)^2}{Z_2 + j\omega L_2} \tag{5.65}$$

- Assume perfect coupling, and that the coils have the same length.
  $\Rightarrow M^2 = L_1L_2$ and $\frac{L_1}{L_2} = \left(\frac{N_1}{N_2}\right)^2$

$$\Rightarrow \frac{V_1}{I_1} = \frac{j\omega L_1 \ Z_2(N_1/N_2)^2}{j\omega L_1 + Z_2(N_1/N_2)^2}. \tag{5.66}$$

- Also, we find $I_2Z_2 = V_1\frac{N_2}{N_1}$, as expected.

- The impedance of the 'input' to the transformer, $Z_i \equiv V_1/I_1$ is the same as that for impedances $Z_2(N_1/N_2)^2$ and $j\omega L_1$ in parallel.

  $\Rightarrow$ The transformer 'looks like' an inductance L_1 in parallel with a 'reflected impedance' $Z_2(N_1/N_2)^2$.

- The transformer can match a low-impedance device with a high-impedance device if the turns ratio is chosen appropriately.

- Usually $\omega L_1 \gg Z_2(N_1/N_2)^2$, so
$$Z_\mathrm{i} \approx Z_2(N_1/N_2)^2. \tag{5.67}$$
So if the load Z_2 is resistive (Z_2 real), then Z_i is 'mainly' real, and the voltages across the two coils oscillate in phase.

5.17 Magnetic Energy

We wish to obtain a general expression for the magnetic energy density.

- For a current I flowing in an inductance L, the stored energy $W = \frac{1}{2}LI^2$.

  Now, $L = \Phi/I$, $\quad \Phi =$ flux through circuit.

  $\Rightarrow W = \frac{1}{2}\Phi I$.

- Now consider two circuits:
$$\begin{aligned} W &= \tfrac{1}{2}L_1 I_1^2 + M_{12}I_1 I_2 + \tfrac{1}{2}L_2 I_2^2 \\ &= \tfrac{1}{2}(L_1 I_1 + M_{12}I_2)I_1 + \tfrac{1}{2}(L_2 I_2 + M_{12}I_1)I_2 \\ &= \tfrac{1}{2}I_1\Phi_1 + \tfrac{1}{2}I_2\Phi_2 \end{aligned} \tag{5.68}$$
since the flux through the first circuit $\Phi_1 = L_1 I_1 + M_{12}I_2$, *etc.*

- If there are many circuits $\{i\}$ (or current filaments within each wire), $W = \sum_i \frac{1}{2}\Phi_i I_i$.

- But $\Phi = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}$ and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential,

  $\Rightarrow \Phi = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}$

  $\Rightarrow W = \frac{1}{2}\sum_i \left(\oint \boldsymbol{A}\cdot(I\,\mathrm{d}\boldsymbol{l})\right)_i$

- Go to the distributed limit, $I\mathrm{d}\boldsymbol{l} \to \boldsymbol{J}\mathrm{d}\tau$, and combine the loop integrals and the sum into one integral:
$$\Rightarrow W = \frac{1}{2}\int \mathrm{d}\tau\, \boldsymbol{A}\cdot\boldsymbol{J} \tag{5.69}$$
where the integral is over a large volume enclosing all the current. W is the total magnetic energy in the volume.

- Ampère/Maxwell $\Rightarrow \boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}$ $\quad$ (for now…)
$$\Rightarrow W = \frac{1}{2}\int \mathrm{d}\tau\,\boldsymbol{A}\cdot\boldsymbol{J} = \frac{1}{2}\int \mathrm{d}\tau\ \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{H}.$$

- To prove the following handy theorem, note that the $\boldsymbol{\nabla}$ acts on both $\boldsymbol{A}$ and $\boldsymbol{H}$, generating two terms (Leibniz's product rule); then use rearrangements of the scalar triple product:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{H}) = \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}). \tag{5.70}$$

- Use this theorem to form the divergence, and convert to a surface integral:

$$\begin{aligned} W &= -\tfrac{1}{2}\int \mathrm{d}\tau\ \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{H}) + \tfrac{1}{2}\int \mathrm{d}\tau\ \boldsymbol{H}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} \\ &= -\tfrac{1}{2}\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{A}\times\boldsymbol{H} + \tfrac{1}{2}\int \mathrm{d}\tau\ \boldsymbol{H}\cdot\boldsymbol{B} \end{aligned} \tag{5.71}$$

- Take the integral over a large surface of radius R:
  $\mathrm{d}\boldsymbol{S} \propto R^2$; $\boldsymbol{A} \propto R^{-1}$; $\boldsymbol{H} \propto R^{-2}$;
  $\Rightarrow$ surface integral $\rightarrow 0$ as $R \rightarrow \infty$.

$$\Rightarrow \boxed{W = \int \mathrm{d}\tau \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}} \tag{5.72}$$

$$\Rightarrow \boxed{\text{Magnetic energy density:} \qquad U_{\mathrm{M}} = \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}} \tag{5.73}$$

5.18 Time-varying Electric Fields

- The electromagnetic equations so far are NOT CONSISTENT with *charge conservation*:

$$\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{J} + \int \mathrm{d}\tau \frac{\partial\rho}{\partial t} = 0 \tag{5.74}$$

$$\Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{(since true for all volumes)} \tag{5.75}$$

  This is the **equation of continuity**, which we derived earlier.

- But we have

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J} \qquad \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \quad (!) \tag{5.76}$$

  ($\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\ldots$ is always zero)

- So we must add **Maxwell's Displacement Current** $\dot{\boldsymbol{D}}$ ($\boldsymbol{D}$ is called the electric displacement)

$$\text{(Maxwell 4)} \qquad \boxed{\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}} \tag{5.77}$$

  This is Maxwell's greatest contribution. It may not seem very important yet, but when this equation is combined with the other equations, very important things happen—waves.

- Take the divergence:

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{J} + \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{D} \tag{5.78}$$

- But $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho$ (Maxwell 1), so everything is now OK.

5.19 Maxwell's Equations in Full

$$\begin{array}{rcl} \boldsymbol{\nabla}\cdot\boldsymbol{D} & = & \rho_{\text{free}} \\ \boldsymbol{\nabla}\times\boldsymbol{E} & = & -\dot{\boldsymbol{B}} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} & = & 0 \\ \boldsymbol{\nabla}\times\boldsymbol{H} & = & \boldsymbol{J}_{\text{free}} + \dot{\boldsymbol{D}} \end{array} \tag{5.79}$$

[The dot, of course,means the partial derivative w.r.t. time.]

Describing Maxwell's equations, Heinrich Hertz once said: "One cannot escape the feeling that...they have an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them."

Einstein (around 1905) was inspired by the invariance of the speed of light in Maxwell's equations and so combined them with the principle of relativity to derive $E = mc^2$.

Dirac (in 1927) applied quantum theory to Maxwell's equations and went on to formulate his equation for the electron and lay down the foundations of quantum electrodynamics.

Maxwell's equations recently topped a poll to find the greatest equations of all time in Physics World (October 2004). They narrowly beat "$1 + 1 = 2$"!

- There are also constitutive relationships for media:

$(\boldsymbol{D}, \boldsymbol{E})$	permittivity
$(\boldsymbol{B}, \boldsymbol{H})$	permeability
$(\boldsymbol{J}, \boldsymbol{E})$	conductivity

- Other useful equations:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \qquad \text{Lorentz force} \tag{5.80}$$

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}; \quad \boldsymbol{E} = -\boldsymbol{\nabla}V - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{(next year)} \tag{5.81}$$