Farey Fractions

Definition

The *set of Farey fractions of order n*, denoted by F_n, is the set of reduced fractions in the closed interval $[0, 1]$ with denominators not exceeding n listed in increasing order.

Examples

$$F_1 \quad \frac{0}{1} \quad \frac{1}{1}$$

$$F_2 \quad \frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}$$

$$F_3 \quad \frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}$$

$$F_4 \quad \frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$$

$$F_5 \quad \frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}$$

$$F_6 \quad \frac{0}{1} \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad \frac{1}{1}$$

$$F_7 \quad \frac{0}{1} \quad \frac{1}{7} \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{2}{7} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{3}{7} \quad \frac{1}{2} \quad \frac{4}{7} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{5}{7} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad \frac{6}{7} \quad \frac{1}{1}$$

Remark

These examples illustrate some general properties of Farey fractions. For example, $F_n \subset F_{n+1}$, so we get F_{n+1} by inserting new fractions in F_n. If $\frac{a}{b} < \frac{c}{d}$ are consecutive in F_n and separated in F_{n+1}, then the fraction $\frac{a+c}{b+d}$ does the separating, and no new ones are inserted between $\frac{a}{b}$ and $\frac{c}{d}$.

This new fraction is called the *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$.

Theorem 1

If $\frac{a}{b} < \frac{c}{d}$, *their mediant* $\frac{a+c}{b+d}$ *lies between them.*

PROOF

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc - ad}{b(b+d)} > 0 \quad \text{and} \quad \frac{c}{d} - \frac{a+c}{b+d} = \frac{bc - ad}{d(b+d)} > 0 \ .$$

Theorem 2

Let $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ *with* $bc - ad = 1$.

Then $\frac{a}{b}$ *and* $\frac{c}{d}$ *are consecutive terms in* F_n *for the following values of* n *:*

$$\max\{b, d\} \le n \le b + d - 1 .$$

PROOF

The condition $bc - ad = 1$ implies that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are reduced.
If $\max\{b, d\} \le n$ then $b \le n$ and $d \le n$ so $\frac{a}{b}$ and $\frac{c}{d}$ are certainly in F_n.

Now we prove that they are consecutive if $n \le b + d - 1$.

If they are not consecutive, then there is another fraction $\frac{h}{k}$ between them, that is $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$.

Since

$$k = k(bc - ad) = b(ck - dh) + d(bh - ak) ,$$

and

$$\frac{a}{b} < \frac{h}{k} < \frac{c}{d} \quad \text{implies that} \quad ck - dh \ge 1 \text{ and } bh - ak \ge 1 ,$$

then

$$k = b(ck - dh) + d(bh - ak) \ge b + d .$$

Therefore, if $n \le b + d - 1$, then $\frac{a}{b}$ and $\frac{c}{d}$ must be consecutive in F_n.

Theorem 3

Let $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ *with* $bc - ad = 1$.

If $\frac{h}{k}$ *is the mediant of* $\frac{a}{b}$ *and* $\frac{c}{d}$, *then* $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$ *and* $bh - ak = 1$, $ck - dh = 1$.

PROOF

Since $\frac{h}{k}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$, then $ck - dh \ge 1$ and $bh - ak \ge 1$. Thus

$$k = b(ck - dh) + d(bh - ak)$$

shows that $k = b + d$ if and only if $ck - dh = 1$, $bh - ak = 1$.

Theorem 4

The set F_{n+1} *includes* F_n.

Each fraction in F_{n+1} *which is not in* F_n *is the mediant of a pair of consecutive fractions in* F_n.

Moreover, if $\frac{a}{b} < \frac{c}{d}$ *are consecutive in any* F_n, *then they satisfy the* <u>*unimodular relation*</u> $bc - ad = 1$.

PROOF

Use induction on *n*.

When $n = 1$, the fractions $\frac{0}{1}$ and $\frac{1}{1}$ are consecutive and satisfy the unimodular relation.
We pass from F_1 to F_2 by inserting $\frac{1}{2}$.

Now suppose $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in F_n and satisfy the unimodular relation $bc - ad = 1$.

Then by Theorem 2, they will be consecutive in F_m for all *m* satisfying

$$\max\{b, d\} \le m \le b + d - 1 \ .$$

Form their mediant $\frac{h}{k}$ where $h = a + c$, $k = b + d$.

By Theorem 3, $bh - ak = 1$, $ck - dh = 1$, so *h* and *k* are relatively prime.

The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in F_m for all *m* satisfying $\max\{b, d\} \le m \le b + d - 1$, but are not consecutive in F_k since $k = b + d$ and $\frac{h}{k}$ lies in F_k between $\frac{a}{b}$ and $\frac{c}{d}$.

But the two new pairs $\frac{a}{b} < \frac{h}{k}$ and $\frac{h}{k} < \frac{c}{d}$ are now consecutive in F_k because

$$k = \max\{b, k\} \text{ and } k = \max\{d, k\}.$$

The two new pairs still satisfy the unimodular relations $bh - ak = 1$, $ck - dh = 1$.

This shows that in passing from F_n to F_{n+1} every new fraction inserted must be the mediant of a consecutive pair in F_n and the new consecutive pairs satisfy the unimodular relations. Therefore F_{n+1} has these properties if F_n does.

Theorem 5

If $\frac{a}{b}$ *and* $\frac{c}{d}$ *are consecutive in* F_n, *then*

$$\left| \frac{a}{b} - \frac{a+c}{b+d} \right| = \frac{1}{b(b+d)} \le \frac{1}{b(n+1)}$$

and

$$\left| \frac{c}{d} - \frac{a+c}{b+d} \right| = \frac{1}{d(b+d)} \le \frac{1}{d(n+1)} .$$

Theorem 6

Given any real number θ *and any positive integer N, there exist rational numbers* $\frac{a}{b}$ *such that*

$$\left| \theta - \frac{a}{b} \right| < \frac{1}{b(N+1)} \quad \text{with} \quad 0 < b \le N .$$

Theorem 7

Given any irrational θ, *there exist infinitely many rational numbers* $\frac{a}{b}$ *such that*

$$\left| \theta - \frac{a}{b} \right| < \frac{1}{b^2} .$$

Theorem 8

If m and n are positive integers then <u>*not both*</u> *of the inequalities* <u>*can hold*</u> :

$$\frac{1}{mn} \ge \frac{1}{\sqrt{5}} \left(\frac{1}{m^2} + \frac{1}{n^2} \right)$$

and

$$\frac{1}{m(m+n)} \ge \frac{1}{\sqrt{5}} \left(\frac{1}{m^2} + \frac{1}{(m+n)^2} \right) .$$

Theorem 9 Hurwitz

Given any irrational θ, *there are infinitely many rational numbers* $\frac{a}{b}$ *such that*

$$\left| \theta - \frac{a}{b} \right| < \frac{1}{\sqrt{5}\, a^2} .$$

Moreover the result is false if $\frac{1}{\sqrt{5}}$ *is replaced by any smaller constant.*