Correspondence of Nicolas Bernoulli concerning the St. Petersburg Game

1.
Nicolas Bernoulli to Montmort.
Basel, 9 September, 1713.
Printed in *Essay d'Analysis,* p. 402

… Fourth Problem. A promises to give a coin to B, if with an ordinary die he achieves 6 points on the first throw, two coins if he achieves 6 on the second throw, 3 coins if he achieves this point on the third throw, 4 coins if he achieves it on the fourth and thus it follows; one asks what is the expectation of B?

Fifth Problem. One asks the same thing if A promises to B to give him some coins in this progression 1, 2, 4, 8, 16 etc. or 1, 3, 9, 27 etc. or 1, 4, 9, 16, 25 etc. or 1, 8, 27, 64 in stead of 1, 2, 3, 4, 5 etc. as beforehand. Although for the most part these problems are not difficult, you will find however something most curious.

2.
Montmort to Nicolas Bernoulli.
(Paris) 15 November 1713.
Printed in *Essay d'Analysis,* p. 407

The two last of your five problems have no difficulty, it is only agreed to find the sum of the series of which the numerators are in the progression of squares, cubes, etc. the denominators in geometric progression: the late Mr. your uncle has given the method to find the sum of these series.

3.
Nicolas Bernoulli to Montmort.
(Basel) 20 February 1714.
pp. 210, 211

… It is true what you say that the last two of my problems have no difficulty, nevertheless you would have done well to find the solution, because it will have furnished you the occasion to make a curious remark. Let the expectation of B be called x in the case of the fourth problem, you will have $x = (1 + 5y)/6$ (I name y the expectation of B after he has missed the six on the first throw); now y is necessarily $x + 1$; because after he has missed the six on the first throw, he expects to receive some coins in this progression 2, 3, 4, 5, 6, of which each term is one unit greater than the corresponding term of that here 1, 2, 3, 4. Substitute therefore $x + 1$ in place of y, and you will have $x = (5x + 6)/6$, and therefore $x = 6$. This which you will find also by the route of infinite series.

But if you follow the same analysis in the examples of the 5th problem as in the example of this progression 1, 2, 4, 8, etc, where you will have $y = 2x$, you will find $x = (1 + 10x)/6 = -1/4$, which is a contradiction. In order to respond to this contradiction, one might say that this fraction, regarded as having a negative denominator and consequently smaller than zero, is greater than $1/0$, and that therefore the expectation of B is more than infinity, which one find also effectively by the method of infinite series. But it will follow from here that B must give to A an infinite sum and likewise more than infinity (if it is permitted to speak thus) in order that he be able to make the advantage to give him some coins in this progression 1, 2, 4, 8, 16 etc. Now it is certain that B in given some sum by right always, since it is morally impossible that B not achieve six in a finite number of throws.

From all this I conclude that the just value of a certain expectation is not always the average that one finds by dividing by the sum of all the possible cases the sum of the products of each expectation by the number of the case which gives it; that which is against our basic rule. The reason for this is that the case which will be a very small probability must be neglected and deemed for null, although they can give a great expectation. For this reason one is able to again doubt if the value of the expectation of B in the case of the 4th problem such as I have found above, it is not too great. Similarly in lotteries where there are one or two great lots, the just value of a single ticket is smaller than the sum of all the money of the lottery divided by the sum of all the tickets, supposing that the number of those that be also very great. This is a remark which merits to be well exhausted.

4.
Montmort to Nicolas Bernoulli.
Paris, 24 March 1714.
Letter 188, p. 2.

…It is very true that in the case of the 4th problem $x = 6$, when Pierre gives to Paul 1, 2, 3, 4, 5, 6, etc coins, according that Paul achieve a 6 either on the 1st or 2nd or 3rd etc throw, but I am not able to believe that in the case that one gives to Paul some coins according to this sequence 1, 2, 4, 8, 16, 32 etc the advantage of Paul be infinite or similarly more than infinity, as similarly marks the fraction $1/-4$. which is well found. I myself am not able to resolve to abandon our lemma, which must be generally true. You must have reason to say that all this merits to be investigated and it is sure without compliment that no person be more capable than you.

5.
Montmort to Nicolas Bernoulli.
Paris, 22 March 1715.
Letter 194, p. 31.

…I would have again to say to you on the two problems that you have proposed page 402. I have thrown well some things into writing, there is enough to make under 16 pages. I have not the strength to begin such a great task, this will be for another time, because it is necessary to leave behind nothing and to end, if it is possible, all our algebraic or philosophical disputes.

6.
Nicolas Bernoulli to Montmort.
Basel, 31 March 1716.
p. 236.

…But regarding this fifth problem I beg you to communicate to me your reflections on the last two of these problems. You have promised to leave nothing behind and to eliminate all our differences.

7.
Montmort to Nicolas Bernoulli.
Monmort, 28 December 1716.
Letter 199, p. 11.

…I will owe you word on all these 5 problems, but I ask you some time, because I am absent-minded and lazy.

8.
Cramer to Nicolas Bernoulli.
London, 21 May 1728.
Letter 52, pp. 2,3.

…I know not if I deceive myself, but I believe to hold the solution of the singular case that you have proposed to Mr. de Montmort in your letter of 9 September 1713, Prob. 5, page 402. In order to render the case more simple I will suppose that A throw in the air a piece of money, B undertakes to give him a coin, if the side of the Heads falls on the first toss, 2, if it is only the second, 4, if it is the 3rd toss, 8, if it is the 4th toss, etc. The paradox consists in this that the calculation gives for the equivalent that A must give to B an infinite sum, which would seem absurd, since no person of good sense, would wish to give 20 coins. One asks the reason for the difference between the mathematical calculation and the vulgar estimate. I believe that it comes from this that the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it. This which renders the mathematical expectation infinite, this is the prodigious sum that I am able to receive, if Heads falls only very late, the 100th or 1000th toss. Now this sum, if I reason as a sensible man, is not more for me, does not make for more pleasure for me, does not engage me more to accept the game, than if it would be only 10 or 20 million coins. Suppose therefore that the total sum beyond 20 millions or (for more ease) beyond $2^{24} = 16777216$ coins, is equal to him or rather that I am never able to receive more than 2^{24} coins, however late comes Heads. And my expectation will be

$$\frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 4 + \text{\&c}\cdots + \frac{1}{2^{25}} 2^{24} + \frac{1}{2^{26}} 2^{24} + \frac{1}{2^{27}} 2^{24} + \text{\&c.}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\, 24 \text{ terms} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \text{ \&c.} = 12 + 1 = 13.$$

Therefore *speaking morally* my expectation is reduced to 13 coins, and my equivalent to as much, which would seem much more reasonable than the made infinite. One will be able to find again a smaller by making some other assumption of the *Moral Value* of the riches. Because what I come to make is not exactly just, since it will be true that 100 millions are more pleasure than 10 millions, although not making 10 times more.

P. S. If one wishes to suppose that the moral value of goods were as the square root of the mathematical quantities, C. to D. as the pleasure that $40\,000\,000$ makes me was double the pleasure $10\ 000\ 000$. So my *moral expectation* will be

$$\frac{1}{2}\sqrt{1} + \frac{1}{4}\sqrt{2} + \frac{1}{8}\sqrt{4} + \frac{1}{16}\sqrt{8} + \text{\&c.} = \frac{1}{2} \times \overline{\frac{1}{1}\sqrt{1} + \frac{1}{2}\sqrt{2} + \frac{1}{4}\sqrt{4} + \frac{1}{8}\sqrt{8} + \text{\&c.}}$$

$$= \frac{1}{2} \times \overline{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{8}} + \text{\&c}} = \frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{2} - 1}.$$

But this quantity is not equivalent. Because the equivalent must be, not equal to my expectation, but such that the sorrow of its loss be equal to the *moral expectation* of the pleasure that I expect to receive in winning. Therefore the equivalent must be (by assumption)

$$\left(\frac{1 \cdot \sqrt{2}}{2(\sqrt{2} - 1)} \right)^2 = \frac{2}{12 - 8\sqrt{2}} = 2,9\ldots = \text{less than } 3,$$

that which is very mediocre and that I believe therefore to approach nearer the vulgar estimate than 13. You will see, Sir, that the assumption is not just. I believe likewise impossible to make of it any one very reasonable. But this suffices nonetheless, in my opinion, in order to make see, by the same calculation, that a sensible man must not give an infinite equivalent. One would be able to draw from there well some corollaries that I omit for fear of boring you.

9.
Nicolas Bernoulli to Cramer.
Basel, 3 July 1728.
pp. 23, 24.

…The response that you give for the solution of the singular case proposed to Mr. de Montmort page 402, Prob. 5 satisfies only part of it; it suffices, as you say, to make see that A must not give to B an infinite equivalent; but it does not demonstrate the true reason for the difference that there is between the mathematical expectation and vulgar estimate; for example in the case of Heads and Tails there is no person of good sense who wished to give 20 coins, not for this reason that the use or the pleasure that one is able to draw from an infinite sum is barely greater than the one which can be taken of a sum of 10, or 20, or 100 millions, but because in giving for example 20 coins one has a very small probability to win something, and that one believes the loss morally certain. The vulgar neither stakes here in the storyline nor of millions, nor of hundreds of coins paying no attention at all to this that the terms of the geometric progression 1, 2, 4, 8, 16, &c becoming fairly great they are able to be considered equal, he is enlisted through this neither to accept nor to refuse the game, it is determined solely by the degree of probability that he has to win or lose; to him a very small probability to win a great sum does not counterbalance a very great probability to lose a small sum, he regards the event of the first case as impossible, and the event of the second as certain. It is necessary therefore, in order to settle the equivalent justly, to determine as far as where the quantity of one probability must diminish, so that it be able to be deemed null; but here is that which is impossible to determine, any assumption that one makes, one encounters always difficulties; the limits of these small probabilities are not precise, but they have a certain latitude what one not be able to fix easily; a probability which for example has $1/100$ certitude must not be reputed null instead that which has $1/99$ certitude. It seems to me therefore that in admitting this assumption that a man of good sense not wish to give 20 coins, because he holds for certain that the sum which will fall to him will be less than 20 coins, one is able to find the equivalent sought by the following reasoning: by hypothesis it is morally impossible that he obtain 20 coins; it will be therefore also morally impossible that he obtain 32 coins or some other number of coins in this progression 32, 64, 128, &c; or the probability to obtain a number of this progression is $1/64 + 1/128 + 1/256 + \text{\&c.} = 1/32$; therefore this man of good sense reputes a probability which does not surpass $1/32$ as null, and a probability which has $31/32$ as a total certitude, consequently his expectation will be worth by the rule $1/2 \cdot 1 + 1/4 \cdot 2 + 1/8 \cdot 4 + 1/16 \cdot 8 + 0 \cdot 16 + 0 \cdot 32 + \text{\&c.} = 2$. But I do not know if this other reasoning will be more just: A man who does not wish to give 20 coins estimates all the cases which give to him a lesser sum than 20 coins possible, and each of the others, which are able to give a greater sum, impossible; he regards therefore only the probabilities less than $1/32$ as null, consequently his expectation will be worth

$$1/2 \cdot 1 + 1/4 \cdot 2 + 1/8 \cdot 4 + 1/16 \cdot 8 + 0 \cdot 16^1 + 0 \cdot 32 + \text{\&c.} = 2.$$

There will be well again some things to say on this matter, but not having the leisure to arrange in order or to develop the ideas which are presented to my spirit, I pass over them in silence.

10.
Nicolas Bernoulli to Daniel Bernoulli.
Basel, 27 October 1728.
p. 14.

…I would wish well to know your sentiment on the 2nd and 5th Problems of those that I have formerly proposed to the late Mr. de Montmort, see the l'Analysis sur les Jeux de hazard page 402, particularly on the last, of which Mr. the Professor Cramer of Geneva has communicated to me a solution, which does not satisfy me entirely; in order to render the case more simple, one is able to suppose, that A throws into the air a piece of money, B engages to give to him a coin if Heads on the first throw; 2, if it is only the second; 4, if it is the 3rd throw; 8, if it is the 4th throw, etc. He agrees to find what must be the equivalent that a must give to B; the calculation gives an infinite sum, that which is absurd, because there is no person of good sense who wished to give 20 coins only.

11.
Cramer to Nicolas Bernoulli.
Leiden, 27 September 1728.
Letter 53, p. 3. Response to No. 9.

…In my solution of the singular case proposed to Mr. de Montmort page 402 I have not pretended to guess what is the reason which urges a man to not wish to grant an infinite equivalent. I have only wished to seek a reason for me to persuade that I must not give an infinite equivalent. Now I believe that this that I give to him must make an impression on a sensible man, likewise when he would consider an expectation very small as some thing that which he must make, although in the remainder it is quite true that the greatest part not make it.

[1] The original reads $1/32 \cdot 16$.

I have moreover the advantage that in my solution, if the case is offered to me, I am able to know how far I am able to give the equivalent, not having for the one that to me to consult to know to what degree of richness it will be very easy to arrive, in stead that I will be able never to fix to what point a probability becomes zero or certitude.

12.
Daniel Bernoulli to Nicolas Bernoulli.
Petersburg, 5 (15) November 1728

… My father will have said to you that I have marked him touching your 5 problems, which are in the book of Mr. de Montmort on the games of chance pp. 401 and 402; I have not put in writing the calculations that I have made thereon, … The 4th and the 5th problems are easy, although their solutions be a little paradox. In the case of the geometric progression 1, 2, 4, etc. the paradox is found on the small probability that there is that the game will last more than 20 or 30 throws.

13.
Nicolas Bernoulli to Daniel Bernoulli.
Basel, 5 February 1729.
p. 14.

… The small probability, which there is, that the games of the last two problems last 20 or 30 throws, forming not only a paradox, but a difficulty that you have not raised; is it just that in the 5th one give an infinite sum for equivalent? I believe not.

14.
Daniel Bernoulli to Johann Bernoulli.
Petersburg, Late autumn 1729.
Lost.

15.
Nicholas Bernoulli to Daniel Bernoulli.
Basel, 4 February 1730.
p. 14 i.

I have seen there is some month that you have written to Mr. your father… I have already responded to you on this that Mr. your father has communicated to me in regard to the 5 problems (on the art of conjecture) in question… As for the last 2 of the 5 problems I do not agree with you that one is able to resolve them in estimating the riches of those with which one is able to play. Suppose that A be obliged to play one time with B in the conditions of the 5th problem, I ask this that it is necessary to make *hic et nunc*, if A would wish to disengage from his obligation without playing, what sum must he pay to B?

16.
Daniel Bernoulli to Nicolas Bernoulli.
Petersburg, January 1731.
Letter 2, p.1.

There is near to a year that I have received your last without that I know myself the cause which was able to delay so much this response… I would know nothing to say to you on the problem of probabilities, because I no longer have the book, where they are recorded… The last two problems are so vague that I have no more to say to you, if you do not believe that it is necessary to know the sum that the other is in position to pay. This method would seem to me therefore again very just, and the reason for which the solution would appear a little paradox, is that a person would not wish to play against another who wished to wager an infinite sum in a game, where there be an infinitely small probability for him. I will be therefore very pleased to know your sentiment on this.

17.
Daniel Bernoulli to Nicolas Bernoulli.

Petersburg, 4 July 1731.

Letter 3, p.1

... As regards problem 5 in that which Mr. de Montmort has inserted in his book, I believe almost to have divined your thought: I have set down my thoughts in a rough draft in a part that I take the liberty to send you as much as it is. I will be very pleased to know your sentiments on this. In every case I believe that my reflections merit some attention, because they agree well with all that a certain instinct naturally inspires, to all the world, I agree therefore that although my reasoning be all done geometrically, one must not therefore take the thing rigorously, the hypothesis is not able to be assumed true geometrically without the particular cases that I consider. If the Mr. Bernoullis who have lost all to the bankruptcy of Mr. Muller would have paid good attention to the same principle that I propose to make, they would not have perhaps lost all.

18.
Nicolas Bernoulli to Daniel Bernoulli.

Basel, 5 April 1732.

p. 14

... I thank you for the effort that you yourself have given to me communicating a copy of your *Specimen theoriae novae metiendi sortem pecuniariam*; I have read it with pleasure, and I have found your theory most ingenious, but permit me to say to you that it does not solve the knot of the problem in question. There is not agreed to measure the use or the pleasure that one derives from a sum that one wins, nor the lack of use or the sorrow that one has by the loss of a sum; there is agreed no longer to seek an equivalent between the things there; but there is agreed to find how a player is obliged in justice or in equity to give to another for the advantage that therein accords him in the game of chance in question, or in other games in general, so that the game is able to be deemed fair, as for example a game is considered fair, when the two players bet an equal sum on a game under equal conditions, although in your theory, and in paying attention to their riches, the pleasure or the advantage of gain in the favorable case is not equal to the sorrow or the disadvantage that one suffers in the contrary case. Mr. Cramer has also tried to resolve the problem by reflecting on use or on pleasure that men are able to derive from money, but without paying attention to the sum of goods that one already possesses. Here is that which he has written to me in 1728 on this matter:

(Here follows an extract from Letter 8 with the exception of the last sentence.)

I have indicated to him next that it would seem to me that in admitting this assumption, that a man of good sense is not willing to give 20 coins, because he estimates all the cases which give him a lesser sum than 20 coins possibles, and each of the others, which are able to give a greater sum, impossible; that in admitting, I say, this assumption, one is able to evaluate his expectation

$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 4+\frac{1}{16}\cdot 8+\frac{1}{32}\cdot 16+0\cdot 32+\&\mathrm{c}=2\frac{1}{2}.$$

I claim that this reasoning is not too exact, but I believe that in matching together your idea and that of Mr. Cramer and my own on that it is necessary to estimate a small probability as null, one is able to determine exactly the sought equivalent. You have been able, it seems to me, to spare the remark you have made, that the Mr. Bernoullis would not have had perhaps so much loss in the bankruptcy of Muller, if they would have paid attention to the principles that you pose in your theory. These principles show only that one risks to permit a greater sorrow in placing a great sum with a single debtor, than in placing the same sum in parts among many debtors; but it does not show that one risks also to make a greater loss. A thousand coins placed with a dealer of which there is a $1/10$ probability that he will make bankruptcy is worth 900 coins, and 2 times 500 coins placed with two dealers equally subject to make bankruptcy worth as such. It is very true and we know it without paying attention to your principle, that one does nonetheless better to place 500 coins in 2 places, than 1000 coins in a single place, because one is not exposed to losing as easily all 1000 coins in the 1st case as in the 2nd, the first case having $1/100$ probability, in stead of the second case has $1/10$. *One must not put too many eggs in one basket*, says our Bâlois. But as you would make if you, have needed to make worth our money in crediting it to a merchant, and if you do not have the occasion to place it by small parts?