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A MATHEMATICAL EXAMINATION OF THE METHODS OF DETERMINING THE ACCURACY OF AN OBSERVATION BY THE MEAN ERROR, AND BY THE MEAN SQUARE ERROR

Author's Note (CMS 2.757a) *

This early paper arose from an examination of a statement by A. S. Eddington in his book, *Stellar Movements*, page 147. It concerns the relative precision of two estimates of the variance of a normal distribution: (*a*) using Bessel's formula, based on the mean square; and (*b*) using Peter's formula, using the mean deviation.

The results include the exact sampling distribution of Bessel's estimate and the mean square of Peter's. The variance of the latter is, in large samples, the larger in the ratio $(\pi-2)$.

Next is considered the class of estimates based on powers of the deviation in general, showing that the precision is maximised when $p = 2$, the variance being 14 per cent greater for $p = 1$ and 9 per cent greater for $p = 3$. For continuous variation of p the precision of the mean square is a true maximum. These results had all been obtained before, without the knowledge of the author or of Eddington.

The most important point of the paper is the consideration of the simultaneous distribution of two estimates. This is examined in detail for the case of four observations, but the more general point is established that for a given value of σ_2 the distribution of σ_1 is independent of σ. Consequently when σ_2, the estimate based on the mean square, is known, a value of σ_1, the estimate based on the mean deviation, gives no additional information as to the true value. It is shown that the same proposition is true if any other estimate is substituted for σ_1, and consequently that the whole of the information respecting the variance which a sample provides is summed up in the single estimate σ_2. I believe this is the first occasion on which attention has been called to this property characteristic of a *sufficient* estimate.

Monthly Notices of the Royal Astronomical Society, 80: 758-770, (1920).

A Mathematical Examination of the Methods of determining the Accuracy of an Observation by the Mean Error, and by the Mean Square Error. By R. A. Fisher, M.A.

1. In estimating the precision of a number of observations two methods are in common use: that of the Mean Square Error, and that of the Mean Error. It is, I believe, usually admitted that the former has the firmer mathematical basis, although it is sometimes asserted that the latter is more accurate. It is not generally recognised that the merits of the two methods may be compared with precision. The case is of interest in itself, and it will be found that the method here outlined is illuminating in all similar cases, where the same quantity may be ascertained by more than one statistical formula.

Suppose the probability distribution of each observation to be centred about a true mean m, with normal distribution and standard deviation σ, so that the chance of any observation falling in the range dx is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

The unknown, σ, is to be determined from n observations, $x_1, x_2, x_3, \ldots, x_n$. Let

$$n\bar{x} = S(x),$$

$$n\sigma_1 = \sqrt{\frac{\pi}{2}} S(|x-\bar{x}|),$$

and

$$n\sigma_2^2 = S(x-\bar{x})^2,$$

then σ_1 is the value obtained by the method of Mean Error, σ_2 that obtained by the method of Mean Square Error, and σ the true value. Both σ_1 and σ_2 may be adjusted by means of appropriate functions of n so as to make the mean value of each of them obtained from a number of samples agree with the true value, but this for the moment is immaterial.

2. The distribution of σ_2.

I have described elsewhere ("Frequency Distribution of the Values of the Correlation Coefficient in Samples from an Indefinitely Large Population," *Biometrika*, **10**, 507) a method by which the frequency distribution of σ_2 may be established. If $x_1, x_2, \ldots, x_n$ are co-ordinates in generalised space of n dimensions, then any sample is represented by a single point having the observed values as co-ordinates. Let O be the origin and C the point at which every observed value is equal to m; then at any point along the line OC, produced indefinitely in both directions, all the co-ordinates are equal.

Let the point P represent the sample; from P draw PM perpendicular to OC, then it is easy to see that M is the point,

$$x_1 = x_2 = x_3 = \ldots x_n = \bar{x},$$

and that PM^2 is

$$S(x-\bar{x})^2 = n\sigma_2^2.$$

Now, since the chance of any observation falling in the range dx is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

the chance of the sample falling in the space $dx_1, dx_2, \ldots dx_n$ is

$$\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{S(x-m)^2}{2\sigma^2}} dx_1, dx_2, \ldots dx_n,$$

but

$$S(x-m)^2 = n(\bar{x}-m)^2 + n\sigma_2^2,$$

and the element of volume is evidently proportional to

$$d\bar{x} \cdot \sigma_2^{n-2} d\sigma_2,$$

so that the chance of a sample falling in the range $d\bar{x}\, d\sigma_2$ is proportional to

† $$e^{-\frac{n(\bar{x}-m)^2}{2\sigma^2}} dx \cdot \sigma_2^{n-2} \quad e^{-\frac{n\sigma_2^2}{2\sigma^2}} d\sigma_2.$$

The distributions of $\bar{x}$ and σ_2 are therefore perfectly independent; the chance of $\bar{x}$ falling in the range $d\bar{x}$ is

$$\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{x}-m)^2}{2\sigma^2}} d\bar{x},$$

and the chance of σ_2 falling in the range $d\sigma_2$ is

$$\frac{n^{\frac{1}{2}(n-1)}}{2^{\frac{1}{2}(n-3)} \cdot \left(\frac{n-3}{2}\right)!} * \cdot \frac{\sigma_2^{n-2} d\sigma_2}{\sigma^{n-1}} \cdot e^{-\frac{n\sigma_2^2}{2\sigma^2}}.$$

We may note at once that the mean value of σ_2 is

$$\sqrt{\frac{2}{n}} \cdot \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!} \sigma,$$

while the mean value of σ_2^2 is

$$\frac{n-1}{n} \sigma^2.$$

As n is increased the curve rapidly tends to the normal form; the mean is approximately

$$\left(1 - \frac{3}{4n}\right)\sigma,$$

* The symbol $x!$ is here used in a sense equivalent to $\Pi(x)$ or $\Gamma(x+1)$, whether x is an integer or not.

† For dx, read $d\bar{x}$

whence it is easy to see that the standard error of σ_2 is

$$\frac{\sigma}{\sqrt{2n}}.$$

If it were desired to bring the mean into coincidence with the true value σ, the value of σ_2 obtained should evidently be multiplied by

$$\sqrt{\frac{n}{2}}\frac{\left(\frac{n-3}{2}\right)!}{\left(\frac{n-2}{2}\right)!}.$$

This correction, if it be so considered, is never of importance; when n is large the value $\frac{3}{4n}$ is of higher order than the standard error. Even when $n=2$, and the correction amounts to an increase of 25 per cent., the standard error is much greater, being 75 per cent. of the mean.

3. Standard error of σ_1.

If the co-ordinates $x_1, x_2, \ldots, x_n$ are the deviations of individuals of a sample from its mean, the representative point lies in a plane space of $(n-1)$ dimensions, in which

$$S(x)=0.$$

The frequency density at any distance, r, from the origin is proportional to

$$e^{-\frac{r^2}{2\sigma^2}},$$

and

$$r^2=S(x^2).$$

The region in which any co-ordinate has an assigned value, x_1, is a plane space of $(n-2)$ dimensions, at a distance

$$x_1\sqrt{\frac{n}{n-1}}$$

from the origin, and the frequency with which x_1 falls into the range dx_1 is therefore proportional to

$$e^{-\frac{nx_1^2}{2(n-1)\sigma^2}}dx_1.$$

Thus deviations from the mean of samples of n of a normal population are themselves normally distributed. The deviations from the mean of the population are, however, independent, while deviations from the mean of the same sample are not. Consider the distribution of pairs of values, x_1 and x_2.

The space in which the representative points lie is parallel to the line

$$x_1+x_2=0$$
$$x_3=x_4=\ldots=x_n=0,$$

while it makes with the line

$$x_1 = x_2$$
$$x_3 = x_4 = \ . \ . \ . \ = x_n = 0$$

an angle the cosine of which is

$$\sqrt{\frac{n-2}{n}}.$$

Consequently the frequency in the range $dx_1 dx_2$ is proportional to

$$e^{-\frac{1}{2\sigma^2}\left\{\frac{(x_1-x_2)^2}{2}+\frac{n}{n-2}\cdot\frac{(x_1+x_2)^2}{2}\right\}}dx_1 dx_2$$
$$= e^{-\frac{n-1}{2(n-2)\sigma^2}\left\{x_1^2+\frac{2x_1x_2}{n-1}+x_2^2\right\}}dx_1 dx_2,$$

showing a surface of normal correlation, with correlation coefficient, $-\frac{1}{n-1}$, between any two deviations.

If the deviation be considered without reference to sign, each is distributed from 0 to ∞ with frequency,

$$\sqrt{\frac{2n}{\pi(n-1)}}e^{-\frac{nx^2}{2(n-1)\sigma^2}} \cdot \frac{dx}{\sigma},$$

and each pair with frequency,

$$\frac{2(n-1)}{\pi\sigma^2\sqrt{n(n-2)}}e^{-\frac{n-1}{2(n-2)\sigma^2}(x_1^2+x_2^2)}\cosh\frac{(x_1x_2)}{(n-2)\sigma^2}dx_1dx_2.$$

* The mean value of x is therefore

$$\sigma\sqrt{\frac{n-1}{n}} \cdot \sqrt{\frac{2}{\pi}},$$

so that the mean value of σ_1 is

$$\sigma\sqrt{\frac{n-1}{n}}$$

as is generally known.

The mean value of x^2 is

$$\frac{n-1}{n}\sigma^2,$$

and that of xy is

$$\frac{2\sigma^2}{\pi n}\left(\sqrt{n(n-2)}+\sin^{-1}\frac{1}{n-1}\right),$$

whence it follows that the mean value of $\sigma_1{}^2$ is

$$\frac{(n-1)\sigma^2}{n^2}\left(\frac{\pi}{2}+\sqrt{n(n-2)}+\sin^{-1}\frac{1}{n-1}\right).$$

* For x, read $|x|$.

As in the case of σ_2, the mean differs from the true value by a term of the order $\frac{1}{n}$. When n is large this is insignificant, for the mean value of $\sigma_1{}^2$ is approximately

$$\frac{n-1}{n}\sigma^2\left\{1+\left(\frac{\pi}{2}-1\right)\frac{1}{n}\right\};$$

hence the standard error is

$$\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\pi-2}{2}}.$$

As n is made large, therefore the standard error of σ_1 tends to bear a constant ratio to that of σ_2. The former is the larger in the ratio $\sqrt{\pi-2}$; in other words, the value of the standard deviation, or probable error, obtained from the Mean Square Deviation of a sample has greater weight by 14 per cent. than that obtained from the Mean Deviation. To obtain a result of equal accuracy by the latter method, the number of observations must be increased by 14 per cent.*

4. The derivate of minimum error.

The correlation between deviations, taken positive, being of order $\frac{1}{n^2}$, does not, as we have seen, affect the expression for the variance of a derivate, when n is large, because this is of order $\frac{1}{n}$. We may, then, in examining the comparative variance of different derivates ignore this correlation, and treat the deviations as though they were independent. The term "variance" is used here as elsewhere to signify the square of the standard deviation or standard error; by the "relative variance" is intended the same quantity divided by the square of the mean.

It is easy to verify that, although the variance is diminished as we pass from the derivate of the first power to that of the second, it is increased as we pass from the second power to the third. It is, therefore, of interest to determine for what power it is actually a minimum.

If $\sigma^p\mu_p$ is the mean value of x^p, then

$$\mu_p=\sqrt{\frac{2}{\pi}}\int_0^\infty t^p e^{-\frac{1}{2}t^2}dt=\left(\frac{p-1}{2}\right)!\,\frac{2^{\frac{1}{2}p}}{\sqrt{\pi}},$$

* Mr. Fisher kindly allows me to correct here an erroneous statement in my book, *Stellar Movements*, p. 147, footnote. I think it accords with the general experience of astronomers that, for the errors commonly occurring in practice, the mean error is a safer criterion of accuracy than the mean square error, especially if any doubtful observations have been rejected; but I was wrong in claiming a theoretical advantage for the mean error in the case of a truly Gaussian distribution. My formulæ were somewhat different from Mr. Fisher's, since I considered the deviations of σ_2 from σ instead of from $\overline{\sigma_2}$; but, as he points out, this correction (as I considered it) is of minor importance, and my mistake arose in the numerical evaluation of the results. —A. S. EDDINGTON.

if, then, σ_p is the derivate of the p^{th} power, we take as definition

$$\sigma_p{}^p = \frac{S(x^p)}{n\mu_p}.$$

It is well known that the variance of the p^{th} moment of a sample of n is

$$\frac{1}{n}\left(\mu_{2p} - \mu_p{}^2\right)$$

when n is large; whence it follows that the relative variance of $\sigma_p{}^p$ is

$$\frac{1}{n}\left(\frac{\mu_{2p}}{\mu_p{}^2} - 1\right),$$

and, therefore, that the relative variance of σ_p is

$$\frac{1}{np^2}\left(\frac{\mu_{2p}}{\mu_p{}^2} - 1\right) = \frac{1}{np^2}\left\{\frac{(p-\frac{1}{2})!}{(\frac{1}{2}p-\frac{1}{2})!^2}\sqrt{\pi} - 1\right\}.$$

Putting $p = 1$, 2, and 3, we have $\frac{\pi - 2}{2n}$, $\frac{1}{2n}$, and $\frac{15\pi - 8}{72n}$, being in the ratio 1·1416 : 1 : 1·0868.

If the relative variance is a minimum for variations of p, then must

$$\frac{2}{p}\left\{\frac{(p-\frac{1}{2})!}{(\frac{1}{2}p-\frac{1}{2})!^2}\sqrt{\pi} - 1\right\} = \frac{(p-\frac{1}{2})!\sqrt{\pi}}{(\frac{1}{2}p-\frac{1}{2})!^2}\left\{\frac{d}{dp}\log(p-\tfrac{1}{2})! - \frac{d}{d(\frac{1}{2}p)}\log(\tfrac{1}{2}p-\tfrac{1}{2})!\right\}.$$

Now, when $p = 2$, the factor outside the right-hand bracket reduces to 3, and the left-hand side of the equation is therefore 2. The right-hand bracket may be evaluated from the definite integral,

$$\frac{d}{dz}\log(z!) = \int_0^\infty\left(\frac{1}{t} - \frac{e^{-tz}}{1-e^{-t}}\right)e^{-t}dt,$$

writing successively $1\frac{1}{2}$ and $\frac{1}{2}$ for z, as in the bracket, there remains

$$\int_0^\infty\left(e^{-\frac{1}{2}t} - e^{-\frac{3}{2}t}\right)\frac{e^{-t}}{1-e^{-t}}dt$$

$$= \int_0^\infty e^{-\frac{3}{2}t}dt = \tfrac{2}{3}.$$

The equation is therefore satisfied, the relative variance having its minimum value when p is 2. The mean square deviation is the derivate with the minimum relative variance.

5. The distribution of pairs of values of σ_1 and σ_2, when $n = 4$.

Full knowledge of the effects of using one rather than another of two derivates can only be obtained from the frequency surface of pairs of values of the two derivates. By integration with respect to one derivate or the other, the two frequency curves can be obtained and compared, in respect of any quality which may be in question; but the additional information supplied by the mutual frequency surface is essential to a thorough examination of the question.

When n is large the problem is simplified by the fact that both curves rapidly approach the normal form centred about the true value as mean; the only possible difference between such curves is in the standard deviation. For small values of n the case is much more complicated; failing a complete expression for the frequency surface of σ_1 and σ_2 in terms of n, it will be best to investigate this surface in the single case, $n = 4$. This single case will be found sufficient to bring out the decisive features of the general surface.

If x_1, x_2, x_3, and x_4 are the four observations, then the chance of all four observations falling into their respective elementary ranges is

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^4 e^{-\frac{S(x-m)^2}{2\sigma^2}} dx_1\, dx_2\, dx_3\, dx_4.$$

As in Article 2, this frequency density is the product of two factors, one depending only on $\bar{x}$,

$$\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{x}-m)^2}{2\sigma^2}} d\bar{x},$$

and the other on σ_2,

$$\frac{1}{(\sigma\sqrt{2\pi})^3} e^{-2\frac{\sigma_2^2}{\sigma^2}} dv,$$

in which dv stands for the element of volume in the plane three-dimensional space,

$$S(x-\bar{x}) = 0.$$

Within this space the values of $(x-\bar{x})$ will be positive or negative according as the representative point lies on one side or the other of four planes, through M, drawn parallel to the faces of a regular tetrahedron. The surface of a sphere with M as centre is therefore divided into 14 areas (all combinations of sign being possible except *all positive* and *all negative*). Of these areas 6 are regular four-sided figures, including the cases in which two deviations are positive and two negative; the remaining 8, regular three-sided figures, include the cases in which one deviation is of opposite sign to the other three. All the sides of these figures are 60° and the angles $\cos^{-1} \pm \frac{1}{3}$. The fraction of the total area included in the four-sided figures is

$$3 - \frac{6}{\pi}\cos^{-1}\tfrac{1}{3} = \cdot 6490,$$

and the remainder in the three-sided areas is

$$\frac{6}{\pi}\cos^{-1}\tfrac{1}{3} - 2 = \cdot 3510.$$

The distribution of σ_1 is different in these two groups of regions.

When two of the deviations are positive and two negative, σ_1 will be constant on a plane space of the type,

$$(x_1-\bar{x})+(x_2-\bar{x})-(x_3-\bar{x})-(x_4-\bar{x})=\sqrt{\frac{2}{\pi}}\cdot 4\sigma_1.$$

This space is at right angles to that which we have considered, namely

$$S(x-\bar{x})=0,$$

and its least distance from the centre therefore lies in that space. This distance is $\sigma_1\sqrt{\frac{8}{\pi}}$, and passes through the centre of figure of the spherical quadrangle in which the representative point lies. If θ be the angular distance of this point from the centre of figure

$$2\sigma_2\cos\theta=\sigma_1\sqrt{\frac{8}{\pi}},$$

i.e.

$$\cos\theta=\sqrt{\frac{2}{\pi}}\cdot\frac{\sigma_1}{\sigma_2} \quad . \quad . \quad . \quad . \quad \text{I A.}$$

In the triangular regions the plane space over which σ_1 is constant, such as

$$(x_1-\bar{x})+(x_2-\bar{x})+(x_3-\bar{x})-(x_4-\bar{x})=\sqrt{\frac{2}{\pi}}\cdot 4\sigma_1,$$

is not perpendicular to the space in which the representative point lies, but makes with it an angle of 60°, the distance from the centre to the plane of intersection is therefore

$$\sqrt{\frac{8}{\pi}}\sigma_1\operatorname{cosec}60^\circ=\sigma_1\sqrt{\frac{32}{3\pi}}$$

whence

$$\cos\theta=\sqrt{\frac{8}{3\pi}}\cdot\frac{\sigma_1}{\sigma_2} \quad . \quad . \quad . \quad . \quad \text{I B.}$$

The frequency distribution of σ_1, for a given value of σ_2, is thus reduced to the frequency of occurrence of different values of θ, in two types of spherical figures. For the quadrangles the greatest possible value of θ is 45°, while the least distance from the perimeter to the centre is $\sin^{-1}\sqrt{\frac{1}{3}}$. From these 6 regions we have

$$\left.\begin{array}{llll} \text{From } 0 \text{ to } \sin^{-1}\frac{1}{\sqrt{3}} & \text{frequency} & 3\sin\theta\,d\theta \\ \text{From } \sin^{-1}\frac{1}{\sqrt{3}} \text{ to } 45^\circ & \text{frequency} & 3\sin\theta\left(1-\frac{4}{\pi}\cos^{-1}\frac{1}{\sqrt{2}\tan\theta}\right)d\theta \end{array}\right\} \text{II A.}$$

The greatest value of θ in the triangles is $\sin^{-1}\frac{1}{\sqrt{3}}$, and the least distance from the perimeter to the centre is $\sin^{-1}\frac{1}{3}$, therefore from 8 such regions we have

$$\left.\begin{array}{lll} \text{from } 0 \text{ to } \sin^{-1}\frac{1}{3} & \text{frequency} & 4\sin\theta d\theta \\ \text{from } \sin^{-1}\frac{1}{3} \text{ to } \sin^{-1}\frac{1}{\sqrt{3}} & \text{frequency} & 4\sin\theta\left(1-\frac{3}{\pi}\cos^{-1}\frac{1}{\sqrt{8}\tan\theta}\right)d\theta \end{array}\right\} \text{IIB.}$$

Substituting for θ in expression II., we obtain for the distribution of σ_1 for a given value of σ_2,

$$\left.\begin{array}{lll} \text{from } \sigma_2\sqrt{\frac{\pi}{2}} \text{ to } \sigma_2\sqrt{\frac{\pi}{3}} & \text{frequency} & 3\sqrt{\frac{2}{\pi}}\frac{d\sigma_1}{\sigma_2} \\ \text{from } \sigma_2\sqrt{\frac{\pi}{3}} \text{ to } \sigma_2\sqrt{\frac{\pi}{4}} & \text{frequency} & 3\sqrt{\frac{2}{\pi}}\left(1-\frac{4}{\pi}\cos^{-1}\frac{\sigma_1}{\sqrt{\pi\sigma_2^2-2\sigma_1^2}}\right)\frac{d\sigma_1}{\sigma_2} \end{array}\right\} \text{IIIA.}$$

for the quadrangles, and for the triangles

$$\left.\begin{array}{lll} \text{from } \sigma_2\sqrt{\frac{3\pi}{8}} \text{ to } \sigma_2\sqrt{\frac{\pi}{3}} & \text{frequency} & 8\sqrt{\frac{2}{3\pi}}\frac{d\sigma_1}{\sigma_2} \\ \text{from } \sigma_2\sqrt{\frac{\pi}{3}} \text{ to } \sigma_2\sqrt{\frac{\pi}{4}} & \text{frequency} & 8\sqrt{\frac{2}{3\pi}}\left(1-\frac{3}{\pi}\cos^{-1}\frac{\sigma_1}{\sqrt{3\pi\sigma_2^2-8\sigma_1^2}}\right)\frac{d\sigma_1}{\sigma_2} \end{array}\right\} \text{IIIB.}$$

These two frequency curves are shown on opposite sides of the same base in fig. 1. The existence of two or more distinct curves according to the partition of the observations by the mean, and the existence of one or more discontinuities are no doubt characteristic of such curves in general. For higher values of n, the right-hand side would no longer be truncated, but would meet the base with increasingly high contact.

From the expressions III., may be obtained the frequency surface. For the frequency in the range $d\sigma_2$, being, by Article 2,

$$\frac{8\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\sigma_2^2}{\sigma^3}e^{-2\frac{\sigma_2^2}{\sigma^2}}d\sigma_2,$$

the frequency in the range $d\sigma_1\,d\sigma_2$ is

$$\left.\begin{array}{ll} \sqrt{\frac{\pi}{2}}>\frac{\sigma_1}{\sigma_2}>\sqrt{\frac{\pi}{3}} & \frac{48}{\pi}e^{-\frac{2\sigma_2^2}{\sigma^2}}\frac{\sigma_2 d\sigma_1 d\sigma_2}{\sigma^3} \\ \sqrt{\frac{\pi}{3}}>\frac{\sigma_1}{\sigma_2}>\sqrt{\frac{\pi}{4}} & \frac{48}{\pi}e^{-\frac{2\sigma_2^2}{\sigma^2}}\left(1-\frac{4}{\pi}\cos^{-1}\frac{\sigma_1}{\sqrt{\pi\sigma_2^2-2\sigma_1^2}}\right)\cdot\frac{\sigma_2 d\sigma_1 d\sigma_2}{\sigma^3} \end{array}\right\} \text{IVA.}$$

when two observations lie on each side of the mean, and

$$\left.\begin{array}{l} \sqrt{\frac{3\pi}{8}} > \frac{\sigma_1}{\sigma_2} > \sqrt{\frac{\pi}{3}} \quad \frac{128}{\pi\sqrt{3}} e^{-\frac{2\sigma_2^2}{\sigma^2}} \cdot \frac{\sigma_2}{\sigma^3} d\sigma_1 d\sigma_2 \\ \sqrt{\frac{\pi}{3}} > \frac{\sigma_1}{\sigma_2} > \sqrt{\frac{\pi}{4}} \quad \frac{128}{\pi\sqrt{3}} e^{-\frac{2\sigma_2^2}{\sigma^2}} \left(1 - \frac{3}{\pi}\cos^{-1}\frac{\sigma_1}{\sqrt{3\pi\sigma_2^2 - 8\sigma_1^2}}\right) \frac{\sigma_2}{\sigma^3} d\sigma_1 d\sigma_2 \end{array}\right\} \text{IV}_{\text{B}}.$$

when three are on one side and one on the other.

FREQUENCY DISTRIBUTION OF σ_1/σ_2.

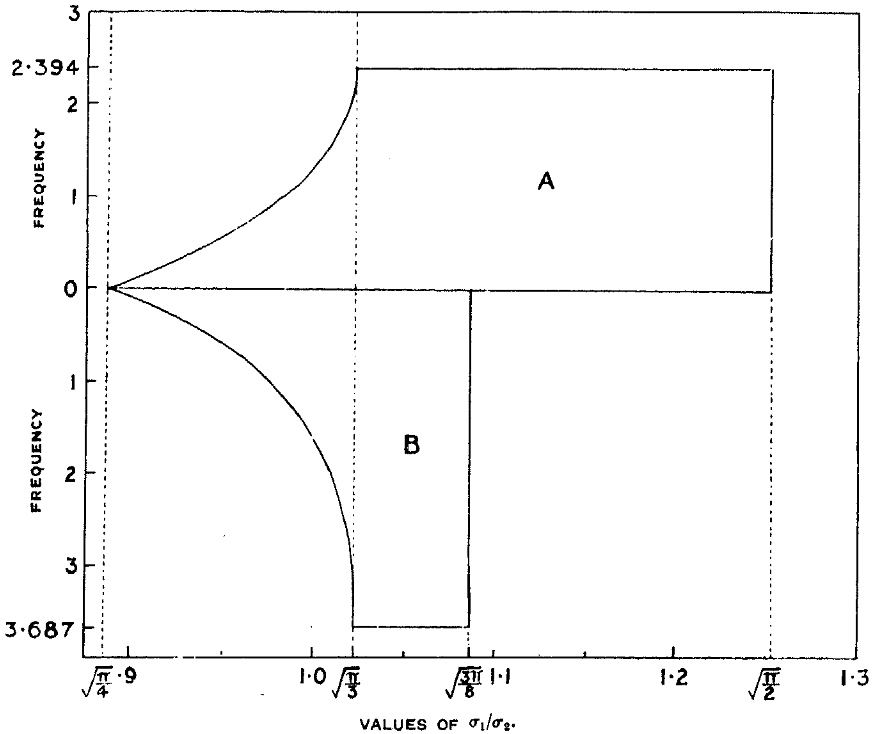

FIG. 1.

In expressions IV. we have the complete account of the facts of random sampling respecting the two variables σ_1 and σ_2 ; it will be of interest first to obtain the frequency curves of σ_1.

6. The distribution of σ_1.

The expressions IV. may be integrated with respect to σ_2, over its whole range of variability from 0 to ∞ ; by so doing we arrive at two frequency curves corresponding to the two partitions of the observations,

$$\frac{12}{\pi} e^{-\frac{4\sigma_1^2}{\pi\sigma^2}} \left\{1 - \frac{4}{\pi} e^{-\frac{2\sigma_1^2}{\pi\sigma^2}} \int_0^1 e^{-\frac{2\sigma_1^2 t^2}{\pi\sigma^2}} \cdot \frac{dt}{t^2+1}\right\} \frac{d\sigma_1}{\sigma} \quad . \quad \text{V}_{\text{A}}.$$

and

$$\frac{32}{\pi\sqrt{3}} e^{-\frac{16\sigma_1^2}{3\pi\sigma^2}} \left\{1 - \frac{3\sqrt{3}}{\pi} e^{-\frac{2\sigma_1^2}{3\pi\sigma^2}} \int_0^1 e^{-\frac{2\sigma_1^2 t^2}{\pi\sigma^2}} \cdot \frac{dt}{3t^2+1}\right\} \frac{d\sigma_1}{\sigma} \quad . \quad \text{V}_{\text{B}}.$$

Of these curves it is easy to calculate the moments, and thence to find those of the compound curve obtained by throwing them together, as is done if we consider the distribution of σ_1 without regard for the manner in which the observations are parted by the mean.

The moments of these three curves and of the corresponding curve for σ_2 are shown in the following table:—

	σ_1. A.	σ_1. B.	σ_1. Total.	σ_2.
μ_0	$3-\frac{6\alpha}{\pi}$	$\frac{6\alpha}{\pi}-2$	1	1
μ_1/σ	$\frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{\pi}}$
μ_2/σ^2	$\frac{3}{8}(\pi-2\alpha)+\frac{1}{2\sqrt{2}}$	$\frac{3}{16}(3\alpha-\pi)+\frac{1}{4\sqrt{2}}$	$\frac{3}{16}(\pi-\alpha)+\frac{3}{4\sqrt{2}}$	$\frac{3}{4}$
μ_3/σ^3	$\frac{\pi}{3\sqrt{3}}+\frac{1}{8}$	$\frac{13\pi}{96\sqrt{3}}+\frac{1}{16}$	$\frac{5\pi\sqrt{3}}{32}+\frac{3}{16}$	$\sqrt{\frac{2}{\pi}}$
μ_4/σ^4	$\frac{9\pi}{64}(\pi-2\alpha)+\frac{29}{64\sqrt{2}}$	$\frac{27\pi}{512}(3\alpha-\pi)+\frac{49}{384\sqrt{2}}$	$\frac{9\pi}{512}(5\pi-7\alpha)+\frac{133}{384\sqrt{2}}$	$\frac{15}{16}$

α here stands for $\cos^{-1}\frac{1}{3}$; of the numerical values we need only cite the coefficient of variation, ·4296 for σ_1 and ·4220 for σ_2. The derivate of the second power is less variable even for small values of n, but the difference in weight in favour of σ_2 is increased fourfold when n is made large. The curve for σ_1 has not only a larger coefficient of variation, it is also more skew, β_1 is ·297 against ·238 and β_2 is 3·28 against 3·11.

7. Unique properties of σ_2.

So far the variables have been compared only in respect of the quantitative characters of their frequency distributions. There exists also in the form of the frequency surface (IV.) a qualitative distinction, which reveals the unique character of σ_2.

From the manner in which the frequency surface has been derived, as in expressions III., it is evident that:—

For a given value of σ_2, the distribution of σ_1 is independent of σ.

On the other hand, it is clear from expressions (IV.) and (V.) that for a given value of σ_1 the distribution of σ_2 does involve σ. In other words, if, in seeking information as to the value of σ, we first determine σ_1, then we can still further improve our estimate by determining σ_2; but if we had first determined σ_2, the frequency curve for σ_1 being entirely independent of σ, the actual value of σ_1 can give us no further information as to the value of σ. The whole of the information to be obtained from σ_1 is included in that supplied by a knowledge of σ_2.

This remarkable property of σ_2, as the methods which we have used to determine the frequency surface demonstrate, follows from the distribution of frequency density in concentric spheres over each of which σ_2 is constant. It therefore holds equally if σ_3 or any other derivate be substituted for σ_1. If this is so, then it must be admitted that:—

The whole of the information respecting σ, which a sample provides, is summed up in the value of σ_2.

This unique superiority of σ_2 is dependent on the form of the normal curve,

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

which leads to a frequency density in generalised space distributed on concentric spheres. Since it is sometimes urged in

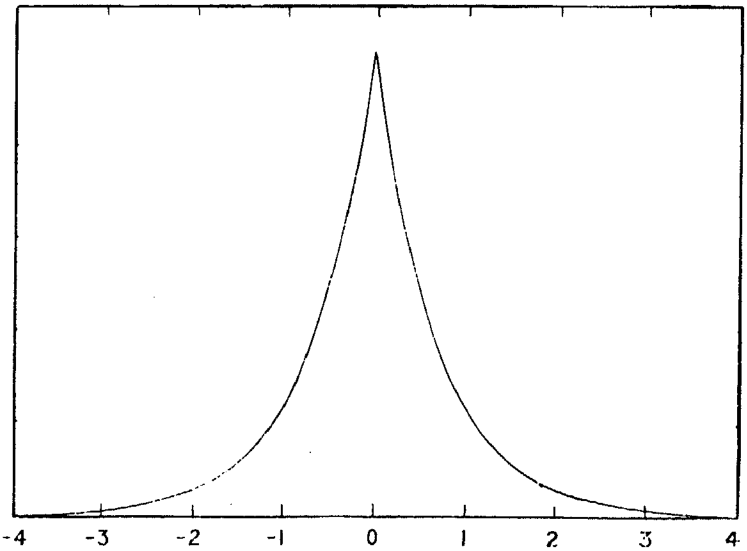

Fig. 2.

favour of the Mean Error that it gives less weight to the large deviations, and that these large deviations do in fact occur in excess of the normal expectation, it is of interest to see if any curve is related to the Mean Error in the same way as is the normal curve to the Mean Square Error.

The somewhat artificial curve

$$\frac{1}{\sigma\sqrt{2}} e^{-\frac{|x-m|}{\sigma}\sqrt{2}} dx$$

replaces the generalised spheres by generalised octahedra, upon the surfaces of which σ_1 is constant, provided $\bar{x}=m$. For large values of n this condition is sufficiently approached and σ_1 may be taken as the ideal measure of σ for curves of this type. When n is small, and allowance has to be made for the aberrations of $\bar{x}$, the figure on which σ_1 is constant is the central section of a generalised octahedron, as was seen in the case $n=4$, where the figure over which σ_1 is constant was found to be bounded by six squares and eight equilateral triangles; while the surface of equal probability is in general an eccentric section in which the squares become rectangles and the triangles are not all equal.

When n is large, however, it does not seem unreasonable to employ σ_1 to samples from curves which resemble the above rather than the normal curve. The value of β_2 (the ratio of the fourth moment to the square of the second moment) seems well fitted to provide a test. If this is near to 3 the Mean Square Error will be required; if, on the other hand, it approaches 6, its value for the double exponential curve, it may be that σ_1 is a more suitable measure of dispersion. It should not be forgotten, however, that the factor $\sqrt{\frac{\pi}{2}}$ in the formula for σ_1 is derived from the normal curve of errors. The corresponding factor for the double exponential curve is $\sqrt{2}$, about 12 per cent. bigger.