CHAPTER I

MATRICES AND VECTORS

1.01 Linear transformations and vectors. In a set of linear equations

$$\begin{aligned}
\eta'_1 &= a_{11}\eta_1 + a_{12}\eta_2 + \cdots + a_{1n}\eta_n \\
\eta'_2 &= a_{21}\eta_1 + a_{22}\eta_2 + \cdots + a_{2n}\eta_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\eta'_n &= a_{n1}\eta_1 + a_{n2}\eta_2 + \cdots + a_{nn}\eta_n
\end{aligned}$$

or

$$\eta'_i = \sum_{j=1}^{n} a_{ij}\eta_j \qquad (i = 1, 2, \cdots, n) \tag{1}$$

the quantities $\eta_1, \eta_2, \cdots, \eta_n$ may be regarded as the coordinates of a point P in n-space and the point $P'(\eta'_1, \eta'_2, \cdots, \eta'_n)$ is then said to be derived from P by the *linear homogeneous transformation* (1). Or, in place of regarding the η's as the coordinates of a point we may look on them as the components of a vector y and consider (1) as defining an operation which transforms y into a new vector y'. We shall be concerned here with the properties of such transformations, sometimes considered abstractly as entities in themselves, and sometimes in conjunction with vectors.

To prevent misconceptions as to their meaning we shall now define a few terms which are probably already familiar to the reader. By a *scalar* or number we mean an element of the field in which all coefficients of transformations and vectors are supposed to lie; unless otherwise stated the reader may assume that a scalar is an ordinary number real or complex.

A *vector*[1] of order n is defined as a set of n scalars $(\xi_1, \xi_2, \cdots, \xi_n)$ given in a definite order. This set, regarded as a single entity, is denoted by a single symbol, say x, and we write

$$x = (\xi_1, \xi_2, \cdots, \xi_n).$$

The scalars $\xi_1, \xi_2, \cdots, \xi_n$ are called the *coordinates* or *components* of the vector.

If $y = (\eta_1, \eta_2, \cdots, \eta_n)$ is also a vector, we say that $x = y$ if, and only if, corresponding coordinates are equal, that is, $\xi_i = \eta_i$ $(i = 1, 2, \cdots, n)$. The vector

$$z = (\zeta_1, \zeta_2, \cdots, \zeta_n) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n)$$

is called the *sum* of x and y and is written $x + y$; it is easily seen that the operation of addition so defined is commutative and associative, and it has a unique inverse if we agree to write 0 for the vector $(0, 0, \cdots, 0)$.

[1] In chapter 5 we shall find it convenient to use the name *hypernumber* for the term vector which is then used in a more restricted sense, which, however, does not conflict with the use made of it here.

If ρ is a scalar, we shall write

$$\rho x = x\rho = (\rho\xi_1, \rho\xi_2, \cdots, \rho\xi_n).$$

This is the only kind of multiplication we shall use regularly in connection with vectors.

1.02 **Linear dependence.** In this section we shall express in terms of vectors the familiar notions of linear dependence.[2] If $x_1, x_2, \cdots, x_r$ are vectors and $\omega_1, \omega_2, \cdots, \omega_r$ scalars, any vector of the form

$$x = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r \tag{2}$$

is said to be *linearly dependent* on $x_1, x_2, \cdots, x_r$; and these vectors are called linearly independent if an equation which is reducible to the form

$$0 = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r$$

can only be true when each $\omega_i = 0$. Geometrically the r vectors determine an r-dimensional subspace of the original n-space and, if $x_1, x_2, \cdots, x_r$ are taken as the coordinate axes, $\omega_1, \omega_2, \cdots, \omega_r$ in (2) are the coordinates of x.

We shall call the totality of vectors x of the form (2) the *linear set* or *subspace* $(x_1, x_2, \cdots, x_r)$ and, when $x_1, x_2, \cdots, x_r$ are linearly independent, they are said to form a *basis* of the set. The number of elements in a basis of a set is called the *order* of the set.

Suppose now that $(x_1, x_2, \cdots, x_r)$, $(y_1, y_2, \cdots, y_s)$ are bases of the same linear set and assume $s \geq r$. Since the x's form a basis, each y can be expressed in the form

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r \qquad (i = 1, 2, \cdots, s) \tag{3}$$

and, since the y's form a basis, we may set

$$x_i = b_{i1}y_1 + b_{i2}y_2 + \cdots + b_{is}y_s \qquad (i = 1, 2, \cdots, r)$$

and therefore from (3)

$$y_i = \sum_{j=1}^{r} a_{ij}x_j = \sum_{j=1}^{r} a_{ij} \sum_{k=1}^{s} b_{jk}y_k = \sum_{k=1}^{s} c_{ik}y_k, \tag{4}$$

where $c_{ik} = \sum_{j=1}^{r} a_{ij}b_{jk}$, which may also be written

$$c_{ik} = \sum_{j=1}^{s} a_{ij}b_{jk} \qquad (i = 1, 2, \cdots, s) \tag{5}$$

if we agree to set $a_{ij} = 0$ when $j > r$. Since the y's are linearly independent, (4) can only hold true if $c_{ii} = 1$, $c_{ik} = 0$ $(i \neq k)$ so that the determinant

[2] See for instance Bôcher, *Introduction to Higher Algebra*, p. 34.

$|c_{ik}| = 1$. But from the rule for forming the product of two determinants it follows from (5) that $|c_{ik}| = |a_{ik}||b_{ik}|$ which implies (i) that $|a_{ik}| \neq 0$ and (ii) that $r = s$, since otherwise $|a_{ik}|$ contains the column $a_{i,\, r+1}$ each element of which is 0. The order of a set is therefore independent of the basis chosen to represent it.

It follows readily from the theory of linear equations (or from §1.11 below) that, if $|a_{ij}| \neq 0$ in (3), then these equations can be solved for the x's in terms of the y's, so that the conditions established above are sufficient as well as necessary in order that the y's shall form a basis.

If e_i denotes the vector whose ith coordinate is 1 and whose other coordinates are 0, we see immediately that we may write

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$$

in place of $x = (\xi_1, \xi_2, \cdots, \xi_n)$. Hence $e_1, e_2, \cdots, e_n$ form a basis of our n-space. We shall call this the *fundamental basis* and the individual vectors e_i the fundamental *unit vectors*.

If $x_1, x_2, \cdots, x_r (r < n)$ is a basis of a subspace of order r, we can always find $n-r$ vectors $x_{r+1}, \cdots, x_n$ such that $x_1, x_2, \cdots, x_n$ is a basis of the fundamental space. For, if x_{r+1} is any vector not lying in $(x_1, x_2, \cdots, x_r)$, there cannot be any relation

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r + \omega_{r+1} x_{r+1} = 0$$

in which $\omega_{r+1} \neq 0$ (in fact every ω must be 0) and hence the order of $(x_1, x_2, \cdots, x_r, x_{r+1})$ is $r + 1$. Since the order of $(e_1, e_2, \cdots, e_n)$ is n, a repetition of this process leads to a basis $x_1, x_2, \cdots, x_r, \cdots, x_n$ of order n after a finite number of steps; a suitably chosen e_i may be taken for x_{r+1}. The $(n-r)$-space $(x_{r+1}, \cdots, x_n)$ is said to be *complementary* to $(x_1, x_2, \cdots, x_r)$; it is of course not unique.

1.03 **Linear vector functions and matrices.** The set of linear equations given in §1.01, namely,

$$\eta'_i = \sum_{j=1}^{n} a_{ij}\eta_j \qquad (i = 1, 2, \cdots, n) \tag{6}$$

define the vector $y' = (\eta'_1, \eta'_2, \cdots, \eta'_n)$ as a linear homogeneous function of the coordinates of $y = (\eta_1, \eta_2, \cdots, \eta_n)$ and in accordance with the usual functional notation it is natural to write $y' = A(y)$; it is usual to omit the brackets and we therefore set in place of (6)

$$y' = Ay.$$

The function or operator A when regarded as a single entity is called a matrix; it is completely determined, relatively to the fundamental basis, when the n^2 numbers a_{ij} are known, in much the same way as the vector y is determined by its coordinates. We call the a_{ij} the *coordinates* of A and write

$$A = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix} \tag{7}$$

or, when convenient, $A = \| a_{ij} \|$. It should be noted that in a_{ij} the first suffix denotes the row in which the coordinate occurs while the second gives the column.

If $B = \| b_{ij} \|$ is a second matrix, $y'' = A(By)$ is a vector which is a linear vector homogeneous function of y, and from (6) we have

$$\eta_i'' = \sum_{p=1}^{n} a_{ip} \sum_{p=1}^{n} b_{pj}\eta_j = \sum_{j=1}^{n} d_{ij}\eta_j,$$

where

$$d_{ij} = \sum_{p=1}^{n} a_{ip}b_{pj}. \tag{8}$$

The matrix $D = \| d_{ij} \|$ is called the *product* of A into B and is written AB. The form of (8) should be carefully noted; in it each element of the ith row of A is multiplied into the corresponding element of the jth column of B and the terms so formed are added. Since the rows and columns are not interchangeable, AB is in general different from BA; for instance

$$\begin{Vmatrix} 1 & 0 \\ 2 & 1 \end{Vmatrix} \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} = \begin{Vmatrix} a & b \\ 2a + c & 2b + d \end{Vmatrix}$$

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ 2 & 1 \end{Vmatrix} = \begin{Vmatrix} a + 2b & b \\ c + 2d & d \end{Vmatrix}.$$

The product defined by (8) is associative; for if $C = \| c_{ij} \|$, the element in the ith row and jth column of $(AB)C$ is

$$\sum_{q=1}^{n} \left(\sum_{p=1}^{n} a_{ip}b_{pq} \right) c_{qj} = \sum_{p=1}^{n} a_{ip} \left(\sum_{q=1}^{n} b_{pq}c_{qj} \right)$$

and the term on the right is the (i, j) coordinate of $A(BC)$.

If we add the vectors Ay and By, we get a vector whose ith coordinate is (cf. (6))

$$\eta_i' = \sum_{j=1}^{n} a_{ij}\eta_j + \sum_{j=1}^{n} b_{ij}\eta_j = \sum_{j=1}^{n} c_{ij}\eta_j$$

where $c_{ij} = a_{ij} + b_{ij}$. Hence $Ay + By$ may be written Cy where $C = \| c_{ij} \|$. We define C to be the *sum* of A and B and write $C = A + B$; two matrices are then added by adding corresponding coordinates just as in the case of vectors. It follows immediately from the definition of sum and product that

$$A + B = B + A, \qquad (A + B) + C = A + (B + C),$$

$$A(B + C) = AB + AC, \qquad (B + C)A = BA + CA,$$

$$A(x + y) = Ax + Ay,$$

A, B, C being any matrices and x, y vectors. Also, if k is a scalar and we set $y' = Ay$, $y'' = ky'$, then

$$y'' = ky' = kA(y) = A(ky)$$

or in terms of the coordinates

$$\eta_i'' = \sum_j k a_{ij} \eta_j.$$

Hence kA may be interpreted as the matrix derived from A by multiplying each coordinate of A by k.

On the analogy of the unit vectors e_i we now define the *fundamental unit matrices* e_{ij} $(i, j = 1, 2, \cdots, n)$. Here e_{ij} is the matrix whose coordinates are all 0 except the one in the ith row and jth column whose value is 1. Corresponding to the form $\Sigma \xi_i e_i$ for a vector we then have

$$A = \sum_{i,j=1}^{n} a_{ij} e_{ij}. \tag{9}$$

Also from the definition of multiplication in (8)

$$e_{ij} e_{jk} = e_{ik}, \qquad e_{ij} e_{pq} = 0, \qquad (j \neq p) \tag{10}$$

a set of relations which might have been made the basis of the definition of the product of two matrices. It should be noted that it follows from the definition of e_{ij} that

$$e_{ij} e_j = e_i, \qquad e_{ij} e_k = 0 \qquad (j \neq k), \tag{11}$$

$$A e_k = \sum_{i,j} a_{ij} e_{ij} e_k = \sum_i a_{ik} e_i. \tag{12}$$

Hence the coordinates of Ae_k are the coordinates of A that lie in the kth column.

1.04 **Scalar matrices.** If k is a scalar, the matrix K defined by $Ky = ky$ is called a *scalar matrix*; from (1) it follows that, if $K = \| k_{ij} \|$, then $k_{ii} = k$ $(i = 1, 2, \cdots, n)$, $k_{ij} = 0$ $(i \neq j)$. The scalar matrix for which $k = 1$ is called the identity matrix of order n; it is commonly denoted by I but, for reasons explained below, we shall here usually denote it by 1, or by 1_n if it is desired to indicate the order. When written at length we have

$$1_n = \begin{Vmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{Vmatrix}, \quad K = \begin{Vmatrix} k & & & & \\ & k & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & k \end{Vmatrix}$$

A convenient notation for the coordinates of the identity matrix was introduced by Kronecker: if δ_{ij} is the numerical function of the integers i, j defined by

$$\delta_{ii} = 1, \qquad \delta_{ij} = 0 \qquad (i \neq j), \tag{13}$$

then $1_n = \| \delta_{ij} \|$. We shall use this Kronecker delta function in future without further comment.

THEOREM 1. *Every matrix is commutative with a scalar matrix.*

Let k be the scalar and $K = \| k_{ij} \| = \| k\delta_{ij} \|$ the corresponding matrix. If $A = \| a_{ij} \|$ is any matrix, then from the definition of multiplication

$$KA = \left\| \sum_p k_{ip}a_{pj} \right\| = \left\| \sum_p k\delta_{ip}a_{pj} \right\| = \| ka_{ij} \|$$

$$AK = \left\| \sum_p a_{ip}k_{pj} \right\| = \left\| \sum_p ka_{ip}\delta_{pj} \right\| = \| ka_{ij} \|$$

so that $AK = KA$.

If k and h are two scalars and K, H the corresponding scalar matrices, then $K + H$ and KH are the scalar matrices corresponding to $k + h$ and kh. Hence the *one-to-one* correspondence between scalars and scalar matrices is maintained under the operations of addition and multiplication, that is, the two sets are simply isomorphic with respect to these operations. So long therefore as we are concerned only with matrices of given order, there is no confusion introduced if we replace each scalar by its corresponding scalar matrix, just as in the theory of ordinary complex numbers, $(a, b) = a + bi$, the set of numbers of the form $(a, 0)$ is identified with the real continuum. We shall therefore as a rule denote $\| \delta_{ij} \|$ by 1 and $\| k\delta_{ij} \|$ by k.

1.05 Powers of a matrix; adjoint matrices. Positive integral powers of $A = \| a_{ij} \|$ are readily defined by induction; thus

$$A^2 = A \cdot A, \qquad A^3 = A \cdot A^2, \cdots, \qquad A^m = A \cdot A^{m-1}.$$

With this definition it is clear that $A^rA^s = A^{r+s}$ for any positive integers r, s. Negative powers, however, require more careful consideration.

Let the determinant formed from the array of coefficients of a matrix be denoted by

$$| A | = \text{det. } A$$

and let α_{qp} be the cofactor of a_{pq} in A, so that from the properties of determinants

$$(14) \qquad \sum_p a_{ip}\alpha_{pj} = | A | \delta_{ij} = \sum_p \alpha_{ip}a_{pj} \qquad (i, j = 1, 2, \cdots, n).$$

The matrix $\| \alpha_{ij} \|$ is called the *adjoint* of A and is denoted by adj A. In this notation (14) may be written

$$(15) \qquad A(\text{adj } A) = | A | = (\text{adj } A)A,$$

so that a matrix and its adjoint are commutative.

If $| A | \neq 0$, we define A^{-1} by

$$(16) \qquad A^{-1} = | A |^{-1} \text{adj } A.$$

Negative integral powers are then defined by $A^{-r} = (A^{-1})^r$; evidently $A^{-r} = (A^r)^{-1}$. We also set $A^0 = 1$, but it will appear later that a different interpretation must be given when $| A | = 0$. Since $AB \cdot B^{-1}A^{-1} = A \cdot BB^{-1} \cdot A^{-1} = AA^{-1} = 1$, the reciprocal of the product AB is

$$(AB)^{-1} = B^{-1}A^{-1}.$$

If A and B are matrices, the rule for multiplying determinants, when stated in our notation, becomes

$$| AB | = | A | | B |.$$

In particular, if $AB = 1$, then $| A | | B | = 1$; hence, if $| A | = 0$, there is no matrix B such that $AB = 1$ or $BA = 1$. The reader should notice that, if k is a scalar matrix of order n, then $| k | = k^n$.

If $A = 0$, A is said to be *singular*; if $A \neq 0$, A is *regular* or non-singular. When A is regular, A^{-1} is the only solution of $AX = 1$ or of $XA = 1$. For, if $AX = 1$, then

$$A^{-1} = A^{-1} \cdot 1 = A^{-1}AX = X.$$

If $AX = 0$, then either $X = 0$ or A is singular; for, if A^{-1} exists,

$$0 = A^{-1}Ax = X.$$

If $A^2 = A \neq 0$, then A is said to be *idempotent*; for example e_{11} and $\begin{Vmatrix} 4 & -2 \\ 6 & -3 \end{Vmatrix}$ are idempotent. A matrix a power of which is 0 is called *nilpotent*. If the lowest power of A which is 0 is A^r, r is called the *index* of A; for example, if $A = e_{12} + e_{23} + e_{34}$, then

$$A^2 = e_{13} + e_{24}, \qquad A^3 = e_{14}, \qquad A^4 = 0,$$

so that the index of A in this case is 4.

1.06 The transverse of a matrix. If $A = \| a_{ij} \|$, the matrix $\| a'_{ij} \|$ in which $a'_{ij} = a_{ji}$ is called the *transverse*[3] of A and is denoted by A'. For instance the transverse of

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \quad \text{is} \quad \begin{Vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{Vmatrix}.$$

The transverse, then, is obtained by the interchange of corresponding rows and columns. It must be carefully noted that this definition is relative to a particular set of fundamental units and, if these are altered, the transverse must also be changed.

THEOREM 2. *The transverse of a sum is the sum of the transverses of the separate terms, and the transverse of a product is the product of the transverses of the separate factors in the reverse order.*

The proof of the first part of the theorem is immediate and is left to the reader. To prove the second it is sufficient to consider two factors. Let $A = \| a_{ij} \|$, $B = \| b_{ij} \|$, $C = AB = \| c_{ij} \|$ and, as above, set $a'_{ij} = a_{ji}$, $b'_{ij} = b_{ji}$, $c'_{ij} = c_{ji}$; then

$$c'_{ij} = c_{ji} = \sum_p a_{jp} b_{pi} = \sum_p b'_{ip} a'_{pj}$$

whence

$$(AB)' = C' = B'A'.$$

The proof for any number of factors follows by induction.

If $A = A'$, A is said to be *symmetric* and, if $A = -A'$, it is called *skew-symmetric* or *skew*. A scalar matrix k is symmetric and the transverse of kA is kA'.

THEOREM 3. *Every matrix can be expressed uniquely as the sum of a symmetric and a skew matrix.*

For if $A = B + C$, $B' = B$, $C' = -C$, then $A' = B' + C' = B - C$ and therefore

$$B = (A + A')/2, \qquad C = (A - A')/2.$$

Conversely $2A = (A + A') + (A - A')$ and $A + A'$ is symmetric, $A - A'$ skew.

[3] It is also called the transposed or conjugate of A. It is sometimes written $\check{A}$.

1.07 Bilinear forms. A scalar bilinear form in two variable vectors, $x = \Sigma\xi_i e_i$, $y = \Sigma\eta_i e_i$, is a function of the form

$$A(x, y) = \sum_{i,j=1}^{n} a_{ij}\xi_i\eta_j. \tag{17}$$

There is therefore a one-to-one correspondence between such forms and matrices, $A = \| a_{ij} \|$ corresponding to $A(x, y)$. The special form for which $A = \| \delta_{ij} \| = 1$ is of very frequent occurrence and we shall denote it by S; it is convenient to omit the brackets and write simply

$$Sxy = \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_n\eta_n \tag{18}$$

and, because of the manner in which it appears in vector analysis, we shall call it the *scalar* of xy. Since S is symmetric, $Sxy = Syx$.

The function (17) can be conveniently expressed in terms of A and S; for we may write $A(x, y)$ in the form

$$A(x, y) = \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} a_{ij}\eta_j \right) = SxAy.$$

It may also be written

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}\xi_i \right) \eta_j = SA'xy = SyA'x;$$

hence

$$SxAy = SyA'x, \tag{19}$$

so that the form (17) is unaltered when x and y are interchanged if at the same time A is changed into A'. This gives another proof of Theorem 2. For

$$Sx(AB)'y = SyABx = SBxA'y = SxB'A'y,$$

which gives $(AB)' = B'A'$ since x and u are independent variables.

1.08 Change of basis. We shall now investigate more closely the effect of a change in the fundamental basis on the coordinates of a vector or matrix. If $f_1, f_2, \cdots, f_n$ is a basis of our n-space, we have seen (§1.02) that the f's are linearly independent. Let

$$f_i = \sum_{j=1}^{n} p_{ji}e_j = Pe_i \qquad (i = 1, 2, \cdots, n) \tag{20}$$
$$P = \| p_{ij} \|.$$

Since the f's form a basis, the e's are linearly expressible in terms of them, say

$$e_i = \sum_{j=1}^{n} q_{ji}f_j, \tag{21}$$

and, if $Q = \| q_{ij} \|$, this may be written

$$e_i = \sum_j q_{ji} \sum_k p_{kj} e_k = PQe_i \qquad (i = 1, 2, \cdots, n). \tag{22}$$

Hence $PQ = 1$, which is only possible if $| P | \neq 0$, $Q = P^{-1}$.

Conversely, if $| P | \neq 0$, $Q = P^{-1}$, and $f_i = Pe_i$ as in (20), then (22) holds and therefore also (21), that is, the e's, and therefore also any vector x, are linearly expressible in terms of the f's. We have therefore the following theorem.

THEOREM 4. *If* $f_i = Pe_i$ $(i = 1, 2, \cdots, n)$, *the vectors* f_i *form a basis if, and only if,* $| P | \neq 0$.

If we have fewer than n vectors, say $f_1, f_2, \cdots, f_r$, we have seen in 1.02 that we can choose $f_{r+1}, \cdots, f_n$ so that $f_1, f_2, \cdots, f_n$ form a basis. Hence

THEOREM 5. *If* $f_1, f_2, \cdots, f_r$ *are linearly independent, there exists at least one non-singular matrix* P *such that* $Pe_i = f_i$ $(i = 1, 2, \cdots, r)$.

We shall now determine how the form Sxy, which was defined relatively to the fundamental basis, is altered by a change of basis. As above let

$$f_i = Pe_i, \qquad e_i = P^{-1}f_i = Qf_i, \qquad | P | \neq 0, \qquad (i = 1, 2, \cdots, n) \tag{23}$$

be a basis and

$$x = \Sigma \xi_i e_i = \Sigma \xi_i' f_i, \qquad y = \Sigma \eta_i e_i = \Sigma \eta_i' f_i$$

variable vectors; then from (23)

$$x = Q\Sigma \xi_i f_i = P\Sigma \xi_i' e_i, \qquad y = Q\Sigma \eta_i f_i = P\Sigma \eta_i' e_i$$

and

$$\Sigma \xi_i' e_i = P^{-1}x = Qx, \qquad \Sigma \eta_i' e_i = Qy.$$

Let us set temporarily $S_e xy$ for Sxy and also put $S_f xy = \Sigma \xi_i' \eta_i'$, the corresponding form with reference to the new basis; then

$$\begin{aligned} S_f xy &= S_e QxQy = S_e xQ'Qy \\ S_e xy &= S_f PxPy. \end{aligned} \tag{24}$$

Consider now a matrix $A = \| a_{ij} \|$ defined relatively to the fundamental basis and let A_1 be the matrix which has the same coördinates when expressed in terms of the new basis as A has in the old. From the definition of A and from $\xi_j = S_e e_j x$ we have

$$Ax = \sum_{i,j} a_{ij} \xi_j e_i = \sum_{i,j} a_{ij} e_i S_e e_j x$$

and hence

$$A_1x = \Sigma a_{ij}\xi'_j f_i = \Sigma a_{ij} f_i S_f f_j x = \Sigma a_{ij} Q^{-1} e_i S_e Q f_j Q x \tag{25}$$
$$= Q^{-1}\Sigma a_{ij} e_i S_e e_j Q x = Q^{-1}AQx.$$

We have therefore, remembering that $Q = P^{-1}$,

THEOREM 6. *If $f_i = Pe_i$ $(i = 1, 2, \cdots, n)$ is a basis and A any matrix, the matrix PAP^{-1} has the same coordinates when expressed in terms of this basis as A has in terms of the fundamental basis.*

The matrix $Q^{-1}AQ$ is said to be *similar* to A and to be the *transform* of A by Q. Obviously the transform of a product (sum) is the product (sum) of the transforms of the individual factors (terms) with the order unaltered. For instance $Q^{-1}ABQ = Q^{-1}AQ \cdot Q^{-1}BQ$.

Theorem 6 gives the transformation of the matric units e_{ij} defined in §1.03 which corresponds to the vector transformation (23); the result is that, if f_{ij} is the unit in the new system corresponding to e_{ij}, then

$$f_{ij} = Pe_{ij}P^{-1}$$

which is readily verified by setting

$$A = e_{ij} = e_i S_e e_j(\ \ ), \qquad A_1 = f_{ij} = f_i S_f f_j(\ \ )$$

in (25). The effect of the change of basis on the form of the transverse is found as follows. Let A^* be defined by

$$S_f x A y = S_f y A^* x;$$

then

$$S_f y A^* x = S_f x A y = S_e Q x Q A y = S_e x Q' Q A y = S_e Q y (Q') A' Q' Q x$$
$$= S_f y (Q'Q) A' Q' Q x.$$

Hence

$$A^* = (Q'Q)A'Q'Q. \tag{26}$$

1.09 **Reciprocal and orthogonal bases.** With the same notation as in the previous section we have $S_f f_i f_j = 0$ $(i \neq j)$, $S_f f_i f_i = 1$. Hence

$$\delta_{ij} = S_f f_i f_j = S_e Q f_i Q f_j = S_e f_i Q' Q f_j.$$

If, therefore, we set

$$f'_j = Q'Qf_j \qquad (j = 1, 2, \cdots, n), \tag{27}$$

we have, on omitting the subscript e in S_e,

$$Sf_i f'_j = \delta_{ij} \qquad (i, j = 1, 2, \cdots, n). \tag{28}$$

Since $|Q'Q| \neq 0$, the vectors $f'_1, f'_2, \cdots, f'_n$ form a basis which we say is *reciprocal* to $f_1, f_2, \cdots, f_n$. This definition is of course relative to the fundamental basis since it depends on the function S but, apart from this the basis (f'_i) is uniquely defined when the basis (f_i) is given since the vectors f_i determine P and $Q = P^{-1}$.

The relation between (f'_i) and (f_i) is a reciprocal one; for

$$f'_j = Q'Qf_j = Q'QPe_j = Q'e_j$$

and, if $R = (Q')^{-1}$, we have $f_j = R'Rf'_j$.

If only the set $(f_1, f_2, \cdots, f_r)$ is supposed given originally, and this set of linearly independent vectors is extended by $f_{r+1}, \cdots, f_n$ to form a basis of the n-space, then $f'_{r+1}, \cdots, f'_n$ individually depend on the choice of $f_{r+1}, \cdots, f_n$. But (28) shows that, if $Sf_ix = 0$ $(i = 1, 2, \cdots, r)$, then x belongs to the linear set $(f'_{r+1}, \cdots, f'_n)$; hence this linear set is uniquely determined although the individual members of its basis are not. We may therefore without ambiguity call $\mathfrak{F}' = (f'_{r+1}, \cdots, f'_n)$ reciprocal to $\mathfrak{F} = (f_1, f_2, \cdots, f_r)$; $\mathfrak{F}'$ is then the set of all vectors x for which $Sxy = 0$ whenever y belongs to $\mathfrak{F}$.

In a later chapter we shall require the following lemma.

Lemma 1. *If $(f_1, f_2, \cdots, f_r)$ and $(f'_{r+1}, \cdots, f'_n)$ are reciprocal, so also are $(B^{-1}f_1, B^{-1}f_2, \cdots, B^{-1}f_r)$ and $(B'f'_{r+1}, B'f'_{r+2}, \cdots, B'f'_n)$ where B is any non-singular matrix.*

For $SB'f'_iB^{-1}f_j = Sf'_iBB^{-1}f_j = Sf'_if_j = \delta_{ij}$.

Reciprocal bases have a close connection with reciprocal or inverse matrices in terms of which they might have been defined. If P is non-singular and $Pe_i = f_i$ as above, then $P = \Sigma f_iSe_i(\ )$ and, if $Q = \Sigma e_iSf'_i(\ )$, then

$$PQ = \Sigma e_iSf'_if_jSe_j(\ ) = \Sigma\delta_{ij}e_jSe_j(\ ) = 1$$

so that $Q = P^{-1}$.

If $QQ' = 1$, the bases (f_i) and (f'_i) are identical and $Sf_if_j = \delta_{ij}$ for all i and j; the basis is then said to be *orthogonal* as is also the matrix Q. The inverse of an orthogonal matrix and the product of two or more orthogonal matrices are orthogonal; for, if $RR' = 1$,

$$(RQ)(RQ)' = RQQ'R' = RR' = 1.$$

Suppose that $h_1, h_2, \cdots, h_r$ are real vectors which are linearly independent and for which $Sh_ih_j = \delta_{ij}$ $(i \neq j)$; since h_i is real, we have $Sh_ih_i \neq 0$. If $r < n$, we can always find a real vector x which is not in the linear set $(h_1, \cdots, h_r)$ and, if we put

$$h_{r+1} = x - \sum_1^r h_iSh_ix/Sh_ih_i,$$

then $h_{r+1} \neq 0$ and $Sh_ih_{r+1} = 0$ $(i = 1, 2, \cdots, r)$. Hence we can extend the original set to form a basis of the fundamental n-space. If we set $f_i = h_i/(Sh_ih_i)^{\frac{1}{2}}$, then $Sf_if_j = \delta_{ij}$ even when $i = j$; this modified basis is called an *orthogonal basis* of the set.

If the vectors h_i are not necessarily real, it is not evident that x can be chosen so that $Sh_{r+1}h_{r+1} \neq 0$ when $Sh_ih_i \neq 0$ $(i = 1, 2, \cdots, r)$. This may be shown as follows. In the first place we cannot have $Syh_{r+1} = 0$ for every y, and hence $Sh_{r+1}h_{r+1} \neq 0$ when $r = n - 1$. Suppose now that for every choice of x we have $Sh_{r+1}h_{r+1} = 0$; we can then choose a basis $h_{r+1}, \cdots, h_n$ supplementary to $h_1, \cdots, h_r$ such that $Sh_ih_i = 0$ $(i = r + 1, \cdots, n)$ and $Sh_ih_j = 0$ $(i = r + 1, \cdots, n; j = 1, 2, \cdots, r)$. Since we cannot have $Sh_{r+1}h_i = 0$ for every h_i of the basis of the n-space, this scalar must be different from 0 for some value of $i > r$, say $r + k$. If we then put $h'_{r+1} = h_{r+1} + h_{r+k}$ in place of h_{r+1}, we have $Sh_ih'_{r+1} = 0$ $(i = 1, 2, \cdots, r)$ as before and also

$$\begin{aligned} Sh'_{r+1}h'_{r+1} &= Sh_{r+1}h_{r+1} + Sh_{r+k}h_{r+k} + 2Sh_{r+1}h_{r+k} \\ &= 2Sh_{r+1}h_{r+k} \neq 0. \end{aligned}$$

We can therefore extend the basis in the manner indicated for real vectors even when the vectors are complex.

When complex coordinates are in question the following lemma is useful; it contains the case discussed above when the vectors used are real.

LEMMA 2. *When a linear set of order r is given, it is always possible to choose a basis $g_1, g_2, \cdots, g_n$ of the fundamental space such that $g_1, \cdots, g_r$ is a basis of the given set and such that $Sg_i\bar{g}_j = \delta_{ij}$ where $\bar{g}_j$ is the vector whose coordinates are the conjugates of the coordinates of g_j when expressed in terms of the fundamental basis.*

The proof is a slight modification of the one already given for the real case. Suppose that $g_1, \cdots, g_s$ are chosen so that $Sg_i\bar{g}_j = \delta_{ij}$ $(i, j = 1, 2, \cdots, s)$ and such that $(g_1, \cdots, g_s)$ lies in the given set when $s < r$ and when $s > r$, then $g_1, \cdots, g_r$ is a basis of this set. We now put

$$g'_{s+1} = x - \sum_1^s g_iSg_ix/S\bar{g}_ig_i$$

which is not 0 provided x is not in $(g_1, \cdots, g_s)$ and, if $s < r$, will lie in the given set provided x does. We may then put

$$g_{s+1} = g'_{s+1}/(Sg'_{s+1}\bar{g}_{s+1})^{\frac{1}{2}}$$

and the lemma follows readily by induction.

If U is the matrix Σe_iSg_i, then $\bar{U} = \Sigma e_iS\bar{g}_i$ and

$$U\bar{U}' = 1. \tag{29}$$

Such a matrix is called a *unitary* matrix and the basis $g_1, g_2, \cdots, g_n$ is called a unitary basis. A real unitary matrix is of course orthogonal.

1.10 **The rank of a matrix.** Let $A = \| a_{ij} \|$ be a matrix and set (cf. (12) §1.03)

$$h_i = Ae_i = a_{ji}e_j;$$

then, if

$$x = \Sigma \xi_i e_i = \Sigma e_i S e_i x$$

is any vector, we have

$$Ax = A\Sigma e_i S e_i x = \Sigma A e_i S e_i x$$

or

$$Ax = \sum_1^n h_i S e_i x. \tag{30}$$

Any expression of the form $Ax = \sum_1^m a_i S b_i x$, where a_i, b_i are constant vectors, is a linear homogeneous vector function of x. Here (30) shows that it is never necessary to take $m > n$, but it is sometimes convenient to do so. When we are interested mainly in the matrix and not in x, we may write $A = \Sigma a_i S b_i(\ )$ or, omitting the brackets, merely

$$A = \Sigma a_i S b_i. \tag{31}$$

It follows readily from the definition of the transverse that

$$A' = \Sigma b_i S a_i. \tag{32}$$

No matter what vector x is, Ax, being equal to $\Sigma a_i S b_i x$. is linearly dependent on $a_1, a_2, \cdots, a_m$ or, if the form (30) is used, on $h_1, h_2, \cdots, h_n$. When $|A| \neq 0$, we have seen in Theorem 4 that the h's are linearly independent but, if A is singular, there are linear relations connecting them, and the order of the linear set $(a_1, a_2, \cdots, a_m)$ is less than n.

Suppose in (31) that the a's are not linearly independent, say

$$a_s = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{s-1} a_{s-1},$$

then on substituting this value of a_s in (31) we have

$$A = a_1 S(b_1 + \alpha_1 b_s) + \cdots + a_{s-1} S(b_{s-1} + \alpha_{s-1} b_s) + \sum_{s+1}^m a_i S b_i,$$

an expression similar to (31) but having at least one term less. A similar reduction can be carried out if the b's are not linearly independent. After a finite number of repetitions of this process we shall finally reach a form

$$A = \sum_1^r c_i S d_i \tag{33}$$

in which $c_1, c_2, \cdots, c_r$ are linearly independent and also $d_1, d_2, \cdots, d_r$. The integer r is called the *rank* of A.

It is clear that the value of r is independent of the manner in which the reduction to the form (33) is carried out since it is the order of the linear set $(Ae_1, Ae_2, \cdots, Ae_n)$. We shall, however, give a proof of this which incidently yields some important information regarding the nature of A.

Suppose that by any method we have arrived at two forms of A

$$A = \sum_1^r c_i S d_i = \sum_1^s p_i S q_i,$$

where $(c_1, c_2, \cdots, c_r)$ and $(d_1, d_2, \cdots, d_r)$ are spaces of order r and $(p_1, p_2, \cdots, p_s)$, $(q_1, q_2, \cdots, q_s)$ spaces of order s, and let $(c'_{r+1}, c'_{r+2}, \cdots, c'_n), \cdots, (q'_{s+1}, q'_{s+2}, \cdots, q'_n)$ be the corresponding reciprocal spaces. Then

$$Aq'_j = \sum_1^s p_i S q_i q'_j = p_j \qquad (j = 1, 2, \cdots, s)$$

and also $Aq'_j = \Sigma\, c_i S d_i q'_j$. Hence each p_j lies in $(c_1, c_2, \cdots, c_r)$. Similarly each c_i lies in $(p_1, p_2, \cdots, p_s)$ so that these two subspaces are the same and, in particular, their orders are equal, that is, $r = s$. The same discussion with A' in place of A shows that $(d_1, d_2, \cdots, d_r)$ and $(q_1, q_2, \cdots, q_s)$ are the same. We shall call the spaces $\mathfrak{G}_l = (c_1, c_2, \cdots, c_r)$, $\mathfrak{G}_r = (d_1, d_2, \cdots, d_r)$ the left and right *grounds* of A, and the total space $\mathfrak{G} = (c_1, \cdots, c_r, d_1, \cdots, d_r)$ will be called the (total) ground of A.

If x is any vector in the subspace $\mathfrak{N}_r = (d'_{r+1}, d'_{r+2}, \cdots, d'_n)$ reciprocal to $\mathfrak{G}_r$, then $Ax = 0$ since $Sd_i d'_j = 0$ $(i \neq j)$. Conversely, if

$$0 = Ax = \Sigma\, c_i S d_i x,$$

each multiplier $Sd_i x$ must be 0 since the c's are linearly independent; hence every solution of $Ax = 0$ lies in $\mathfrak{N}_r$. Similarly every solution of $A'x = 0$ lies in $\mathfrak{N}_l = (c'_{r+1}, c'_{r+2}, \cdots, c'_n)$. We call $\mathfrak{N}_r$ and $\mathfrak{N}_l$ the right and left *nullspaces* of A; their order, $n - r$, is called the *nullity* of A.

We may summarize these results as follows.

THEOREM 7. *If a matrix A is expressed in the form $\sum_1^r a_i S b_i$, where $\mathfrak{G}_l = (a_1, a_2, \cdots, a_r)$ and $\mathfrak{G}_r = (b_1, b_2, \cdots, b_r)$ define spaces of order r, then, no matter how the reduction to this form is carried out, the spaces $\mathfrak{G}_l$ and $\mathfrak{G}_r$ are always the same. Further, if $\mathfrak{N}_l$ and $\mathfrak{N}_r$ are the spaces of order $n - r$ reciprocal to $\mathfrak{G}_l$ and $\mathfrak{G}_r$, respectively, every solution of $Ax = 0$ lies in $\mathfrak{N}_r$ and every solution of $A'x = 0$ in $\mathfrak{N}_l$.*

The following theorem is readily deduced from Theorem 7 and its proof is left to the reader.

THEOREM 8. *If A, B are matrices of rank r, s, the rank of $A + B$ is not greater than $r + s$ and the rank of AB is not greater than the smaller of r and s.*

1.11 Linear dependence. The definition of the rank of a matrix in the preceding section was made in terms of the linear dependence of vectors associated with the matrix. In this section we consider briefly the theory of linear dependence introducing incidentally a notation which we shall require later.

Let $x_i = \sum_{j=1}^{n} \xi_{ij} e_j$ $(i = 1, 2, \cdots, r; r \leq n)$ be a set of r vectors. From the rectangular array of their coordinates

$$\begin{matrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \xi_{r1} & \xi_{r2} & \cdots & \xi_{rn} \end{matrix} \tag{34}$$

there can be formed $n!/r!(n - r)!$ different determinants of order r by choosing r columns out of (34), these columns being taken in their natural order. If these determinants are arranged in some definite order, we may regard them as the coordinates of a vector in space of order $n!/r!(n - r)!$ and, when this is done, we shall denote this vector by[4]

$$| x_1 x_2 \cdots x_r | \tag{35}$$

and call it a *pure vector* of *grade r*. It follows from this definition that $| x_1 x_2 \cdots x_r |$ has many of the properties of a determinant; its sign is changed if two x's are interchanged, it vanishes when two x's are equal and, if λ and μ are scalars,

$$| (\lambda x_1 + \mu x_1') \, x_2 \cdots x_r | = \lambda \, | x_1 x_2 \cdots x_r | + \mu \, | x_1' x_2 \cdots x_r |. \tag{36}$$

If we replace the x's in (35) by r different units $e_{i_1}, e_{i_2}, \cdots, e_{i_r}$, the result is clearly not 0: we thus obtain $\binom{n}{r}$ vectors which we shall call the fundamental unit vectors of grade r; and any linear combination of these units, say

$$\Sigma \, \xi_{i_1 i_2 \cdots i_r} | e_{i_1} e_{i_2} \cdots e_{i_r} |,$$

is called a vector of grade r. It should be noticed that not every vector is a pure vector except when r equals 1 or n.

If we replace x_i by $\Sigma \, \xi_{ij} e_j$ in (35), we get

$$| x_1 x_2 \cdots x_r | = \Sigma \, \xi_{1 j_1} \xi_{2 j_2} \cdots \xi_{r j_r} | e_{j_1} e_{j_2} \cdots e_{j_r} |,$$

where the summation extends over all permutations $j_1, j_2, \cdots, j_r$ of $1, 2, \cdots, n$ taken r at a time. This summation may be effected by grouping together the

[4] If it had been advisable to use here the indeterminate product of Grassmann, (35) would appear as a determinant in much the ordinary sense (cf. §5.09).

sets $j_1, j_2, \cdots, j_r$ which are permutations of the same combination $i_1, i_2, \cdots, i_r$, whose members may be taken to be arranged in natural order, and then summing these partial sums over all possible combinations $i_1, i_2, \cdots, i_r$. Taking the first step only we have

$$\Sigma\ \xi_{1j_1}\xi_{2j_2} \cdots \xi_{rj_r} \,|\, e_{j_1}e_{j_2} \cdots e_{j_r} \,| = \Sigma\ \delta^{i_1 \cdots i_r}_{j_1 \cdots j_r}\, \xi_{1j_1} \cdots \xi_{rj_r} \,|\, e_{i_1}e_{i_2} \cdots e_{i_r} \,|$$

where $\delta^{i_1 \cdots i_r}_{j_1 \cdots j_r}$ is the sign corresponding to the permutations $\binom{i_1 i_2 \cdots i_r}{j_1 j_2 \cdots j_r}$ and this equals $|\, \xi_{1i_1} \cdots \xi_{ri_r} \,|\,|\, e_{i_1} \cdots e_{i_r} \,|$. We have therefore

$$|\, x_1x_2 \cdots x_r \,| = \sum_{(i)}^{*} |\, \xi_{1i_1}\xi_{2i_2} \cdots \xi_{ri_r} \,|\,|\, e_{i_1}e_{i_2} \cdots e_{i_r} \,|, \tag{37}$$

where the asterisk on Σ indicates that the sum is taken over all r-combinations of $1, 2, \cdots, n$ each combination being arranged in natural order.

THEOREM 9 $|\, x_1x_2 \cdots x_r \,| = 0$ *if, and only if,* $x_1, x_2, \cdots, x_r$ *are linearly dependent.*

The first part of this theorem is an immediate consequence of (36). To prove the converse it is sufficient to show that, if $|\, x_1x_2 \cdots x_{r-1} \,| \neq 0$, then there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$ such that

$$x_r = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_{r-1}x_{r-1}.$$

Let $x_i = \sum_j \xi_{ij}e_j$. Since $|\, x_1x_2 \cdots x_{r-1} \,| \neq 0$, at least one of its coordinates is not 0, and for convenience we may suppose without loss of generality that

$$|\, \xi_{11}\xi_{22} \cdots \xi_{r-1,\, r-1} \,| \neq 0. \tag{38}$$

Since $|\, x_1x_2 \cdots x_r \,| = 0$, all its coordinates equal 0 and in particular

$$|\, \xi_{11}\xi_{22} \cdots \xi_{r-1,\, r-1}\xi_{ri} \,| = 0 \qquad (i = 1, 2, \cdots, n).$$

If we expand this determinant according to the elements of its last column, we get a relation of the form

$$\beta_1\xi_{ri} + \beta_2\xi_{1i} + \cdots + \beta_r\xi_{r-1,\, i} = 0$$

where the β's are independent of i and $\beta_1 \neq 0$ by (38). Hence we may write

$$\xi_{ri} = \alpha_1\xi_{1i} + \cdots + \alpha_{r-1}\,\xi_{r-1,\, i} \qquad (i = 1, 2, \cdots, n) \tag{39}$$

the α's being independent of i. Multiplying (39) by e_i and summing with regard to i, we have

$$x_r = \alpha_1x_1 + \cdots + \alpha_{r-1}\,x_{r-1},$$

which proves the theorem.

If $(a_1, a_2, \cdots, a_m)$ is a linear set of order r, then some set of r a's form a basis, that is, are linearly independent while each of the other a's is linearly dependent on them. By a change of notation, if necessary, we may take $a_1, a_2, \cdots, a_r$ as this basis and write

$$a_{r+i} = \sum_{j=1}^{r} \beta_{ij} a_j, \qquad (i = 1, 2, \cdots, m - r). \tag{40}$$

We shall now discuss the general form of all linear relations among the a's in terms of the special relations (40); and in doing so we may assume the order of the space to be equal to or greater than m since we may consider any given space as a subspace of one of arbitrarily higher dimensionality.

Let

$$\sum_{1}^{m} \gamma_j a_j = 0 \tag{41}$$

be a relation connecting the a's and set

$$c = \sum_{1}^{m} \gamma_j e_j.$$

Then (40), considered as a special case of (41), corresponds to setting for c

$$c_i = -\sum_{j=1}^{r} \beta_{ij} e_j + e_{r+i}, \qquad (i = 1, 2, \cdots, m - r); \tag{42}$$

and there is clearly no linear relation connecting these vectors so that they define a linear set of order $m - r$. Using (40) in (41) we have

$$\sum_{j=1}^{r} \left(\gamma_j + \sum_{i=1}^{m-r} \gamma_{r+i} \beta_{ij} \right) a_j = 0$$

and, since $a_1, a_2, \cdots, a_r$ are linearly independent, we have

$$j = -\sum_{i=1}^{m-r} \beta_{ij} \gamma_{r+i} \qquad (j = 1, 2, \cdots, r)$$

whence

$$c = \sum_{1}^{m} \gamma_j e_j = -\sum_{i=1}^{m-r} \gamma_{r+i} \sum_{j=1}^{r} \beta_{ij} e_j + \sum_{i=1}^{m-r} \gamma_{r+i} e_{r+i} = \sum_{i=1}^{m-r} \gamma_{r+i} c_i, \tag{43}$$

so that c is linearly dependent on $c_1, c_2, \cdots, c_{m-r}$. Conversely, on retracing these steps in the reverse order we see that, if c is linearly dependent on these vectors, so that γ_{r+i} $(i = 1, 2, \cdots, m - r)$ are known, then from (43) the γ_j $(j = 1, 2, \cdots, r)$ are defined in such a way that $c = \sum_{1}^{m} \gamma_j e_j$ and $\sum_{1}^{m} \gamma_j a_j = 0$. We have therefore the following theorem.

THEOREM 10. *If* $a_1, a_2, \cdots, a_m$ *is a linear set of order* r, *there exist* $m - r$ *linear relations* $\sum_{j=1}^{m} \gamma_{ij} a_j = 0$ $(i = 1, 2, \cdots, m - r)$ *such that* (*i*) *the vectors* $c_i = \sum_{j=1}^{m} \gamma_{ij} e_j$ *are linearly independent and* (*ii*) *if* $\Sigma \gamma_j a_j = 0$ *is any linear relation connecting the* a'*s, and if* $c = \Sigma \gamma_j e_j$, *then* c *belongs to the linear set* $(c_1, c_2, \cdots, c_{m-r})$.

This result can be translated immediately in terms concerning the solution of a system of ordinary linear equations or in terms of matrices. If $a_j = \sum_i a_{ji} e_i$, then (41) may be written

$$
\begin{array}{c}
a_{11}\gamma_1 + a_{21}\gamma_2 + \cdots + a_{m1}\gamma_m = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{1n}\gamma_1 + a_{2n}\gamma_2 + \cdots + a_{mn}\gamma_m = 0
\end{array} \tag{44}
$$

a system of linear homogeneous equations in the unknowns $\gamma_1, \gamma_2, \cdots, \gamma_m$. Hence (44) has solutions for which some $\gamma_i \neq 0$ if, and only if, the rank r of the array

$$
\begin{array}{cccc}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\cdot & \cdot & \cdots & \cdot \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{array} \tag{45}
$$

is less then m and, when this condition is satisfied, every solution is linearly dependent on the set of $m - r$ solutions given by (42) which are found by the method given in the discussion of Theorem 9.

Again, if we make (45) a square array by the introduction of columns or rows of zeros and set $A = \| a_{ij} \|$, $c = \Sigma \gamma_i e_i$, then (41) becomes $A'c = 0$ and Theorem 10 may therefore be interpreted as giving the properties of the null-space of A' which were derived in §1.10.