


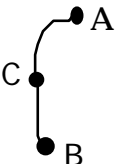
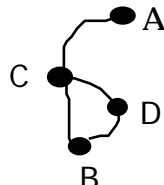
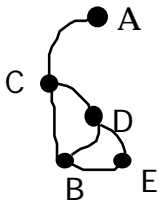
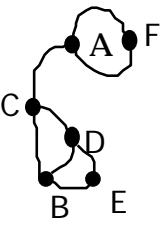
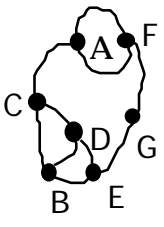
SPROUTS:
Analyzing a Simple Game
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A very simple game which has its origins in the field of *Discrete Mathematics* is a game called SPROUTS. You play SPROUTS according to the following rules:

1. Begin with a given number of dots.
2. An arc can be drawn connecting any two dots.
3. An arc can be drawn connecting a dot to itself.
4. On each *new* arc, a *new* dot is placed at its midpoint.
5. No dot can have more than three arcs coming from it.
6. No arc can cross any other arc.
7. The winner is the player who makes the *last valid move*.

A game of SPROUTS which begins with two dots is played out below.

Start	1st Move	2nd Move	3rd Move	4th Move	5th move
					
	Player 1 connects point A to point B and puts in midpoint C.	Player 2 connects point B to point C and puts in midpoint D.	Player 1 connects point C to point D and puts in midpoint E.	Player 2 connects point A to itself and puts in midpoint F.	Player 1 wins by connecting point E to point F and putting in point G.

Player 2 can begin an arc at point G, but there is no point available for its other endpoint. Every dot, except G, already has 3 arcs joining it. The game ends with Player 1 the winner.

Replay the game starting with Player 1 connecting point A to itself.

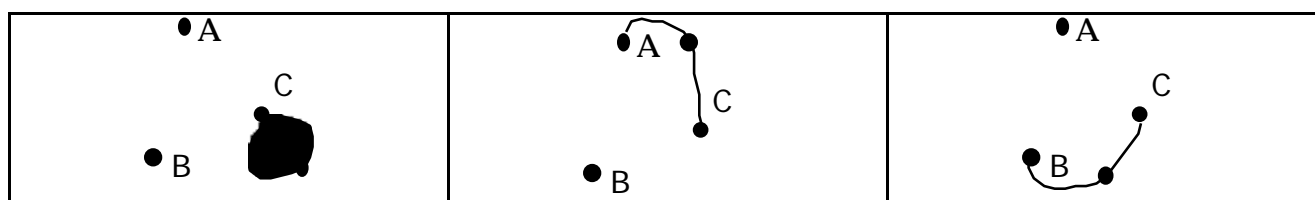
- How many moves does the game have?
- Who wins?
- Can you see that there would be another (virtually identical) game if you begin by connecting point B to itself?

For every 2-dot game, three initial moves are possible. Try to play out all unique possible 2-dot games. Now, play a few games starting with 3 dots. How many initial moves are possible for a 3-dot games? For each game, count the number of moves it takes to determine a winner. Record who wins each game.

Do you understand the game well enough to analyze it? If so, complete the following table.

# of dots at start of game	# initial possible moves	# moves to determine winner	winner
1	1	2	Player 2
2	3	5	Player 1
3	.	.	.
4	.	.	.
5	.	.	.
6	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
n	.	.	.

Having trouble? You might think about what happens as you *add a dot (C) to the 2 dot game*. Three initial possible moves are added, one which connects point C to itself and two additional moves each of which connects point C to one of the other two points.



When a *fourth* dot is added, again it can be connected to itself and also to the three pre-existing dots adding four more initial possible moves. When you add the *n*th dot, how many more initial possible moves are added?

This kind of thinking is called *recursive thinking*.

Recursive Thinking

- We know the number of initial possible moves for a 1-dot game.
- As long as we know the number of initial possible moves for a k-dot game and we know how many more initial moves are added when one more dot is added, we can figure out the number of initial possible moves for any game which begins with more than k dots.

# of dots at start of game	# initial possible moves
1	1
2	$1 + (1 + 1) = 3$
3	$3 + (1 + 2) = 6$
4	$6 + (1 + 3) = 10$
5	$10 + (1 + 4) = 15$
.	.
.	.
.	.
n	$\frac{(n-1)n}{2} + (1 + (n-1)) = \frac{n(n+1)}{2}$

You may recognize the number of initial possible moves as the *triangular* numbers (see Hamberg, IMSAMJ, Fall, 1992, page 7). The n th triangular number is usually represented by the expression $\frac{n(n+1)}{2}$. Why would $\frac{(n-1)n}{2}$ represent the $(n-1)$ st triangular number?

Formula Thinking

A second way to obtain the same result is to immediately consider the n -dot game. For the n -dot game, each dot may initially be connected to itself (n initial possible moves), and one additional initial move is possible for each *pair* of dots which can be chosen from the set.

In mathematical notation this method of determining the number of initial possible moves can be represented as

$$\begin{aligned} n + {}_n C_2 &= n + \frac{n!}{2!(n-2)!} \\ &= n + \frac{n(n-1)(n-2)!}{2(n-2)!} \\ &= n + \frac{(n-1)n}{2} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

How Many Moves to Determine a Winner?

To complete the original table, we need to find the maximum number of moves for a winner to be determined. Consider that when we begin with 2 dots, there are initially 6 arcs possible which will have one of the first two dots as an endpoint. (Remember, each dot may have at most 3 arcs coming from it.)

Consider what happens on the first move of the game:

- One dot is added, but it has two arcs already coming from it, so only *one additional arc is added*.
- Each of the two original dots *loses* one available arc.

There is a *net loss* of one arc from the initial 6. Since one arc is lost for each move, the maximum number of moves in a 2-dot game is 5. Now, you should be able to complete the original table.

A Problem for Analysis

What if the game is changed so that each arc may have **at most 4 arcs** coming from it?

Does the number of initial moves change?

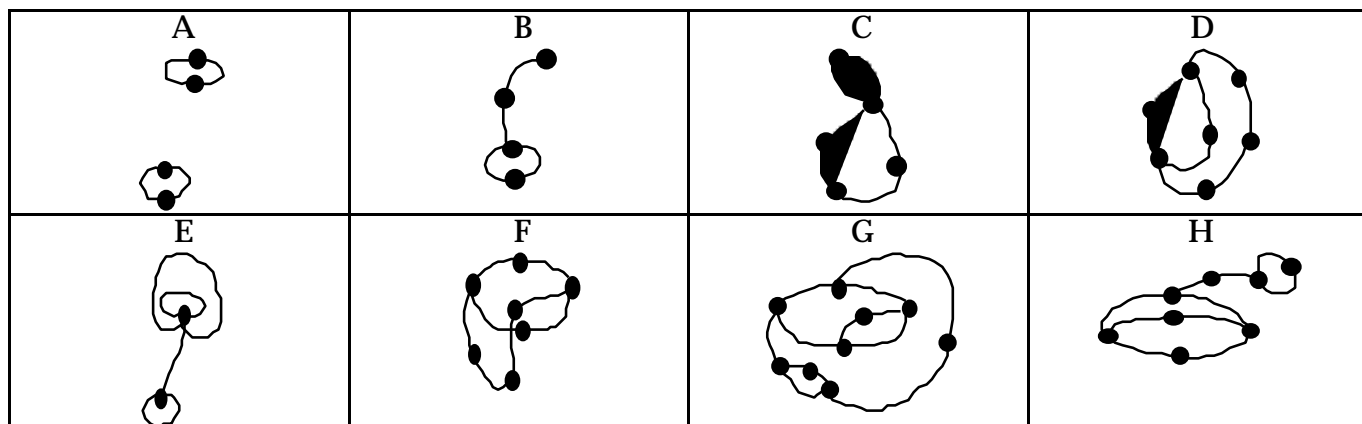
Does the number of moves to determine a winner change?

What if the game is changed so that each dot may have at most 3 arcs coming from it, however, **two new dots are placed on each new arc**?

Does the number of initial moves change?

Does the number of moves to determine a winner change?

To be certain that you understand the game, determine in each of the following cases whether the game situation *could* occur. If it is *not* a possible scenario, explain why not. 🍰



The following table summarizes results for games that start with two, three, four, and five dots and generalizes the number pattern for n dots.

# of dots at start of game	# initial possible moves	# moves to determine winner
2	3	5
3	6	8
4	10	11
5	15	14
.	.	.
.	.	.
.	.	.
n	$\frac{n(n+1)}{2}$	$3n - 1$

Reference: S. Krulik, "Networks", *Student Math Notes*, September, 1984.

A: Possible.	B: Possible.	C: Not possible, one point is the endpoint for 4 arcs.
D: Possible.	E: Not possible, one point is the endpoint for 5 arcs.	F: Not possible, two arcs intersect.
G: Possible.	H: Possible.	