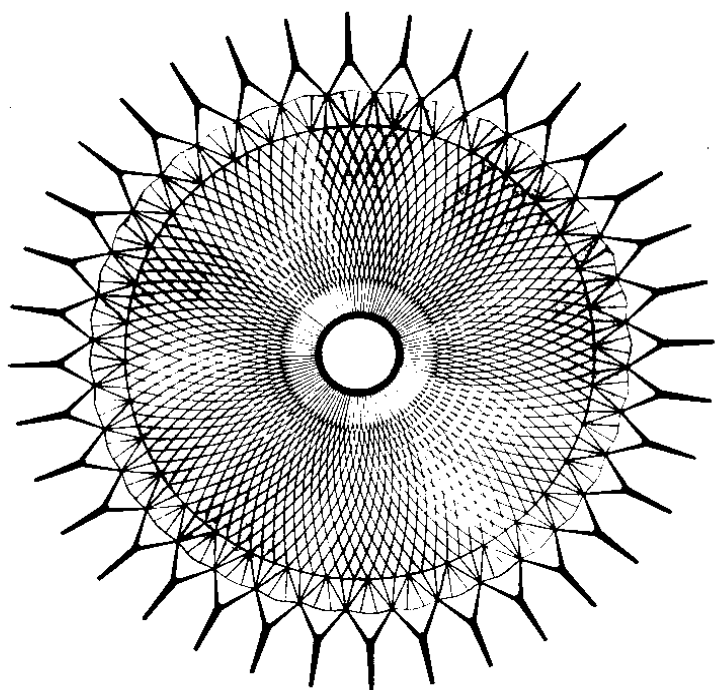
DRAFT

Lecture Notes in:

STRUCTURAL ENGINEERING

Analysis and Design



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PREFACE

Whereas there are numerous excellent textbooks covering Structural Analysis, or Structural Design, I felt that there was a need for a single reference which

- Provides a **succinct, yet rigorous**, coverage of Structural Engineering.
- **Combines**, as much as possible, Analysis with Design.
- Presents numerous, **carefully selected, example problems**.

in a properly type set document.

As such, and given the reluctance of undergraduate students to go through extensive verbage in order to capture a key concept, I have opted for an unusual format, one in which each key idea is clearly distinguishable. In addition, such a format will hopefully foster group learning among students who can easily reference misunderstood points.

Finally, whereas all problems have been taken from a variety of references, I have been very careful in not only properly selecting them, but also in enhancing their solution through appropriate figures and LaTeX typesetting macros.

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Part I

ANALYSIS

Chapter 1

A BRIEF HISTORY OF STRUCTURAL ARCHITECTURE

> *If I have been able to see a little farther than some others, it was because I stood on the shoulders of giants.*
>
> Sir Isaac Newton

1 More than any other engineering discipline, Architecture/Mechanics/Structures is the proud outcome of a of a long and distinguished history. Our profession, second oldest, would be better appreciated if we were to develop a sense of our evolution.

1.1 Before the Greeks

2 Throughout antiquity, structural engineering existing as an art rather than a science. No record exists of any rational consideration, either as to the strength of structural members or as to the behavior of structural materials. The builders were guided by rules of thumbs and experience, which were passed from generation to generation, guarded by secrets of the guild, and seldom supplemented by new knowledge. Despite this, structures erected before Galileo are by modern standards quite phenomenal (pyramids, Via Appia, aqueducs, Colisseums, Gothic cathedrals to name a few).

3 The first structural engineer in history seems to have been **Imhotep**, one of only two commoners to be deified. He was the builder of the step pyramid of Sakkara about 3,000 B.C., and yielded great influence over ancient Egypt.

4 Hamurrabi's code in Babylonia (1750 BC) included among its 282 laws penalties for those "architects" whose houses collapsed, Fig. 1.1.

1.2 Greeks

5 The greek philosopher **Pythagoras** (born around 582 B.C.) founded his famous school, which was primarily a secret religious society, at Crotona in southern Italy. At his school he allowed

Figure 1.2: Archimed

conqueror of Syracuse.

1.3 Romans

10 Science made much less progress under the Romans than under the Greeks. The Romans apparently were more practical, and were not as interested in abstract thinking though they were excellent fighters and builders.

11 As the roman empire expanded, the Romans built great roads (some of them still in use) such as the Via Appia, Cassia, Aurelia; Also they built great bridges (such as the third of a mile bridge over the Rhine built by Caesars), and stadium (Colliseum).

12 One of the most notable Roman construction was the **Pantheon**, Fig. 1.3. It is the best-

Figure 1.3: Pantheon

preserved major edifice of ancient Rome and one of the most significant buildings in architectural history. In shape it is an immense cylinder concealing eight piers, topped with a dome and fronted by a rectangular colonnaded porch. The great vaulted dome is 43 m (142 ft) in diameter, and the entire structure is lighted through one aperture, called an *oculus*, in the center of the dome. The Pantheon was erected by the Roman emperor Hadrian between AD 118 and 128.

Chapter 2

INTRODUCTION

2.1 Structural Engineering

1 Structural engineers are responsible for the detailed analysis and design of:

Architectural structures: Buildings, houses, factories. They must work in close cooperation with an architect who will ultimately be responsible for the design.

Civil Infrastructures: Bridges, dams, pipelines, offshore structures. They work with transportation, hydraulic, nuclear and other engineers. For those structures they play the leading role.

Aerospace, Mechanical, Naval structures: aeroplanes, spacecrafts, cars, ships, submarines to ensure the structural safety of those important structures.

2.2 Structures and their Surroundings

2 Structural design is affected by various environmental constraints:

1. Major movements: For example, elevator shafts are usually shear walls good at resisting lateral load (wind, earthquake).
2. Sound and structure interact:
   - A **dome** roof will concentrate the sound
   - A **dish** roof will diffuse the sound
3. Natural light:
   - A flat roof in a building may not provide adequate light.
   - A Folded plate will provide adequate lighting (analysis more complex).
   - A bearing and shear wall building may not have enough openings for daylight.
   - A Frame design will allow more light in (analysis more complex).
4. Conduits for cables (electric, telephone, computer), HVAC ducts, may dictate type of floor system.
5. Net clearance between columns (unobstructed surface) will dictate type of framing.

2.6 Structural Analysis

12 Given an **existing** structure subjected to a certain load determine internal forces (axial, shear, flexural, torsional; or stresses), deflections, and verify that no unstable failure can occur.

13 Thus the basic structural requirements are:

Strength: stresses should not exceed critical values: $\sigma < \sigma_f$

Stiffness: deflections should be controlled: $\Delta < \Delta_{max}$

Stability: buckling or cracking should also be prevented

2.7 Structural Design

14 Given a set of forces, **dimension** the structural element.

Steel/wood Structures Select appropriate section.

Reinforced Concrete: Determine dimensions of the element and internal reinforcement (number and sizes of reinforcing bars).

15 For **new structures**, **iterative** process between analysis and design. A preliminary design is made using **rules of thumbs** (best known to Engineers with design experience) and analyzed. Following design, we check for

Serviceability: deflections, crack widths under the applied load. Compare with acceptable values specified in the design code.

Failure: and compare the failure load with the applied load times the appropriate factors of safety.

If the design is found not to be acceptable, then it must be modified and reanalyzed.

16 For **existing structures rehabilitation**, or verification of an old infrastructure, analysis is the most important component.

17 In summary, analysis is always required.

2.8 Load Transfer Elements

18 From Strength of Materials, Fig. 2.1

Axial: cables, truss elements, arches, membrane, shells

Flexural: Beams, frames, grids, plates

Torsional: Grids, 3D frames

Shear: Frames, grids, shear walls.

Chapter 3

EQUILIBRIUM & REACTIONS

To every action there is an equal and opposite reaction.

Newton's third law of motion

3.1 Introduction

1 In the analysis of structures (hand calculations), it is often easier (but not always necessary) to start by determining the reactions.

2 Once the reactions are determined, internal forces are determined next; finally, deformations (deflections and rotations) are determined last[1].

3 Reactions are necessary to determine **foundation load**.

4 Depending on the type of structures, there can be different types of support conditions, Fig. 3.1.

Roller: provides a restraint in only one direction in a 2D structure, in 3D structures a roller may provide restraint in one or two directions. A roller will allow rotation.

Hinge: allows rotation but no displacements.

Fixed Support: will prevent rotation and displacements in all directions.

3.2 Equilibrium

5 Reactions are determined from the appropriate equations of static equilibrium.

6 Summation of forces and moments, **in a static system** must be equal to zero[2].

[1] This is the sequence of operations in the **flexibility** method which lends itself to hand calculation. In the **stiffness** method, we determine displacements firsts, then internal forces and reactions. This method is most suitable to computer implementation.

[2] In a dynamic system $\Sigma F = ma$ where m is the mass and a is the acceleration.

Structure Type	**Equations**					
Beam, no axial forces		ΣF_y				ΣM_z
2D Truss, Frame, Beam	ΣF_x	ΣF_y				ΣM_z
Grid			ΣF_z	ΣM_x	ΣM_y	
3D Truss, Frame	ΣF_x	ΣF_y	ΣF_z	ΣM_x	ΣM_y	ΣM_z
Alternate Set						
Beams, no axial Force	ΣM_z^A	ΣM_z^B				
2 D Truss, Frame, Beam	ΣF_x	ΣM_z^A	ΣM_z^B			
	ΣM_z^A	ΣM_z^B	ΣM_z^C			

Table 3.1: Equations of Equilibrium

3. The right hand side of the equation should be zero

If your reaction is negative, then it will be in a direction opposite from the one assumed.

16 Summation of all external forces (including reactions) is not necessarily zero (except at hinges and at points outside the structure).

17 Summation of external forces is equal and **opposite** to the internal ones. Thus the net force/moment is equal to zero.

18 The external forces give rise to the (non-zero) shear and moment diagram.

3.3 Equations of Conditions

19 If a structure has an **internal hinge** (which may connect two or more substructures), then this will provide an additional equation ($\Sigma M = 0$ at the hinge) which can be exploited to determine the reactions.

20 Those equations are often exploited in trusses (where each connection is a hinge) to determine reactions.

21 In an **inclined roller** support with S_x and S_y horizontal and vertical projection, then the reaction R would have, Fig. 3.2.

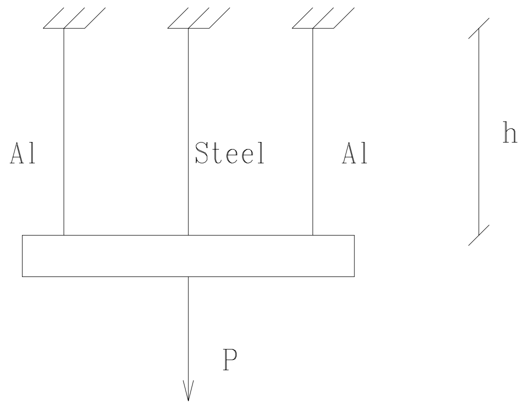
$$\boxed{\frac{R_x}{R_y} = \frac{S_y}{S_x}} \tag{3.3}$$

3.4 Static Determinacy

22 In statically determinate structures, reactions depend only on the geometry, boundary conditions and loads.

23 If the reactions can not be determined simply from the equations of static equilibrium (and equations of conditions if present), then the reactions of the structure are said to be **statically indeterminate**.

A rigid plate is supported by two aluminum cables and a steel one. Determine the force in each cable[5].



If the rigid plate supports a load P, determine the stress in each of the three cables. **Solution:**

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.

$$\begin{array}{rcl} \Sigma M_z = 0; & \Rightarrow & P_{Al}^{\text{left}} = P_{Al}^{\text{right}} \\ \Sigma F_y = 0; & \Rightarrow & 2P_{Al} + P_{St} = P \end{array}$$

   Thus we effectively have two unknowns and one equation.

2. We need to have a third equation to solve for the three unknowns. This will be derived from the **compatibility of the displacements** in all three cables, i.e. all three displacements must be equal:

$$\left.\begin{array}{rcl} \sigma & = & \frac{P}{A} \\ \varepsilon & = & \frac{\Delta L}{L} \\ \varepsilon & = & \frac{\sigma}{E} \end{array}\right\} \Rightarrow \Delta L = \frac{PL}{AE}$$

$$\underbrace{\frac{P_{Al}L}{E_{Al}A_{Al}}}_{\Delta_{Al}} = \underbrace{\frac{P_{St}L}{E_{St}A_{St}}}_{\Delta_{St}} \Rightarrow \frac{P_{Al}}{P_{St}} = \frac{(EA)_{Al}}{(EA)_{St}}$$

   or $-(EA)_{St}P_{Al} + (EA)_{Al}P_{St} = 0$

3. Solution of this system of two equations with two unknowns yield:

$$\begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix} \left\{ \begin{array}{c} P_{Al} \\ P_{St} \end{array} \right\} = \left\{ \begin{array}{c} P \\ 0 \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{c} P_{Al} \\ P_{St} \end{array} \right\} = \begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix}^{-1} \left\{ \begin{array}{c} P \\ 0 \end{array} \right\}$$

$$= \underbrace{\frac{1}{2(EA)_{Al} + (EA)_{St}}}_{\text{Determinant}} \begin{bmatrix} (EA)_{Al} & -1 \\ (EA)_{St} & 2 \end{bmatrix} \left\{ \begin{array}{c} P \\ 0 \end{array} \right\}$$

■

[5] This example problem will be the only statically indeterminate problem analyzed in CVEN3525.

Chapter 4

TRUSSES

4.1 Introduction

4.1.1 Assumptions

1 Cables and trusses are 2D or 3D structures composed of an assemblage of simple one dimensional components which transfer only **axial** forces along their axis.

2 Cables can carry only tensile forces, trusses can carry tensile and compressive forces.

3 Cables tend to be **flexible**, and hence, they tend to oscillate and therefore must be stiffened.

4 Trusses are extensively used for bridges, long span roofs, electric tower, space structures.

5 For trusses, it is assumed that

1. Bars are **pin-connected**
2. Joints are frictionless hinges[1].
3. Loads are applied at the **joints only**.

6 A truss would typically be composed of triangular elements with the bars on the **upper chord** under compression and those along the **lower chord** under tension. Depending on the **orientation of the diagonals**, they can be under either tension or compression.

7 Fig. 4.1 illustrates some of the most common types of trusses.

8 It can be easily determined that in a Pratt truss, the diagonal members are under tension, while in a Howe truss, they are in compression. Thus, the Pratt design is an excellent choice for steel whose members are slender and long diagonal member being in tension are not prone to buckling. The vertical members are less likely to buckle because they are shorter. On the other hand the Howe truss is often preferred for for heavy timber trusses.

9 In a truss analysis or design, we seek to determine the internal force along each member, Fig. 4.2

[1]In practice the bars are riveted, bolted, or welded directly to each other or to gusset plates, thus the bars are not free to rotate and so-called **secondary bending moments** are developed at the bars. Another source of secondary moments is the dead weight of the element.

4.1.2 Basic Relations

Sign Convention: Tension positive, compression negative. On a truss the axial forces are indicated as forces acting on the joints.

Stress-Force: $\sigma = \frac{P}{A}$

Stress-Strain: $\sigma = E\varepsilon$

Force-Displacement: $\varepsilon = \frac{\Delta L}{L}$

Equilibrium: $\Sigma F = 0$

4.2 Trusses

4.2.1 Determinacy and Stability

10 Trusses are **statically determinate** when all the bar forces can be determined from the equations of **statics** alone. Otherwise the truss is **statically indeterminate**.

11 A truss may be statically/externally determinate or indeterminate with respect to the reactions (more than 3 or 6 reactions in 2D or 3D problems respectively).

12 A truss may be internally determinate or indeterminate, Table 4.1.

13 If we refer to j as the number of joints, R the number of reactions and m the number of members, then we would have a total of $m + R$ unknowns and $2j$ (or $3j$) equations of statics (2D or 3D at each joint). If we do not have enough equations of statics then the problem is indeterminate, if we have too many equations then the truss is unstable, Table 4.1.

	2D	3D
Static Indeterminacy		
External	$R > 3$	$R > 6$
Internal	$m + R > 2j$	$m + R > 3j$
Unstable	$m + R < 2j$	$m + R < 3j$

Table 4.1: Static Determinacy and Stability of Trusses

14 If $m < 2j - 3$ (in 2D) the truss is not internally stable, and it will not remain a rigid body when it is detached from its supports. However, when attached to the supports, the truss will be rigid.

15 Since each joint is pin-connected, we can apply $\Sigma M = 0$ at each one of them. Furthermore, summation of forces applied on a joint must be equal to zero.

16 For 2D trusses the external equations of equilibrium which can be used to determine the reactions are $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma M_Z = 0$. For 3D trusses the available equations are $\Sigma F_X = 0$, $\Sigma F_Y = 0$, $\Sigma F_Z = 0$ and $\Sigma M_X = 0$, $\Sigma M_Y = 0$, $\Sigma M_Z = 0$.

17 For a 2D truss we have 2 equations of equilibrium $\Sigma F_X = 0$ and $\Sigma F_Y = 0$ which can be applied at each joint. For 3D trusses we would have three equations: $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma F_Z = 0$.

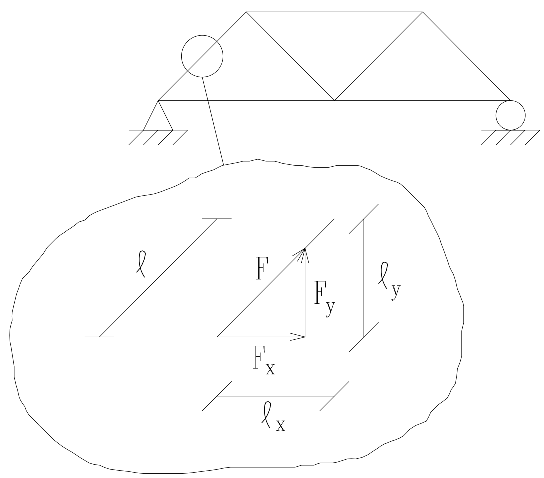


Figure 4.4: X and Y Components of Truss Forces

23 In truss analysis, there is **no sign convention**. A member is **assumed** to be under tension (or compression). If after analysis, the force is found to be negative, then this would imply that the wrong assumption was made, and that the member should have been under compression (or tension).

24 On a **free body diagram**, the internal forces are represented by arrow acting **on the joints** and not as end forces on the element itself. That is for tension, the arrow is pointing away from the joint, and for compression toward the joint, Fig. 4.5.

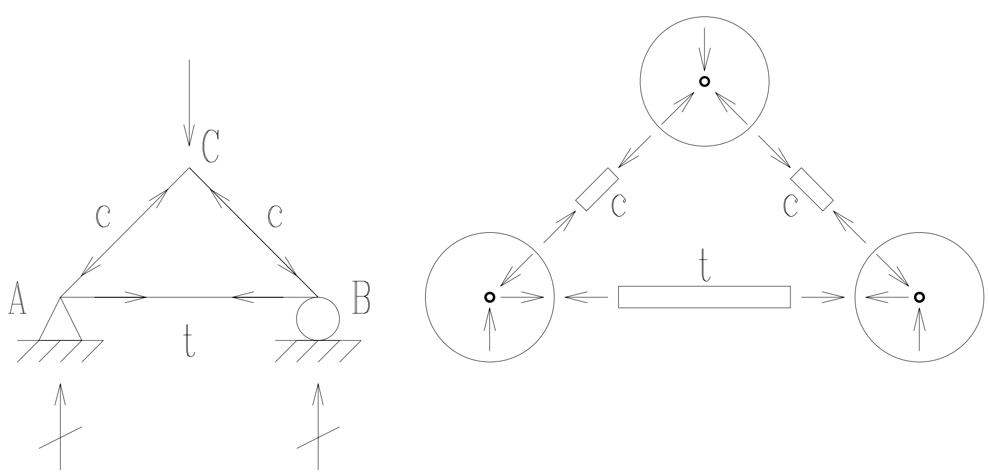
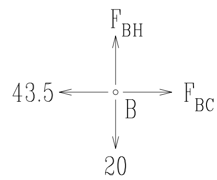


Figure 4.5: Sign Convention for Truss Element Forces

■ **Example 4-1: Truss, Method of Joints**

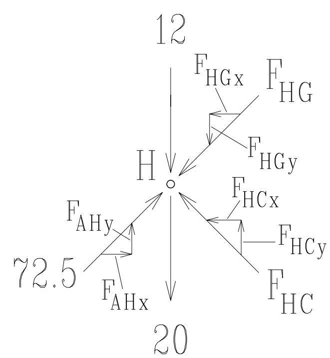
Using the method of joints, analyze the following truss

Node B:



$$(+\rightarrow)\ \Sigma F_x = 0;\ \Rightarrow\ F_{BC} = \boxed{43.5 \text{ k Tension}}$$
$$(+\uparrow)\ \Sigma F_y = 0;\ \Rightarrow\ F_{BH} = \boxed{20 \text{ k Tension}}$$

Node H:



$$\begin{aligned}(+\rightarrow)\ \Sigma F_x = 0;\ \Rightarrow\ & F_{AH_x} - F_{HC_x} - F_{HG_x} = 0\\ & 43.5 - \frac{24}{\sqrt{24^2+32^2}}(F_{HC}) - \frac{24}{\sqrt{24^2+10^2}}(F_{HG}) = 0\\ (+\uparrow)\ \Sigma F_y = 0;\ \Rightarrow\ & F_{AH_y} + F_{HC_y} - 12 - F_{HG_y} - 20 = 0\\ & 58 + \frac{32}{\sqrt{24^2+32^2}}(F_{HC}) - 12 - \frac{10}{\sqrt{24^2+10^2}}(F_{HG}) - 20 = 0\end{aligned}$$

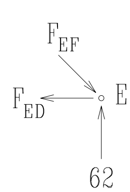
This can be most conveniently written as

$$\begin{bmatrix} 0.6 & 0.921 \\ -0.8 & 0.385 \end{bmatrix} \begin{Bmatrix} F_{HC} \\ F_{HG} \end{Bmatrix} = \begin{Bmatrix} -7.5 \\ 52 \end{Bmatrix} \tag{4.2}$$

Solving we obtain

$$F_{HC} = \boxed{-7.5 \text{ k Tension}}$$
$$F_{HG} = \boxed{52 \text{ k Compression}}$$

Node E:



$$\Sigma F_y = 0;\ \Rightarrow\ F_{EF_y} = 62\ \ \Rightarrow F_{EF} = \frac{\sqrt{24^2+32^2}}{32}(62) = \boxed{77.5 \text{ k}}$$
$$\Sigma F_x = 0;\ \Rightarrow\ F_{ED} = F_{EF_x}\ \ \Rightarrow F_{ED} = \frac{24}{32}(F_{EF_y}) = \frac{24}{32}(62) = \boxed{46.5 \text{ k}}$$

Chapter 5

CABLES

5.1 Funicular Polygons

1 A cable is a slender flexible member with zero or negligible flexural stiffness, thus it can only transmit **tensile** forces[1].

2 The tensile force at any point acts in the direction of the tangent to the cable (as any other component will cause bending).

3 Its strength stems from its ability to undergo extensive changes in slope at the point of load application.

4 Cables resist vertical forces by undergoing **sag** (h) and thus developing tensile forces. The horizontal component of this force (H) is called **thrust**.

5 The distance between the cable supports is called the **chord**.

6 The sag to span ratio is denoted by

$$r = \frac{h}{l} \tag{5.1}$$

7 When a set of concentrated loads is applied to a cable of negligible weight, then the cable deflects into a series of linear segments and the resulting shape is called the **funicular polygon**.

8 If a cable supports vertical forces only, then the horizontal component H of the cable tension T remains constant.

■ **Example 5-1: Funicular Cable Structure**

Determine the reactions and the tensions for the cable structure shown below.

[1] Due to the zero flexural rigidity it will buckle under axial compressive forces.

$$= \frac{H}{\cos\theta_B} = \frac{51}{0.999} = \boxed{51.03 \text{ k}} \quad (5.6\text{-d})$$

$$T_{CD}; \qquad \tan\theta_C = \frac{4.6}{30} = 0.153 \Rightarrow \theta_C = 8.7\,\text{deg} \quad (5.6\text{-e})$$

$$= \frac{H}{\cos\theta_C} = \frac{51}{0.988} = \boxed{51.62 \text{ k}} \quad (5.6\text{-f})$$

■

5.2 Uniform Load

5.2.1 qdx; Parabola

9 Whereas the forces in a cable can be determined from statics alone, its configuration must be derived from its deformation. Let us consider a cable with distributed load $p(x)$ **per unit horizontal projection** of the cable length[2]. An infinitesimal portion of that cable can be assumed to be a straight line, Fig. 31.1 and in the absence of any horizontal load we have

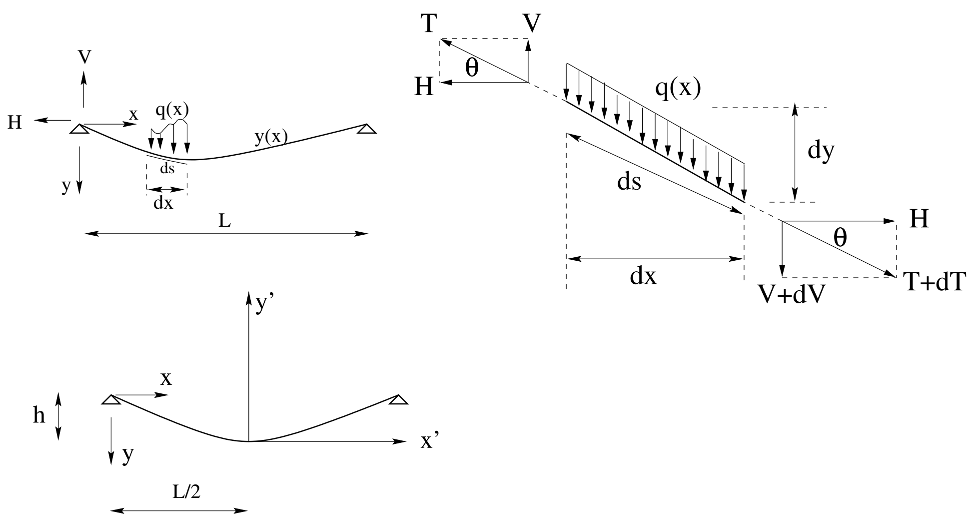


Figure 5.1: Cable Structure Subjected to $q(x)$

H =constant. Summation of the vertical forces yields

$$(+\downarrow)\,\Sigma F_y = 0 \Rightarrow -V + qdx + (V + dV) = 0 \quad (5.7\text{-a})$$

$$dV + qdx = 0 \quad (5.7\text{-b})$$

where V is the vertical component of the cable tension at x[3]. Because the cable must be tangent to T, we have

$$\tan\theta = \frac{V}{H} \quad (5.8)$$

[2] Thus neglecting the weight of the cable

[3] Note that if the cable was subjected to its own weight then we would have qds instead of pdx.

15 Combining Eq. 31.7 and 31.8 we obtain

$$y = \frac{4hx}{L^2}(L-x) \tag{5.18}$$

16 If we shift the origin to midspan, and reverse y, then

$$\boxed{y = \frac{4h}{L^2}x^2} \tag{5.19}$$

Thus the cable assumes a parabolic shape (as the moment diagram of the applied load).

17 The maximum tension occurs at the support where the vertical component is equal to $V = \frac{qL}{2}$ and the horizontal one to H, thus

$$T_{max} = \sqrt{V^2 + H^2} = \sqrt{\left(\frac{qL}{2}\right)^2 + H^2} = H\sqrt{1 + \left(\frac{qL/2}{H}\right)^2} \tag{5.20}$$

Combining this with Eq. 31.8 we obtain[5].

$$\boxed{T_{max} = H\sqrt{1+16r^2} \approx H(1+8r^2)} \tag{5.21}$$

5.2.2 † qds; Catenary

18 Let us consider now the case where the cable is subjected to its own weight (plus ice and wind if any). We would have to replace qdx by qds in Eq. 31.1-b

$$dV + qds = 0 \tag{5.22}$$

The differential equation for this new case will be derived exactly as before, but we substitute qdx by qds, thus Eq. 31.5 becomes

$$\boxed{\frac{d^2y}{dx^2} = -\frac{q}{H}\frac{ds}{dx}} \tag{5.23}$$

19 But $ds^2 = dx^2 + dy^2$, hence:

$$\frac{d^2y}{dx^2} = -\frac{q}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{5.24}$$

solution of this differential equation is considerably more complicated than for a parabola.

20 We let $dy/dx = p$, then

$$\frac{dp}{dx} = -\frac{q}{H}\sqrt{1+p^2} \tag{5.25}$$

[5] Recalling that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdot$ or $(1+b)^n = 1 + nb + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \cdots$; Thus for $b^2 << 1$, $\sqrt{1+b} = (1+b)^{\frac{1}{2}} \approx 1 + \frac{b}{2}$

Chapter 6

INTERNAL FORCES IN STRUCTURES

1 This chapter will start as a review of shear and moment diagrams which you have studied in both *Statics* and *Strength of Materials*, and will proceed with the analysis of statically determinate frames, arches and grids.

2 By the end of this lecture, you should be able to draw the shear, moment and torsion (when applicable) diagrams for each member of a structure.

3 Those diagrams will subsequently be used for member design. For instance, for flexural design, we will consider the section subjected to the highest moment, and make sure that the internal moment is equal and opposite to the external one. For the ASD method, the basic beam equation (derived in Strength of Materials) $\sigma = \frac{MC}{I}$, (where M would be the design moment obtained from the moment diagram) would have to be satisfied.

4 Some of the examples first analyzed in chapter 3 (Reactions), will be revisited here. Later on, we will determine the deflections of those same problems.

6.1 Design Sign Conventions

5 Before we (re)derive the Shear-Moment relations, let us *arbitrarily* define a sign convention.

6 The sign convention adopted here, is the one commonly used for design purposes[1].

7 With reference to Fig. 6.1

2D:

Load Positive along the beam's local y axis (assuming a right hand side convention), that is positive upward.

Axial: tension positive.

Flexure A positive moment is one which causes tension in the lower fibers, and compression in the upper ones. Alternatively, moments are drawn on the compression side (useful to keep in mind for frames).

[1] Later on, in more advanced analysis courses we will use a different one.

6.2 Load, Shear, Moment Relations

8 Let us (re)derive the basic relations between load, shear and moment. Considering an infinitesimal length dx of a beam subjected to a positive load[2] $w(x)$, Fig. 6.3. The infinitesimal

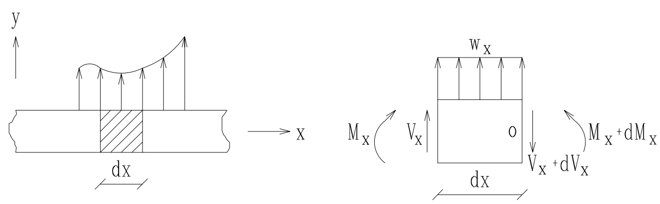


Figure 6.3: Free Body Diagram of an Infinitesimal Beam Segment

section must also be in equilibrium.

9 There are no axial forces, thus we only have two equations of equilibrium to satisfy $\Sigma F_y = 0$ and $\Sigma M_z = 0$.

10 Since dx is infinitesimally small, the small variation in load along it can be neglected, therefore we assume $w(x)$ to be constant along dx.

11 To denote that a small change in shear and moment occurs over the length dx of the element, we add the differential quantities dV_x and dM_x to V_x and M_x on the right face.

12 Next considering the first equation of equilibrium

$$(+\uparrow)\,\Sigma F_y = 0 \Rightarrow V_x + w_x dx - (V_x + dV_x) = 0$$

or

$$\boxed{\frac{dV}{dx} = w(x)} \tag{6.1}$$

> The slope of the shear curve at any point along the axis of a member is given by the load curve at that point.

13 Similarly

$$(+\circlearrowleft)\,\Sigma M_O = 0 \Rightarrow M_x + V_x dx - w_x dx\frac{dx}{2} - (M_x + dM_x) = 0$$

Neglecting the dx^2 term, this simplifies to

$$\boxed{\frac{dM}{dx} = V(x)} \tag{6.2}$$

> The slope of the moment curve at any point along the axis of a member is given by the shear at that point.

[2] In this derivation, as in all other ones we should assume all quantities to be positive.

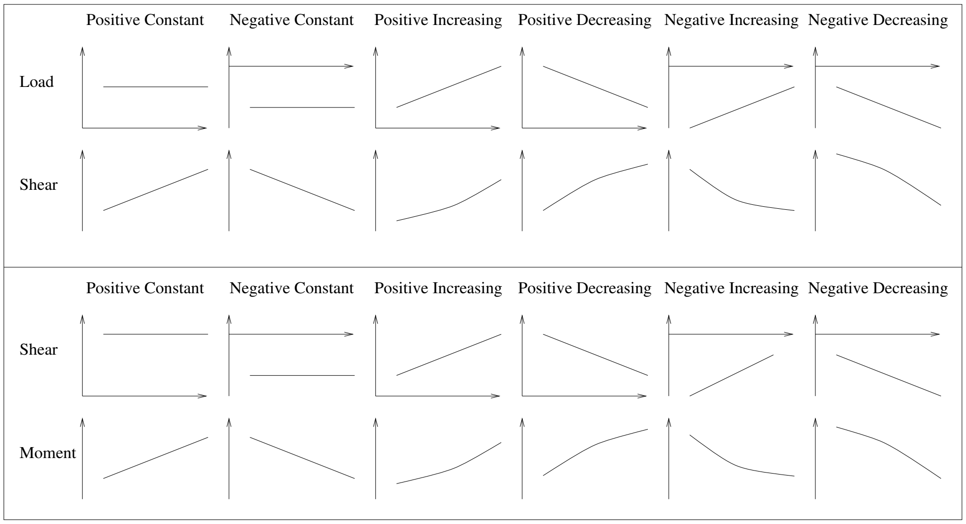


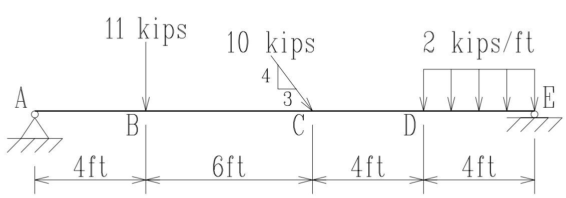
Figure 6.5: Slope Relations Between Load Intensity and Shear, or Between Shear and Moment

6.4 Examples

6.4.1 Beams

■ **Example 6-1: Simple Shear and Moment Diagram**

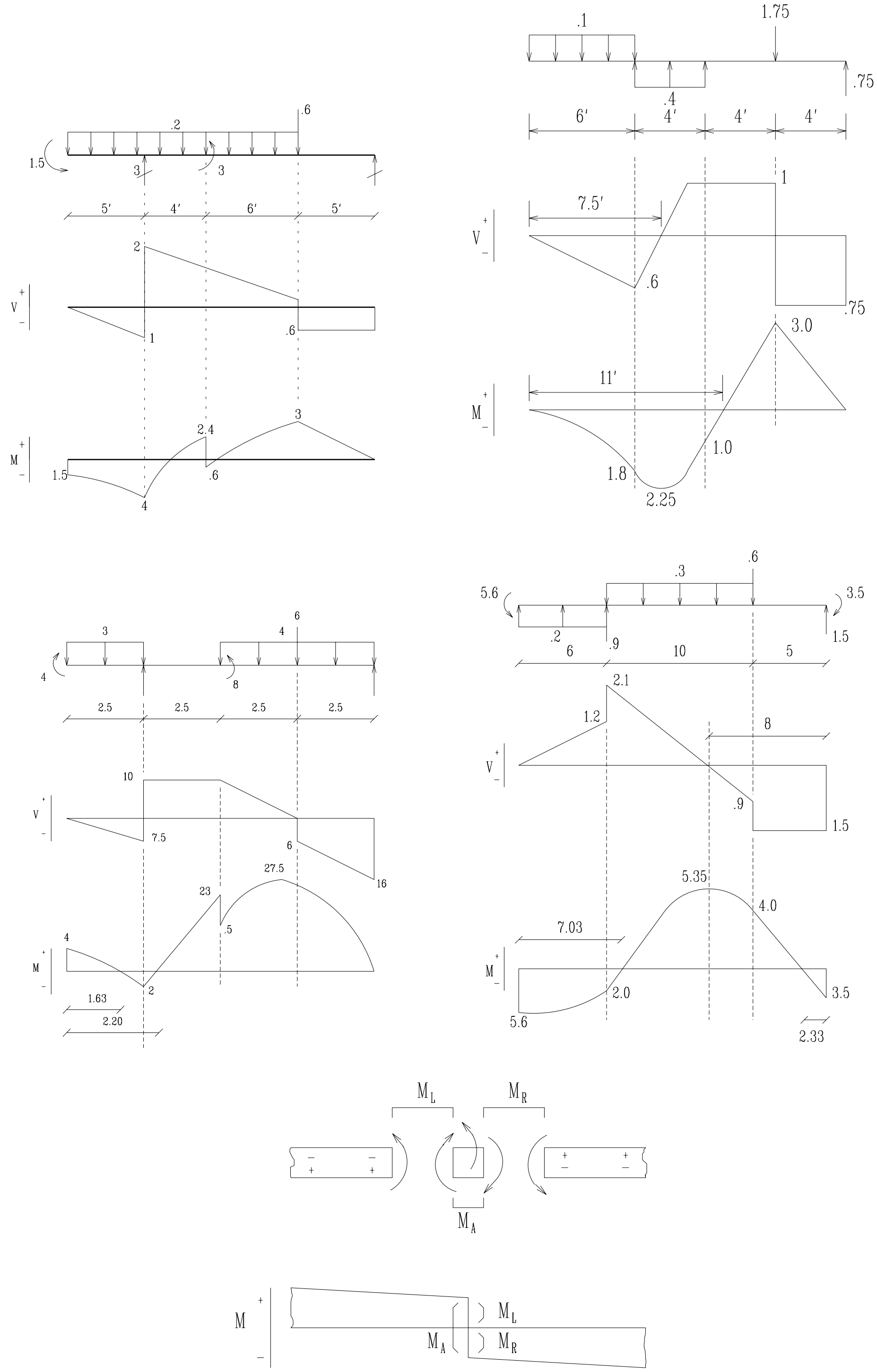
Draw the shear and moment diagram for the beam shown below



Solution:
The free body diagram is drawn below

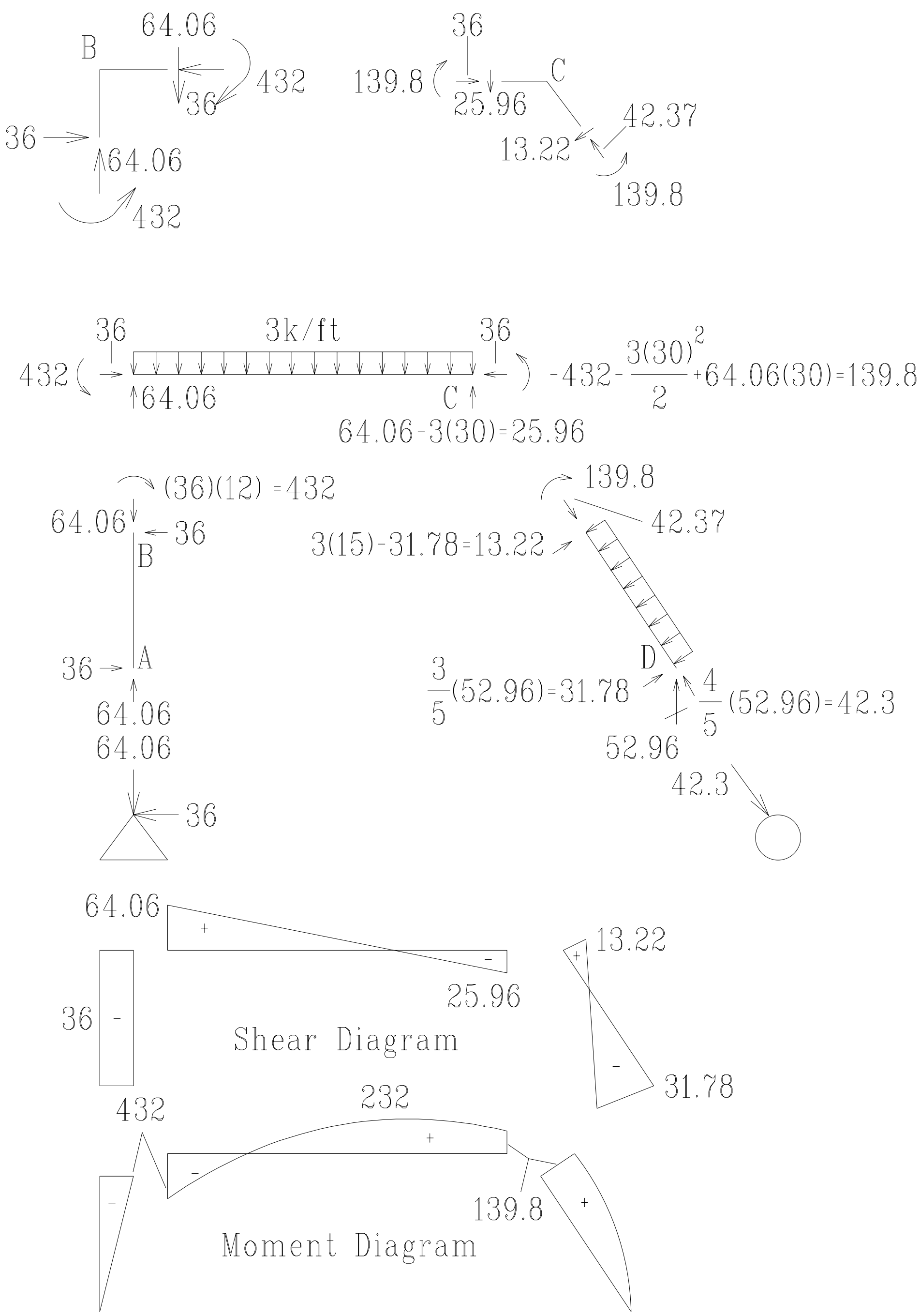
■ **Example 6-2: Sketches of Shear and Moment Diagrams**

For each of the following examples, sketch the shear and moment diagrams.



■

Solution:


B
64.06
432
36
36
64.06
432
36
139.8
25.96
C
42.37
13.22
139.8
36
3k/ft
36
432
64.06
C
-432-3(30)^2/2+64.06(30)=139.8
64.06-3(30)=25.96
(36)(12) =432
64.06
36
B
36
A
64.06
64.06
36
139.8
42.37
3(15)-31.78=13.22
D
3/5(52.96)=31.78
4/5(52.96)=42.3
52.96
42.3
64.06
25.96
36
Shear Diagram
13.22
31.78
432
232
139.8
Moment Diagram

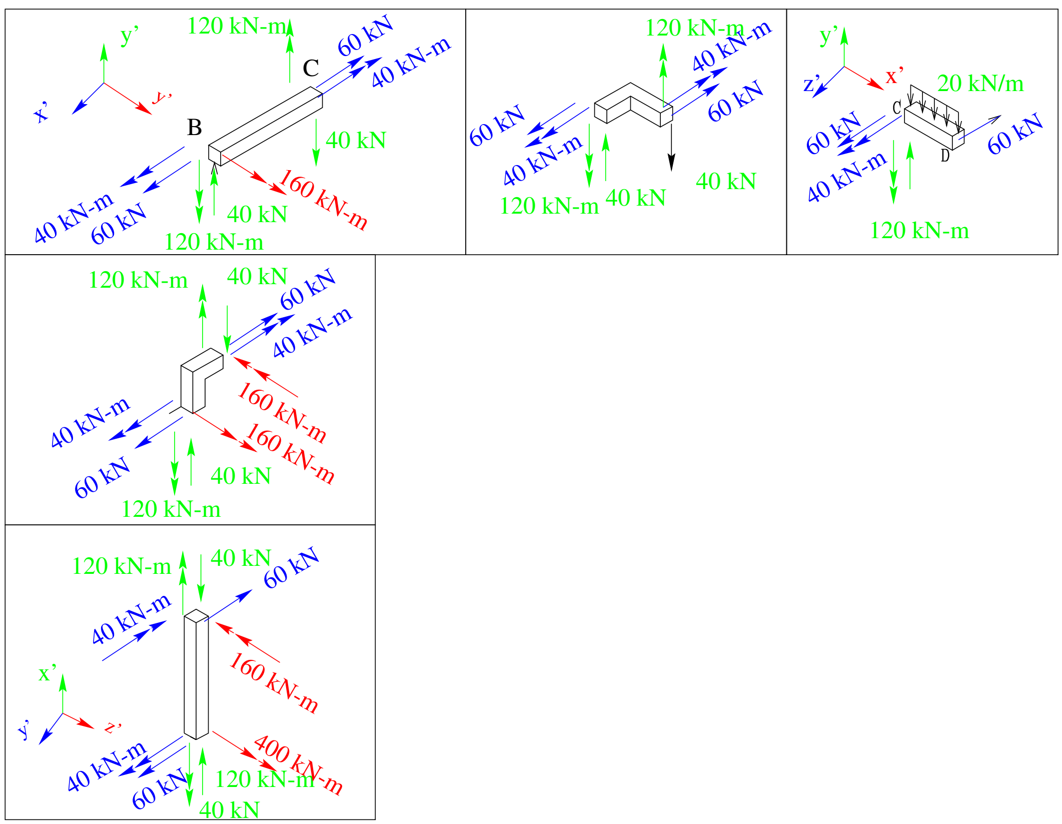
B-C

$$
\begin{array}{llllll}
\Sigma F_{x'} = 0 & \Rightarrow & N_{x'}^{B} & = V_{z'}^{C} & & = -60\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^{B} & = V_{y'}^{C} & & = +40\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^{B} & = M_{y'}^{C} & & = -120\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^{B} & = {V'}_{y}^{C}(4) & = (40)(4) & = +160\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T_{x'}^{B} & = -M_{z'}^{C} & & = -40\text{kN.m}
\end{array}
$$

A-B

$$
\begin{array}{lllll}
\Sigma F_{x'} = 0 & \Rightarrow & N_{x'}^{A} & = V_{y'}^{B} & = +40\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^{A} & = N_{x'}^{B} & = +60\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^{A} & = T_{x'}^{B} & = +40\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^{A} & = M_{z'}^{B} + N_{x'}^{B}(4) = 160 + (60)(4) & = +400\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T_{x'}^{A} & = M_{y'}^{B} & = -120\text{kN.m}
\end{array}
$$

The interaction between axial forces N and shear V as well as between moments M and torsion T is clearly highlighted by this example.



Chapter 7

ARCHES and CURVED STRUCTURES

1 This chapter will concentrate on the analysis of arches.

2 The concepts used are identical to the ones previously seen, however the major (and only) difference is that equations will be written in polar coordinates.

3 Like cables, arches can be used to reduce the bending moment in long span structures. Essentially, an arch can be considered as an inverted cable, and is transmits the load primarily through axial compression, but can also resist flexure through its flexural rigidity.

4 A parabolic arch uniformly loaded will be loaded in compression only.

5 A semi-circular arch unifirmly loaded will have some flexural stresses in addition to the compressive ones.

7.1 Arches

6 In order to optimize dead-load efficiency, long span structures should have their shapes approximate the coresponding moment diagram, hence an arch, suspended cable, or tendon configuration in a prestressed concrete beam all are nearly parabolic, Fig. 7.1.

7 Long span structures can be built using flat construction such as girders or trusses. However, for spans in excess of 100 ft, it is often more economical to build a curved structure such as an arch, suspended cable or thin shells.

8 Since the dawn of history, mankind has tried to span distances using arch construction. Essentially this was because an arch required materials to resist compression only (such as stone, masonary, bricks), and labour was not an issue.

9 The basic issues of static in arch design are illustrated in Fig. 7.2 where the vertical load is per unit horizontal projection (such as an external load but not a self-weight). Due to symmetry, the vertical reaction is simply $V = \frac{wL}{2}$, and there is no shear across the midspan of the arch (nor a moment). Taking moment about the crown,

$$M = Hh - \frac{wL}{2}\left(\frac{L}{2} - \frac{L}{4}\right) = 0 \tag{7.1}$$

Solving for H

$$\boxed{H = \frac{wL^2}{8h}} \tag{7.2}$$

We recall that a similar equation was derived for arches., and H is analogous to the $C - T$ forces in a beam, and h is the overall height of the arch, Since h is much larger than d, H will be much smaller than $C - T$ in a beam.

10 Since equilibrium requires H to remain constant across thee arch, a parabolic curve would theoretically result in no moment on the arch section.

11 Three-hinged arches are statically determinate structures which shape can acomodate support settlements and thermal expansion without secondary internal stresses. They are also easy to analyse through statics.

12 An arch carries the vertical load across the span through a combination of axial forces and flexural ones. A well dimensioned arch will have a small to negligible moment, and relatively high normal compressive stresses.

13 An arch is far more efficient than a beam, and possibly more economical and aesthetic than a truss in carrying loads over long spans.

14 If the arch has only two hinges, Fig. 7.3, or if it has no hinges, then bending moments may exist either at the crown or at the supports or at both places.

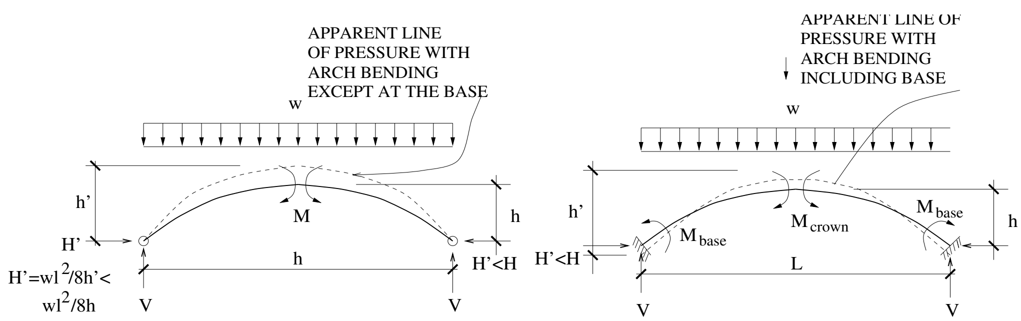


Figure 7.3: Two Hinged Arch, (Lin and Stotesbury 1981)

15 Since H varies inversely to the rise h, it is obvious that one should use as high a rise as possible. For a combination of aesthetic and practical considerations, a span/rise ratio ranging from 5 to 8 or perhaps as much as 12, is frequently used. However, as the ratio goes higher, we may have buckling problems, and the section would then have a higher section depth, and the arch advantage diminishes.

16 In a parabolic arch subjected to a uniform horizontal load there is no moment. However, in practice an arch is not subjected to uniform horizontal load. First, the depth (and thus the weight) of an arch is not usually constant, then due to the inclination of the arch the actual self weight is not constant. Finally, live loads may act on portion of the arch, thus the line of action will not necessarily follow the arch centroid. This last effect can be neglected if the live load is small in comparison with the dead load.

Solving those four equations simultaneously we have:

$$\begin{bmatrix} 140 & 26.25 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 80 & 60 & 0 & 0 \end{bmatrix} \begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix} = \begin{Bmatrix} 2,900 \\ 80 \\ 50 \\ 3,000 \end{Bmatrix} \Rightarrow \begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix} = \begin{Bmatrix} 15.1 \text{ k} \\ 29.8 \text{ k} \\ 34.9 \text{ k} \\ 50.2 \text{ k} \end{Bmatrix} \tag{7.4}$$

We can check our results by considering the summation with respect to b from the right:

$$(+\circlearrowleft)\ \Sigma M_z^B = 0; -(20)(20) - (50.2)(33.75) + (34.9)(60) = 0\surd \tag{7.5}$$

■

■ **Example 7-2: Semi-Circular Arch, (Gerstle 1974)**

Determine the reactions of the three hinged statically determined semi-circular arch under its own dead weight w (per unit arc length s, where $ds = rd\theta$). 7.6

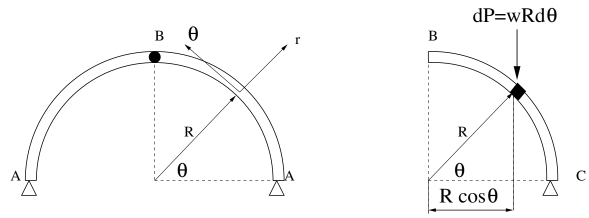


Figure 7.6: Semi-Circular three hinged arch

Solution:

I Reactions The reactions can be determined by **integrating** the load over the entire structure

1. **Vertical Reaction** is determined first:

$$(+\circlearrowright)\ \Sigma M_A = 0; -(C_y)(2R) + \int_{\theta=0}^{\theta=\pi} \underbrace{wRd\theta}_{dP}\ \underbrace{R(1+\cos\theta)}_{\text{moment arm}} = 0 \tag{7.6-a}$$

$$\Rightarrow C_y = \frac{wR}{2}\int_{\theta=0}^{\theta=\pi}(1+\cos\theta)d\theta = \frac{wR}{2}[\theta - \sin\theta]\,|_{\theta=0}^{\theta=\pi}$$

$$= \frac{wR}{2}[(\pi - \sin\pi) - (0 - \sin 0)]$$

$$= \boxed{\tfrac{\pi}{2}wR} \tag{7.6-b}$$

2. **Horizontal Reactions** are determined next

$$(+\circlearrowleft)\ \Sigma M_B = 0; -(C_x)(R) + (C_y)(R) - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \underbrace{wRd\theta}_{dP}\ \underbrace{R\cos\theta}_{\text{moment arm}} = 0 \tag{7.7-a}$$

Chapter 8

DEFLECTION of STRUCTRES; Geometric Methods

1 Deflections of structures must be determined in order to satisfy serviceability requirements i.e. limit deflections under *service* loads to acceptable values (such as $\frac{\Delta}{L} \leq 360$).

2 Later on, we will see that deflection calculations play an important role in the analysis of statically indeterminate structures.

3 We shall focus on flexural deformation, however the end of this chapter will review axial and torsional deformations as well.

4 Most of this chapter will be a *review* of subjects covered in *Strength of Materials*.

5 This chapter will examine deflections of structures based on geometric considerations. Later on, we will present a more pwerful method based on energy considerations.

8.1 Flexural Deformation

8.1.1 Curvature Equation

6 Let us consider a segment (between point 1 and point 2), Fig. 8.1 of a beam subjected to flexural loading.

7 The **slope** is denoted by θ, the change in slope per unit length is the **curvature** κ, the **radius of curvature** is ρ.

8 From *Strength of Materials* we have the following relations

$$ds = \rho d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho} \tag{8.1}$$

9 We also note by extension that $\Delta s = \rho \Delta \theta$

10 As a first order approximation, and with $ds \approx dx$ and $\frac{dy}{dx} = \theta$ Eq. 8.1 becomes

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2 y}{dx^2} \tag{8.2}$$

14 Thus the slope θ, curvature κ, radius of curvature ρ are related to the y displacement at a point x along a flexural member by

$$\boxed{\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}} \tag{8.9}$$

15 If the displacements are very small, we will have $\frac{dy}{dx} << 1$, thus Eq. 8.9 reduces to

$$\boxed{\kappa = \frac{d^2y}{dx^2} = \frac{1}{\rho}} \tag{8.10}$$

8.1.2 Differential Equation of the Elastic Curve

16 Again with reference to Figure 8.1 a positive $d\theta$ at a positive y (upper fibers) will cause a *shortening* of the upper fibers

$$\Delta u = -y\Delta\theta \tag{8.11}$$

17 This equation can be rewritten as

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = -y \lim_{\Delta s \to 0} \frac{\Delta \theta}{\Delta s} \tag{8.12}$$

and since $\Delta s \approx \Delta x$

$$\underbrace{\frac{du}{dx}}_{\varepsilon} = -y\frac{d\theta}{dx} \tag{8.13}$$

Combining this with Eq. 8.10

$$\boxed{\frac{1}{\rho} = \kappa = -\frac{\varepsilon}{y}} \tag{8.14}$$

This is the fundamental relationship between curvature (κ), elastic curve (y), and linear strain (ε).

18 Note that so far we made no assumptions about material properties, i.e. it can be elastic or inelastic.

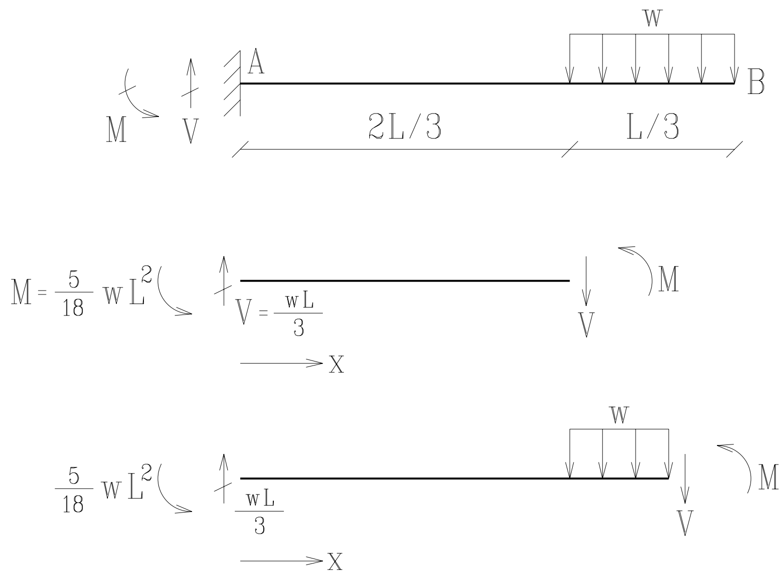
19 For the elastic case:

$$\left.\begin{array}{rcl}\varepsilon_x & = & \frac{\sigma}{E} \\ \sigma & = & -\frac{My}{I}\end{array}\right\} \varepsilon = -\frac{My}{EI} \tag{8.15}$$

Combining this last equation with Eq. 8.14 yields

$$\boxed{\frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI}} \tag{8.16}$$

This fundamental equation relates moment to curvature.



Solution:

At: $0 \leq x \leq \frac{2L}{3}$

1. Moment Equation
$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 \tag{8.22}$$
2. Integrate once
$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x + C_1 \tag{8.23}$$
However we have at $x = 0$, $\frac{dy}{dx} = 0$, $\Rightarrow C_1 = 0$
3. Integrate twice
$$EIy = \frac{wL}{18}x^3 - \frac{5wL^2}{36}x^2 + C_2 \tag{8.24}$$
Again we have at $x = 0$, $y = 0$, $\Rightarrow C_2 = 0$

At: $\frac{2L}{3} \leq x \leq L$

1. Moment equation
$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 - w(x - \frac{2L}{3})(\frac{x - \frac{2L}{3}}{2}) \tag{8.25}$$
2. Integrate once
$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x - \frac{w}{6}(x - \frac{2L}{3})^3 + C_3 \tag{8.26}$$
Applying the boundary condition at $x = \frac{2L}{3}$, we must have $\frac{dy}{dx}$ equal to the value coming from the left, $\Rightarrow C_3 = 0$

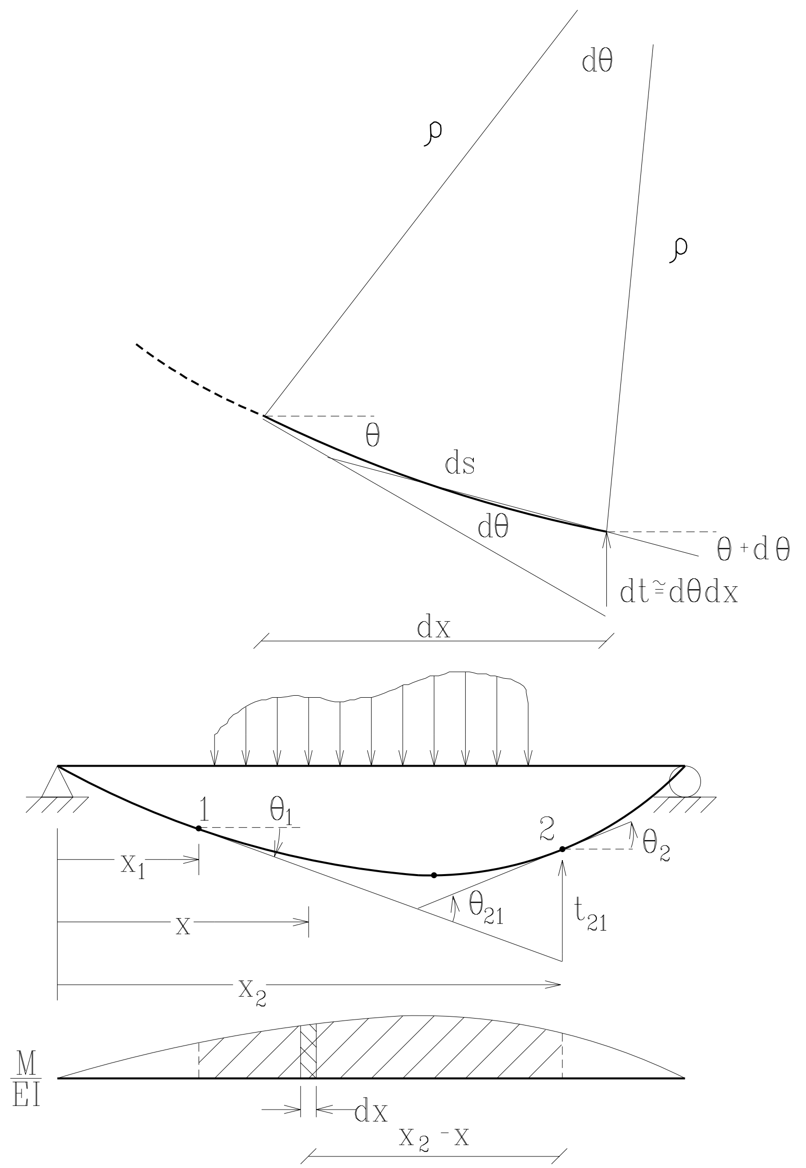


Figure 8.2: Moment Area Theorems

Chapter 9

ENERGY METHODS; Part I

9.1 Introduction

1 Energy methods are powerful techniques for both formulation (of the stiffness matrix of an element[1]) and for the analysis (i.e. deflection) of structural problems.

2 We shall explore two techniques:

1. Real Work
2. Virtual Work (Virtual force)

9.2 Real Work

3 We start by revisiting the first law of thermodynamics:

> The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time.

$$\boxed{\frac{d}{dt}(K+U) = W_e + H} \tag{9.1}$$

where K is the kinetic energy, U the internal strain energy, W_e the external work, and H the heat input to the system.

4 For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \tag{9.2}$$

5 Simply stated, the first law stipulates that the external work must be equal to the internal strain energy due to the external load.

[1] More about this in *Matrix Structural Analysis*.

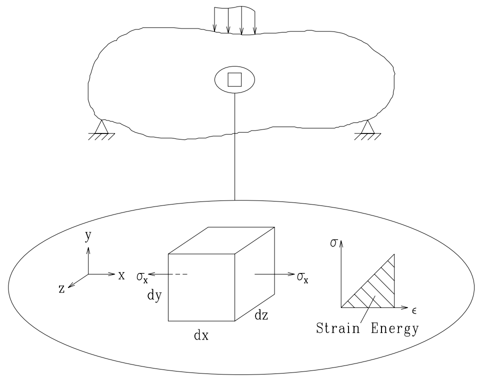


Figure 9.2: Strain Energy Definition

9.2.2 Internal Work

9 Considering an infinitesimal element from an arbitrary structure subjected to uniaxial state of stress, the strain energy can be determined with reference to Fig. 9.2. The net force acting on the element while deformation is taking place is $P = \sigma_x dydz$. The element will undergo a displacement $u = \varepsilon_x dx$. Thus, for a linear elastic system, the strain energy density is $dU = \frac{1}{2}\sigma\varepsilon$. And the total strain energy will thus be

$$\boxed{U = \frac{1}{2}\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol}} \tag{9.7}$$

10 When this relation is applied to various structural members it would yield:

Axial Members:

$$\left.\begin{array}{rcl} U & = & \int_{\text{Vol}} \frac{\varepsilon\sigma}{2} d\text{Vol} \\ \sigma & = & \frac{P}{A} \\ \varepsilon & = & \frac{P}{AE} \\ dV & = & Adx \end{array}\right\} \boxed{U = \int_0^L \frac{P^2}{2AE}dx} \tag{9.8}$$

Torsional Members:

$$\left.\begin{array}{rcl} U & = & \frac{1}{2}\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol} \\ U & = & \frac{1}{2}\int_{\text{Vol}} \gamma_{xy} \underbrace{G\gamma_{xy}}_{\tau_{xy}} d\text{Vol} \\ \tau_{xy} & = & \frac{Tr}{J} \\ \gamma_{xy} & = & \frac{\tau_{xy}}{G} \\ d\text{Vol} & = & rd\theta drdx \\ J & = & \int_o^r \int_0^{2\pi} r^2 d\theta\, dr \end{array}\right\} \boxed{U = \int_0^L \frac{T^2}{2GJ}dx} \tag{9.9}$$

9.3 Virtual Work

11 A severe limitation of the method of real work is that only deflection along the externally applied load can be determined.

12 A more powerful method is the virtual work method.

13 The principle of Virtual Force (VF) relates *force* systems which satisfy the requirements of *equilibrium*, and *deformation* systems which satisfy the requirement of *compatibility*

14 In any application the force system could either be the actual set of **external** loads $d\mathbf{p}$ or some **virtual** force system which happens to satisfy the condition of *equilibrium* $\delta\overline{\mathbf{p}}$. This set of external forces will induce internal actual forces $d\boldsymbol{\sigma}$ or internal virtual forces $\delta\overline{\boldsymbol{\sigma}}$ compatible with the externally applied load.

15 Similarly the deformation could consist of either the actual joint deflections $d\mathbf{u}$ and compatible internal deformations $d\boldsymbol{\varepsilon}$ of the structure, or some **virtual** external and internal deformation $\delta\overline{\mathbf{u}}$ and $\delta\overline{\boldsymbol{\varepsilon}}$ which satisfy the conditions of *compatibility*.

16 It is often simplest to assume that the **virtual load is a unit load**.

17 Thus we may have 4 possible combinations, Table 9.1: where: d corresponds to the actual,

	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$		
2	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$	$\delta\overline{U}^*$	Flexibility
3	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$	$\delta\overline{U}$	Stiffness
4	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$		

Table 9.1: Possible Combinations of Real and Hypothetical Formulations

and δ (with an overbar) to the hypothetical values.

This table calls for the following observations

1. The second approach is the same one on which the method of virtual or unit load is based. It is simpler to use than the third as a internal force distribution compatible with the assumed virtual force can be easily obtained for statically determinate structures. This approach will yield exact solutions for statically determinate structures.

2. The third approach is favored for statically indeterminate problems or in conjunction with approximate solution. It requires a proper "guess" of a displacement shape and is the basis of the stiffness method.

18 Let us consider an arbitrary structure and load it with both real and virtual loads in the following sequence, Fig. 9.4. For the sake of simplicity, let us assume (or consider) that this structure develops only axial stresses.

1. If we apply the virtual load, then

$$\frac{1}{2}\delta\overline{P}\delta\overline{\Delta} = \frac{1}{2}\int_{d\mathrm{Vol}} \delta\overline{\sigma}\delta\overline{\varepsilon}\, d\mathrm{Vol} \qquad (9.12)$$

Chapter 10

STATIC INDETERMINANCY; FLEXIBILITY METHOD

`All the examples in this chapter are taken verbatim from White, Gergely and Sexmith`

10.1 Introduction

1 A statically indeterminate structure has more unknowns than equations of equilibrium (and equations of conditions if applicable).

2 The advantages of a statically indeterminate structures are:

1. Lower internal forces
2. Safety in redundancy, i.e. if a support or members fails, the structure can *redistribute* its internal forces to accomodate the changing B.C. without resulting in a sudden failure.

3 Only disadvantage is that it is more complicated to analyse.

4 Analysis mehtods of statically indeterminate structures *must satisfy* three requirements

Equilibrium

Force-displacement (or stress-strain) relations (linear elastic in this course).

Compatibility of displacements (i.e. no discontinuity)

5 This can be achieved through two classes of solution

Force or Flexibility method;

Displacement or Stiffness method

6 The flexibility method is first illustrated by the following problem of a statically indeterminate cable structure in which a rigid plate is supported by two aluminum cables and a steel one. We seek to determine the force in each cable, Fig. 10.1

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.

6. We observe that the solution of this sproblem, contrarily to statically determinate ones, depends on the elastic properties.

7 Another example is the propped cantilever beam of length L, Fig. 10.2

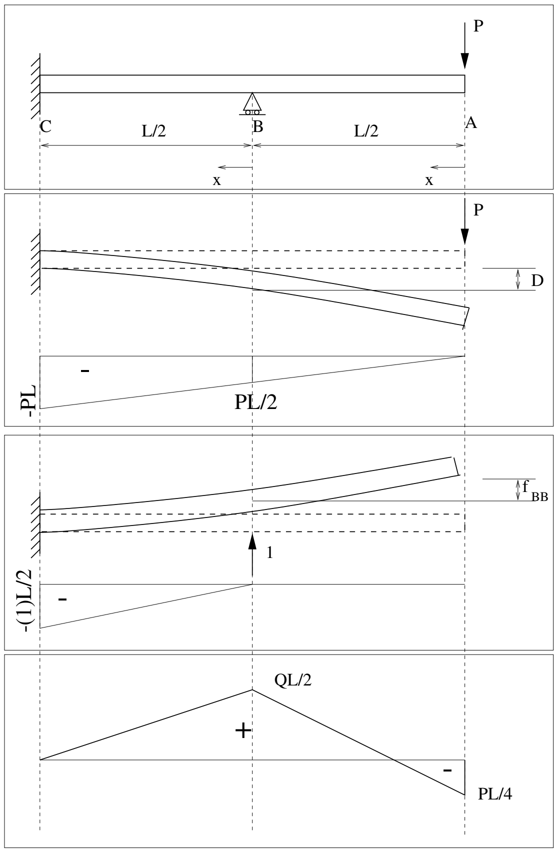


Figure 10.2: Propped Cantilever Beam

1. First we remove the roller support, and are left with the cantilever as a primary structure.
2. We then determine the deflection at point B due to the applied load P using the virtual force method

$$1.D = \int \delta\overline{M}\frac{M}{EI}dx \quad (10.6\text{-a})$$

$$= \int_0^{L/2} 0\frac{-px}{EI}dx + \int_0^{L/2} -\left(\frac{PL}{2} + Px\right)(-x)dx \quad (10.6\text{-b})$$

$$= \frac{1}{EI}\int_0^{L/2}\left(\frac{PL}{2}x + Px^2\right)dx \quad (10.6\text{-c})$$

$$= \frac{1}{EI}\left[\frac{PLx^2}{4} + \frac{Px^3}{3}\right]\Bigg|_0^{L/2} \quad (10.6\text{-d})$$

$$= \frac{5}{48}\frac{PL^3}{EI} \quad (10.6\text{-e})$$

Note that $\mathbf{D}_i^0$ is the vector of initial displacements, which is usually zero unless we have an initial displacement of the support (such as support settlement).

7. The reactions are then obtained by simply inverting the flexibility matrix.

9 Note that from Maxwell-Betti's reciprocal theorem, the flexibility matrix $[\mathbf{f}]$ is always symmetric.

10.3 Short-Cut for Displacement Evaluation

10 Since deflections due to flexural effects must be determined numerous times in the flexibility method, Table 10.1 may simplify some of the evaluation of the internal strain energy. *You are strongly discouraged to use this table while still at school!.*

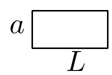
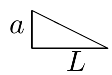
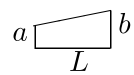
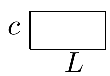
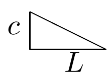
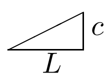
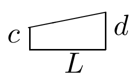
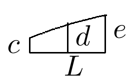
	$g_2(x)$		
$g_1(x)$			
	Lac	$\frac{Lac}{2}$	$\frac{Lc(a+b)}{2}$
	$\frac{Lac}{2}$	$\frac{Lac}{3}$	$\frac{Lc(2a+b)}{6}$
	$\frac{Lac}{2}$	$\frac{Lac}{6}$	$\frac{Lc(a+2b)}{6}$
	$\frac{La(c+d)}{2}$	$\frac{La(2c+d)}{6}$	$\frac{La(2c+d)+Lb(c+2d)}{6}$
	$\frac{La(c+4d+e)}{6}$	$\frac{La(c+2d)}{6}$	$\frac{La(c+2d)+Lb(2d+e)}{6}$

Table 10.1: Table of $\int_0^L g_1(x)g_2(x)dx$

10.4 Examples

■ Example 10-1: Steel Building Frame Analysis, (White et al. 1976)

A small, mass-produced industrial building, Fig. 10.3, is to be framed in structural steel with a typical cross section as shown below. The engineer is considering three different designs for the frame: (a) for poor or unknown soil conditions, the foundations for the frame may not be able to develop any dependable horizontal forces at its bases. In this case the idealized base conditions are a hinge at one of the bases and a roller at the other; (b) for excellent

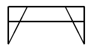
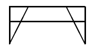
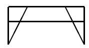
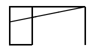
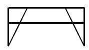
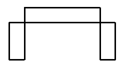
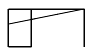
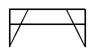
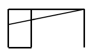
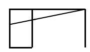
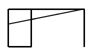
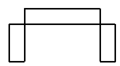
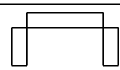
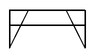
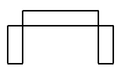
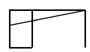
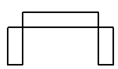
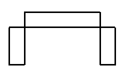
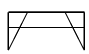
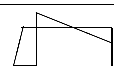
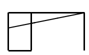
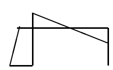
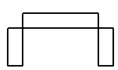
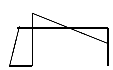
D	$\delta\overline{M}$	M	$\int \delta\overline{M} M dx$			$\int \delta\overline{M} \frac{M}{EI} dx$
			AB	BC	CD	Total
f_{11}			$+\frac{h^3}{3}$	$+Lh^2$	$+\frac{h^3}{3}$	$+\frac{2h^3}{3EI_c} + \frac{Lh^2}{EI_b}$
f_{12}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{13}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{21}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{22}			$+L^2h$	$+\frac{L^3}{3}$	0	$+\frac{L^2h}{EI_c} + \frac{L^3}{3EI_b}$
f_{23}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{31}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{32}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{33}			$+h$	$+L$	$+h$	$+\frac{2h}{EI_c} + \frac{L}{EI_b}$
D_{1Q}			$-\frac{h^2(2h+15L-30)}{6}$	$+\frac{Lh(20-5L)}{2}$	0	$-\frac{h^2(2h+15L-30)}{6EI_c} + \frac{Lh(20-5L)}{2EI_b}$
D_{2Q}			$-\frac{Lh(2h+10L-20)}{2}$	$+\frac{L^2(30-10L)}{6}$	0	$-\frac{Lh(2h+10L-20)}{2EI_c} + \frac{L^2(30-10L)}{6EI_b}$
D_{3Q}			$+\frac{h(2h+10L-20)}{2}$	$-\frac{L(20-5L)}{2}$	0	$-\frac{h(2h+10L-20)}{2EI_c} - \frac{L(20-5L)}{2EI_b}$

Table 10.3: Displacement Computations for a Rectangular Frame

3. In the following discussion the contributions to displacements due to axial strain are denoted with a single prime ($\prime$) and those due to curvature by a double prime ($\prime\prime$).
4. Consider the axial strain first. A unit length of frame member shortens as a result of the temperature decrease from 85°F to 45°F at the middepth of the member. The strain is therefore

$$\alpha\Delta T = (0.0000055)(40) = 0.00022 \tag{10.56}$$

5. The effect of axial strain on the relative displacements needs little analysis. The horizontal member shortens by an amount $(0.00022)(20) = 0.0044$ ft. The shortening of the vertical members results in no relative displacement in the vertical direction 2. No rotation occurs.
6. We therefore have $D'_{1\Delta} = -0.0044$ ft, $D'_{2\Delta} = 0$, and $D'_{3\Delta} = 0$.

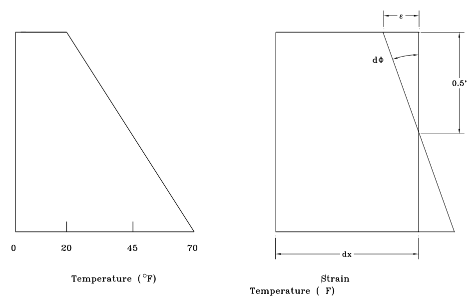


Figure 10.16:

7. The effect of curvature must also be considered. A frame element of length dx undergoes an angular strain as a result of the temperature gradient as indicated in Figure 10.16. The change in length at an extreme fiber is

$$\epsilon = \alpha\Delta T dx = 0.0000055(25)dx = 0.000138dx \tag{10.57}$$

8. with the resulting real rotation of the cross section

$$d\phi = \epsilon/0.5 = 0.000138dx/0.5 = 0.000276dx \text{ radians} \tag{10.58}$$

9. The relative displacements of the primary structure at D are found by the virtual force method.
10. A virtual force $\delta\overline{Q}$ is applied in the direction of the desired displacement and the resulting moment diagram $\delta\overline{M}$ determined.
11. The virtual work equation

$$\delta\overline{Q}D = \int \delta\overline{M}d\phi \tag{10.59}$$

is used to obtain each of the desired displacements D.
12. The results, which you should verify, are

$$D''_{1\Delta} = \boxed{0.0828 \text{ ft}} \tag{10.60-a}$$
$$D''_{2\Delta} = \boxed{0.1104 \text{ ft}} \tag{10.60-b}$$
$$D''_{3\Delta} = \boxed{-0.01104 \text{ radians}} \tag{10.60-c}$$

with units in kips and feet.
21. A moment diagram may now be constructed, and other internal force quantities computed from the now known values of the redundants. The redundants have been valuated separately for effects of temperature and foundation settlement. These effects may be combined with those due to loading using the principle of superposition. ■

■ Example 10-9: Braced Bent with Loads and Temperature Change, (White et al. 1976)

The truss shown in Figure 10.17 reperesents an internal braced bent in an enclosed shed, with lateral loads of 20 kN at the panel points. A temperature drop of 30°C may occur on the outer members (members 1-2, 2-3, 3-4, 4-5, and 5-6). We wish to analyze the truss for the loading and for the temperature effect.
Solution:

1. The first step in the analysis is the definition of the two redundants. The choice of forces in diagonals 2-4 and 1-5 as redundants facilitates the computations because some of the load effects are easy to analyze. Figure **??**-b shows the definition of R_1 and R_2.
2. The computations are organized in tabular form in Table 10.4. The first column gives the bar forces P in the primary structure caused by the actual loads. Forces are in kN. Column 2 gives the force in each bar caused by a unit load (1 kN) corresponding to release 1. These are denoted p_1 and also represent the bar force $\bar{q}_1/\delta\overline{Q}_1$ caused by a virtual force $\delta\overline{Q}_1$ applied

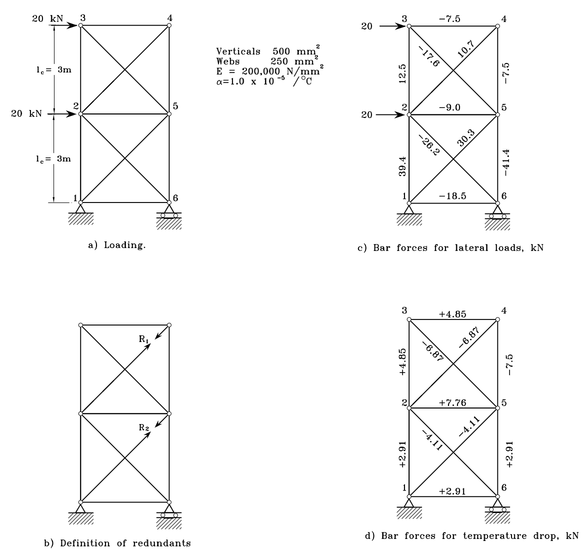


Figure 10.17:

at the same location. Column 3 lists the same quantity for a unit load and for a virtual force $\delta\overline{Q}_2$ applied at release 2. These three columns constitute a record of the truss analysis needed for this problem.

					D_{1Q}	D_{2Q}	f_{11}	f_{21}	f_{12}	f_{22}		$D_{1\Delta}$	$D_{2\Delta}$
	P	p_1	p_2	L/EA	$\bar{q}_1PL/EA$	$\bar{q}_2PL/EA$	$\bar{q}_1p_1L/EA$	$\bar{q}_2p_2L/EA$	$\bar{q}p_2L/EA$	$\bar{q}_2p_2L/EA$	$\Delta/_{temp}$	$\bar{q}_1\Delta/$	$\bar{q}_2\Delta/$
multiply by				L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c	$10^{-4}L_c$	$10^{-4}L_c$
1-2	60.0	0	−0.707	1	0	−42.42	0	0	0	0.50	−0.0003	0	2.12
2-3	20.00	−0.707	0	1	−14.14	0	0.50	0	0	0	−0.0003	2.12	0
3-4	0	−0.707	0	2	0	0	1.00	0	0	0	−0.0003	2.12	0
4-5	0	−0.707	0	1	0	0	0.50	0	0	0	−0.0003	2.12	0
5-6	−20.00	0	−0.707	1	0	14.14	0	0	0	0.50	−0.0003	0	2.12
6-1	40.00	0	−0.707	2	0	−56.56	0	0	0	1.00	0	0	0
2-5	20.00	−0.707	−0.707	2	−28.28	−28.28	1.00	1.00	1.00	1.00	0	0	0
1-5	0	0	1.00	2.828	0	0	0	0	0	2.83	0	0	0
2-6	−56.56	0	1.00	2.828	0	−160.00	0	0	0	2.83	0	0	0
2-4	0	1.00	0	2.828	0	0	2.83	0	0	0	0	0	0
3-5	−28.28	1.00	0	2.838	−80.00	0	2.83	0	0	0	0	0	0
					−122.42	−273.12	8.66	1.00	1.00	8.66		6.36	4.24

Chapter 11

APPROXIMATE FRAME ANALYSIS

1 Despite the widespread availability of computers, approximate methods of analysis are justified by

1. Inherent assumption made regarding the validity of a linear elastic analysis *vis a vis* of an ultimate failure design.
2. Ability of structures to redistribute internal forces.
3. Uncertainties in load and material properties

2 Vertical loads are treated separately from the horizontal ones.

3 We use the design sign convention for moments (+ve tension below), and for shear (ccw +ve).

4 Assume girders to be numbered from left to right.

5 In all free body diagrams assume positivee forces/moments, and take algeebraic sums.

6 The key to the approximate analysis method is our ability to sketch the deflected shape of a structure and identify inflection points.

7 We begin by considering a uniformly loaded beam and frame. In each case we consider an extreme end of the restraint: a) free or b) restrained. For the frame a relativly flexible or stiff column would be analogous to a free or fixed restrain on the beam, Fig. 11.1.

11.1 Vertical Loads

8 With reference to Fig. 11.1, we now consider an *intermediary* case as shown in Fig. 11.2.

9 With the location of the inflection points identified, we may now determine all the reactions and internal forces from statics.

10 If we now consider a multi-bay/multi-storey frame, the girders at each floor are assumed to be continuous beams, and columns are assumed to resist the resulting unbalanced moments from the girders, we may make the following assumptions

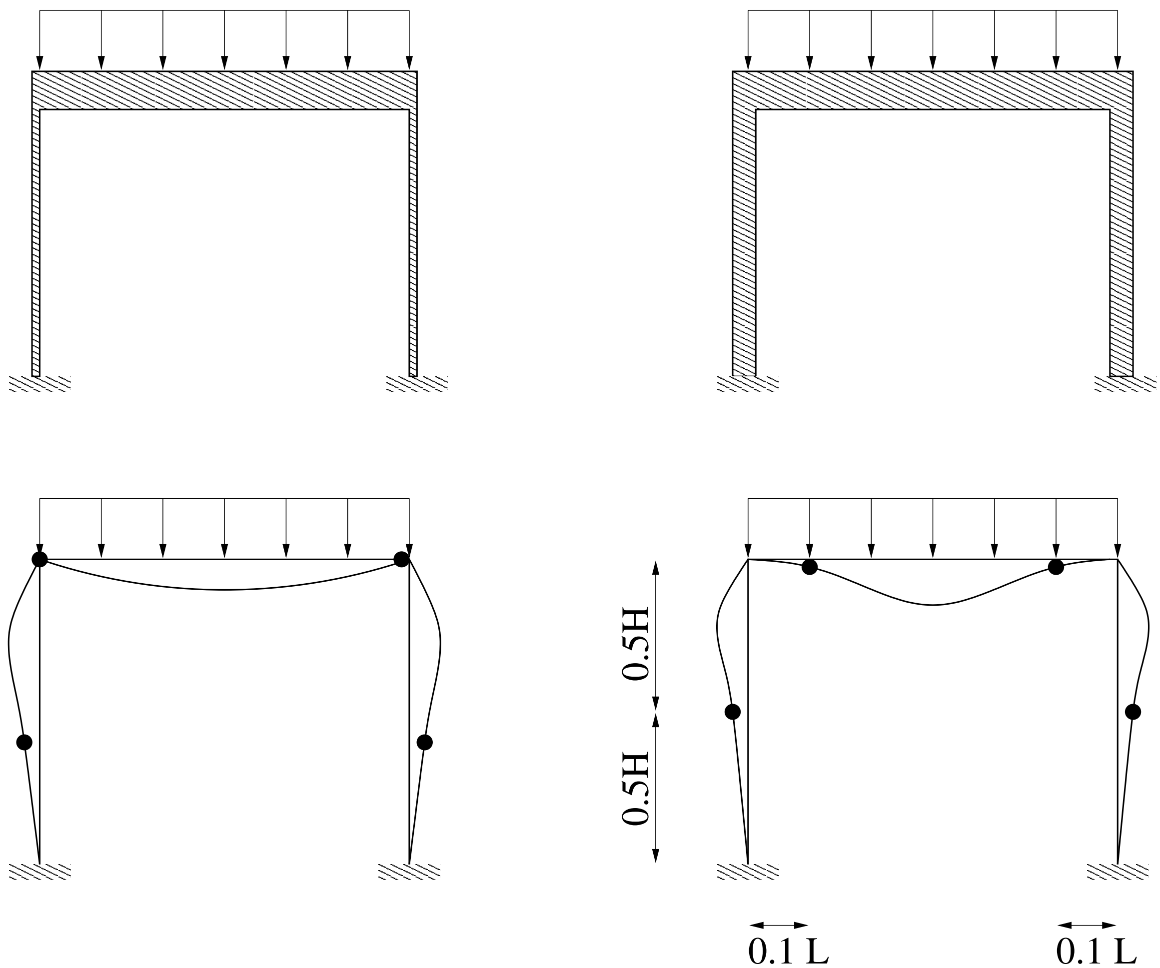


Figure 11.2: Uniformly Loaded Frame, Approximate Location of Inflection Points

$$M^{+} = \frac{1}{8}wL_s^2 = w\frac{1}{8}(0.8)^2L^2 = 0.08wL^2 \tag{11.1}$$

Maximum negative moment at each end of the girder is given by, Fig. 33.9

$$M^{left} = M^{rgt} = -\frac{w}{2}(0.1L)^2 - \frac{w}{2}(0.8L)(0.1L) = -0.045wL^2 \tag{11.2}$$

Girder Shear are obtained from the free body diagram, Fig. 33.10

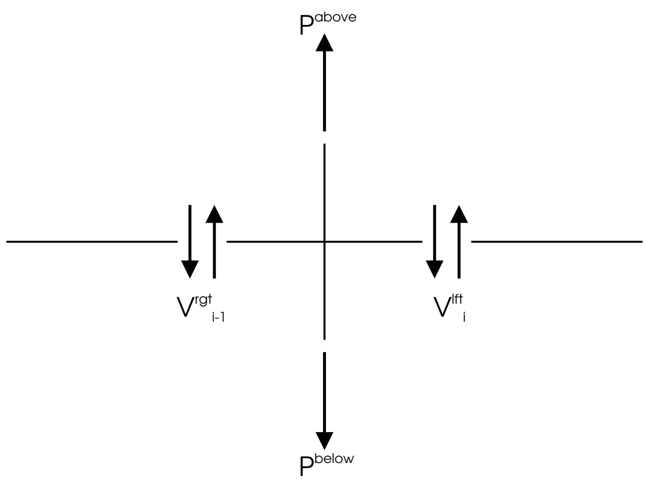


Figure 11.4: Approximate Analysis of Frames Subjected to Vertical Loads; Column Axial Forces

$$V^{lft} = \frac{wL}{2} \qquad V^{rgt} = -\frac{wL}{2} \tag{11.3}$$

Column axial force is obtained by summing all the girder shears to the axial force transmitted by the column above it. Fig. 33.10

$$P^{dwn} = P^{up} + V_{i-1}^{rgt} - V_i^{lft} \tag{11.4}$$

Column Moment are obtained by considering the free body diagram of columns Fig. 33.11

$$M^{top} = M_{above}^{bot} - M_{i-1}^{rgt} + M_i^{lft} \qquad M^{bot} = -M^{top} \tag{11.5}$$

Horizontal Loads, Portal Method

1. Column Shears

$$\begin{array}{lllll} V_5 & = & \frac{15}{(2)(3)} & = & 2.5 \text{ k} \\ V_6 & = & 2(V_5) = (2)(2.5) & = & 5 \text{ k} \\ V_7 & = & 2(V_5) = (2)(2.5) & = & 5 \text{ k} \\ V_8 & = & V_5 & = & 2.5 \text{ k} \\ V_1 & = & \frac{15+30}{(2)(3)} & = & 7.5 \text{ k} \\ V_2 & = & 2(V_1) = (2)(7.5) & = & 15 \text{ k} \\ V_3 & = & 2(V_1) = (2)(2.5) & = & 15 \text{ k} \\ V_4 & = & V_1 & = & 7.5 \text{ k} \end{array}$$

2. Top Column Moments

$$\begin{array}{lllll} M_5^{top} & = & \frac{V_1 H_5}{2} = \frac{(2.5)(14)}{2} & = & 17.5 \text{ k.ft} \\ M_5^{bot} & = & -M_5^{top} & = - & 17.5 \text{ k.ft} \\ M_6^{top} & = & \frac{V_6 H_6}{2} = \frac{(5)(14)}{2} & = & 35.0 \text{ k.ft} \\ M_6^{bot} & = & -M_6^{top} & = - & 35.0 \text{ k.ft} \\ M_7^{top} & = & \frac{V_7^{up} H_7}{2} = \frac{(5)(14)}{2} & = & 35.0 \text{ k.ft} \\ M_7^{bot} & = & -M_7^{top} & = - & 35.0 \text{ k.ft} \\ M_8^{top} & = & \frac{V_8^{up} H_8}{2} = \frac{(2.5)(14)}{2} & = & 17.5 \text{ k.ft} \\ M_8^{bot} & = & -M_8^{top} & = - & 17.5 \text{ k.ft} \end{array}$$

3. Bottom Column Moments

$$\begin{array}{lllll} M_1^{top} & = & \frac{V_1^{dwn} H_1}{2} = \frac{(7.5)(16)}{2} & = & 60 \text{ k.ft} \\ M_1^{bot} & = & -M_1^{top} & = - & 60 \text{ k.ft} \\ M_2^{top} & = & \frac{V_2^{dwn} H_2}{2} = \frac{(15)(16)}{2} & = & 120 \text{ k.ft} \\ M_2^{bot} & = & -M_2^{top} & = - & 120 \text{ k.ft} \\ M_3^{top} & = & \frac{V_3^{dwn} H_3}{2} = \frac{(15)(16)}{2} & = & 120 \text{ k.ft} \\ M_3^{bot} & = & -M_3^{top} & = - & 120 \text{ k.ft} \\ M_4^{top} & = & \frac{V_4^{dwn} H_4}{2} = \frac{(7.5)(16)}{2} & = & 60 \text{ k.ft} \\ M_4^{bot} & = & -M_4^{top} & = - & 60 \text{ k.ft} \end{array}$$

4. Top Girder Moments

$$\begin{array}{lllll} M_{12}^{lft} & = & M_5^{top} & = & 17.5 \text{ k.ft} \\ M_{12}^{rgt} & = & -M_{12}^{lft} & = - & 17.5 \text{ k.ft} \\ M_{13}^{lft} & = & M_{12}^{rgt} + M_6^{top} = -17.5 + 35 & = & 17.5 \text{ k.ft} \\ M_{13}^{rgt} & = & -M_{13}^{lft} & = - & 17.5 \text{ k.ft} \\ M_{14}^{lft} & = & M_{13}^{rgt} + M_7^{top} = -17.5 + 35 & = & 17.5 \text{ k.ft} \\ M_{14}^{rgt} & = & -M_{14}^{lft} & = - & 17.5 \text{ k.ft} \end{array}$$

6. Top Girder Shear

$$\begin{aligned}
V_{12}^{lft} &= -\frac{2M_{12}^{lft}}{L_{12}} = -\frac{(2)(17.5)}{20} &= -1.75 \text{ k}\\
V_{12}^{rgt} &= +V_{12}^{lft} &= -1.75 \text{ k}\\
V_{13}^{lft} &= -\frac{2M_{13}^{lft}}{L_{13}} = -\frac{(2)(17.5)}{30} &= -1.17 \text{ k}\\
V_{13}^{rgt} &= +V_{13}^{lft} &= -1.17 \text{ k}\\
V_{14}^{lft} &= -\frac{2M_{14}^{lft}}{L_{14}} = -\frac{(2)(17.5)}{24} &= -1.46 \text{ k}\\
V_{14}^{rgt} &= +V_{14}^{lft} &= -1.46 \text{ k}
\end{aligned}$$

7. Bottom Girder Shear

$$\begin{aligned}
V_{9}^{lft} &= -\frac{2M_{12}^{lft}}{L_{9}} = -\frac{(2)(77.5)}{20} &= -7.75 \text{ k}\\
V_{9}^{rgt} &= +V_{9}^{lft} &= -7.75 \text{ k}\\
V_{10}^{lft} &= -\frac{2M_{10}^{lft}}{L_{10}} = -\frac{(2)(77.5)}{30} &= -5.17 \text{ k}\\
V_{10}^{rgt} &= +V_{10}^{lft} &= -5.17 \text{ k}\\
V_{11}^{lft} &= -\frac{2M_{11}^{lft}}{L_{11}} = -\frac{(2)(77.5)}{24} &= -6.46 \text{ k}\\
V_{11}^{rgt} &= +V_{11}^{lft} &= -6.46 \text{ k}
\end{aligned}$$

8. Top Column Axial Forces (+ve tension, -ve compression)

$$\begin{aligned}
P_5 &= -V_{12}^{lft} &= -(-1.75) \text{ k}\\
P_6 &= +V_{12}^{rgt} - V_{13}^{lft} = -1.75 - (-1.17) &= -0.58 \text{ k}\\
P_7 &= +V_{13}^{rgt} - V_{14}^{lft} = -1.17 - (-1.46) &= 0.29 \text{ k}\\
P_8 &= V_{14}^{rgt} = -1.46 \text{ k}
\end{aligned}$$

9. Bottom Column Axial Forces (+ve tension, -ve compression)

$$\begin{aligned}
P_1 &= P_5 + V_9^{lft} = 1.75 - (-7.75) &= 9.5 \text{ k}\\
P_2 &= P_6 + V_{10}^{rgt} + V_9^{lft} = -0.58 - 7.75 - (-5.17) &= -3.16 \text{ k}\\
P_3 &= P_7 + V_{11}^{rgt} + V_{10}^{lft} = 0.29 - 5.17 - (-6.46) &= 1.58 \text{ k}\\
P_4 &= P_8 + V_{11}^{rgt} = -1.46 - 6.46 &= -7.66 \text{ k}
\end{aligned}$$

Design Parameters On the basis of the two approximate analyses, vertical and lateral load, we now seek the design parameters for the frame, Table 33.2.

■

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	**PORTAL METHOD**																		
2	**# of Bays**					**3**			**L1**				**L2**				**L3**		
3									**20**				**30**				**24**		
4								**MOMENTS**											
5	**# of Storeys**					**2**		**Bay 1**				**Bay 2**				**Bay 3**			
6			Force		Shear			Col	Beam		Column		Beam		Column		Beam		Col
7		**H**	**Lat.**	Tot	Ext	Int			Lft	Rgt			Lft	Rgt			Lft	Rgt	
8									=+H9	=-I8			=+J8+K9	=-M8			=+N8+O9	=-Q8	
9	**H1**	**14**	**15**	=+C9	=+D9/(2*F2)	=2*E9		=+E9*B9/2			=+F9*B9/2				=+K9				=+H9
10								=-H9			=-K9				=+K10				=+H10
11									=+H12-H10	=-I11			=+K12-K10+J11	=-M11			=+O12-O10+N11	=-Q11	
12	**H2**	**16**	**30**	=SUM(C9:C12)	=+D12/(2*F2)	=2*E12		=+E12*B12/2			=+F12*B12/2				=+K12				=+H12
13								=-H12			=-K12				=+K13				=+H13
14								**SHEAR**											
15								**Bay 1**				**Bay 2**				**Bay 3**			
16								Col	Beam		Column		Beam		Column		Beam		Col
17									Lft	Rgt			Lft	Rgt			Lft	Rgt	
18									=-2*I8/I$3	=+I18			=-2*M8/M$3	=+M18			=-2*Q8/Q$3	=+Q18	
19								=+E9			=+F9				=+F9				=+E9
20								=+H19			=+K19				=+O19				=+S19
21									=-2*I11/I$3	=+I21			=-2*M11/M$3	=+M21			=-2*Q11/Q$3	=+Q21	
22								=+E12			=+F12				=+F12				=+E12
23								=+H22			=+K22				=+O22				=+S22
24								**AXIAL FORCE**											
25								**Bay 1**				**Bay 2**				**Bay 3**			
26								Col	Beam		Column		Beam		Column		Beam		Col
27									0				0				0		
28								=-I18			=+J18-M18				=+N18-Q18				=+R18
29									0				0				0		
30								=+H28-I21			=+K28+J21-M21				=+O28+N21-Q21				=+S28+R21

Figure 11.19: Portal Method; Equations in Spread-Sheet

Mem.		Vert.	Hor.	**Design Values**
9	-ve Moment	9.00	77.50	**86.50**
	+ve Moment	16.00	0.00	**16.00**
	Shear	5.00	7.75	**12.75**
10	-ve Moment	20.20	77.50	**97.70**
	+ve Moment	36.00	0.00	**36.00**
	Shear	7.50	5.17	**12.67**
11	-ve Moment	13.0	77.50	**90.50**
	+ve Moment	23.00	0.00	**23.00**
	Shear	6.00	6.46	**12.46**
12	-ve Moment	4.50	17.50	**22.00**
	+ve Moment	8.00	0.00	**8.00**
	Shear	2.50	1.75	**4.25**
13	-ve Moment	10.10	17.50	**27.60**
	+ve Moment	18.00	0.00	**18.00**
	Shear	3.75	1.17	**4.92**
14	-ve Moment	6.50	17.50	**24.00**
	+ve Moment	11.50	0.00	**11.50**
	Shear	3.00	1.46	**4.46**

Table 11.2: Girders Combined Approximate Vertical and Horizontal Loads

Chapter 12

KINEMATIC INDETERMINANCY; STIFFNESS METHOD

12.1 Introduction

12.1.1 Stiffness vs Flexibility

1 There are two classes of structural analysis methods, Table 12.1:

Flexibility: where the primary unknown is a force, where equations of equilibrium are the starting point, static indeterminancy occurs if there are more unknowns than equations, and displacements of the entire structure (usually from virtual work) are used to write an equation of compatibility of displacements in order to solve for the redundant forces.

Stiffness: method is the counterpart of the flexibility one. Primary unknowns are displacements, and we start from expressions for the forces written in terms of the displacements (at the element level) and then apply the equations of equilibrium. The structure is considered to be *kinematically indeterminate* to the nth degree where n is the total number of independent displacements. From the displacements, we then compute the internal forces.

	Flexibility	**Stiffness**
Primary Variable (d.o.f.)	Forces	Displacements
Indeterminancy	Static	Kinematic
Force-Displacement	Displacement(Force)/Structure	Force(Displacement)/Element
Governing Relations	Compatibility of displacement	Equilibrium
Methods of analysis	"Consistent Deformation"	Slope Deflection; Moment Distribution

Table 12.1: Stiffness vs Flexibility Methods

2 In the flexibility method, we started by releasing as many redundant forces as possible in order to render the structure *statically determinate*, and this made it quite flexible. We then

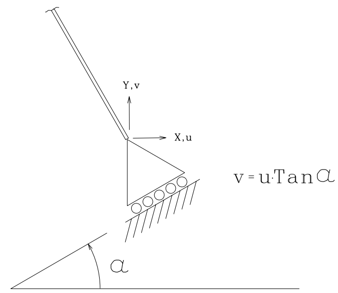


Figure 12.2: Independent Displacements

Type		Node 1	Node 2
1 Dimensional			
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4
2 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6
3 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1,	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}

Table 12.2: Degrees of Freedom of Different Structure Types Systems

12.2.1 Methods of Analysis

12 There are three methods for the stiffness based analysis of a structure

Slope Deflection: (Mohr, 1892) Which results in a system of n linear equations with n unknowns, where n is the degree of kinematic indeterminancy (i.e. total number of independent displacements/rotation).

Moment Distribution: (Cross, 1930) which is an iterative method to solve for the n displacements and corresponding internal forces in flexural structures.

Direct Stiffness method: ($\tilde{1}$960) which is a formal statement of the stiffness method and cast in matrix form is by far the most powerful method of structural analysis.

The first two methods lend themselves to hand calculation, and the third to a computer based analysis.

12.3 Kinematic Relations

12.3.1 Force-Displacement Relations

13 Whereas in the flexibility method we sought to obtain a displacement in terms of the forces (through virtual work) for an entire structure, our starting point in the stiffness method is to develop a set of relationship for the *force in terms of the displacements for a single element*.

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \right\} = \left[\begin{array}{cccc} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array} \right] \left\{ \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \right\} \tag{12.1}$$

14 We start from the differential equation of a beam, Fig. 12.4 in which we have all positive known displacements, we have from strength of materials

$$M = -EI\frac{d^2v}{dx^2} = M_1 - V_1x + m(x) \tag{12.2}$$

where $m(x)$ is the moment applied due to the applied load only. It is positive when counterclockwise.

15 Integrating twice

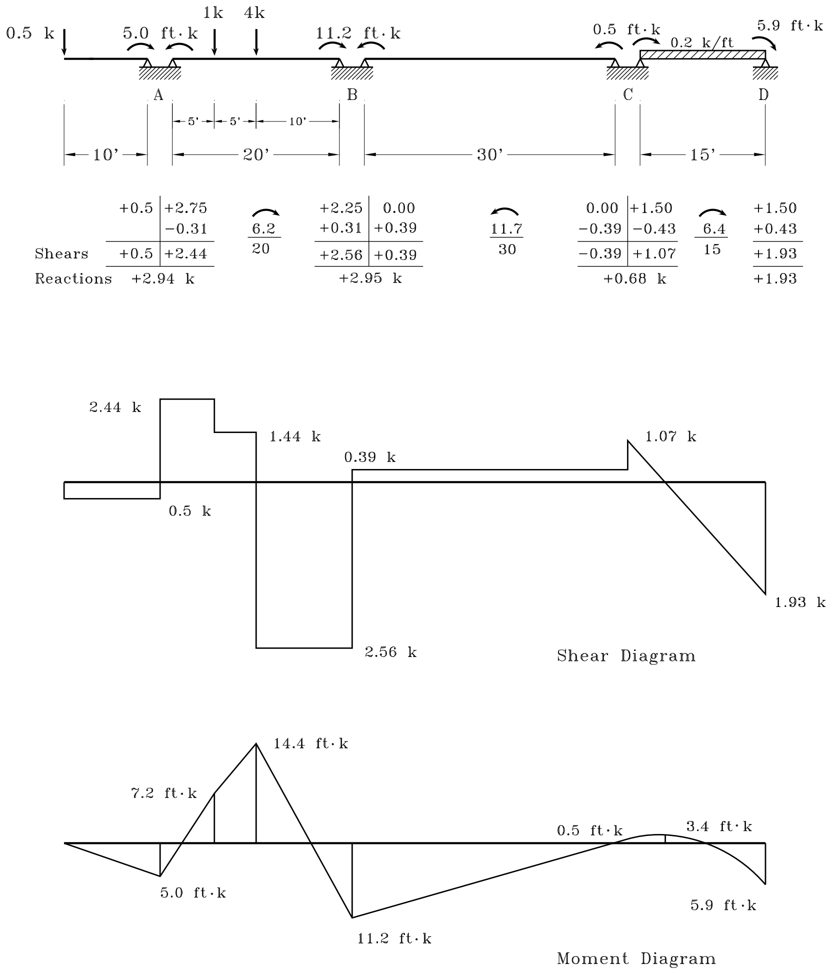
$$-EIv' = M_1x - \frac{1}{2}V_1x^2 + f(x) + C_1 \tag{12.3}$$

$$-EIv = \frac{1}{2}M_1x^2 - \frac{1}{6}V_1x^3 + g(x) + C_1x + C_2 \tag{12.4}$$

where $f(x) = \int m(x)dx$, and $g(x) = \int f(x)dx$.

16 Applying the boundary conditions at $x = 0$

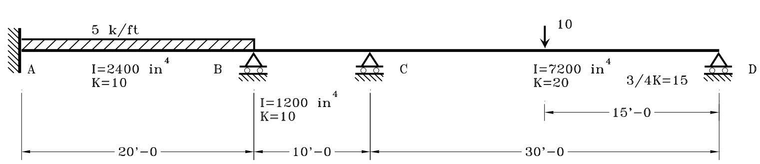
$$\left. \begin{array}{rcl} v' & = & \theta_1 \\ v & = & v_1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} C_1 & = & -EI\theta_1 \\ C_2 & = & -EIv_1 \end{array} \right. \tag{12.5}$$



■

■ Example 12-7: Continuous Beam, Initial Settlement, (Kinney 1957)

For the following beam find the moments at A, B, and C by moment distribution. The support at C settles by 0.1 in. Use $E = 30,000$ k/in^2.

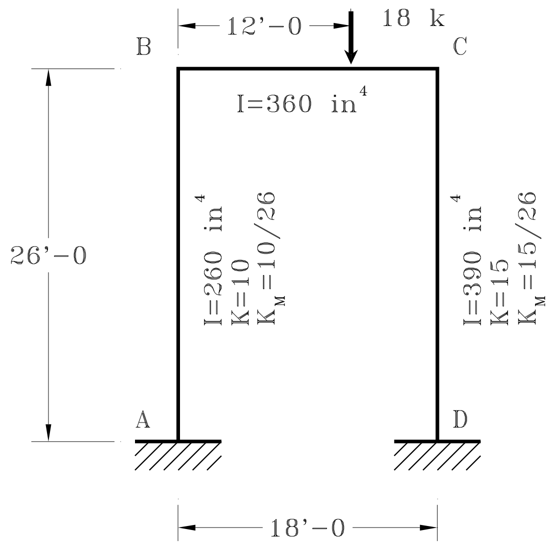


Solution:

1. Fixed-end moments: Uniform load:

$$M_{AB}^F = \frac{wL^2}{12} = \frac{(5)(20^2)}{12} = +167 \text{ k.ft} \quad (12.68\text{-a})$$

$$M_{BA}^F = -167 \text{ k.ft} \quad (12.68\text{-b})$$



Solution:

1. The first step is to perform the usual moment distribution. The reader should fully understand that this balancing operation adjusts the internal moments at the ends of the members by a series of corrections as the joints are considered to rotate, until $\Sigma M = 0$ at each joint. The reader should also realize that *during this balancing operation no translation of any joint is permitted*.
2. The fixed-end moments are

$$M^F_{BC} = \frac{(18)(12)(6^2)}{18^2} = +24 \text{ k.ft} \tag{12.71-a}$$

$$M^F_{CB} = \frac{(18)(6)(12^2)}{18^2} = -48 \text{ k.ft} \tag{12.71-b}$$

3. Moment distribution

Joint	*A*	*B*		*C*		*D*	Balance	CO
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM			+24.0	-48.0			FEM	
			+13.7	+27.4	+20.6	+10.3	C	DC; BC
	-6.3	-12.6	-25.1	-12.5			B	AB; CB
			+3.6	+7.1	+5.4	+2.7	C	BC; DC
	-0.6	-1.2	-2.4	-1.2			B	AB; CB
			+0.3	+0.7	+0.5	+0.02	C	BC; DC
		-0.1	-0.2				B	
Total	**-6.9**	**-13.9**	**+13.9**	**-26.5**	**+26.5**	**+13.2**		

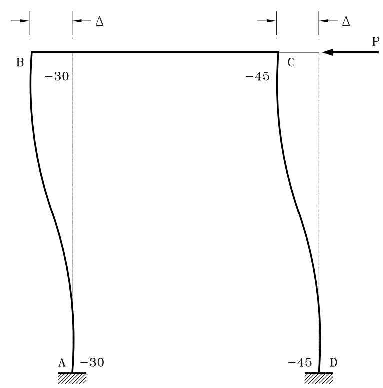
4. The *final moments listed in the table are correct only if there is no translation of any joint.* It is therefore necessary to determine whether or not, with the above moments existing, there is any tendency for side lurch of the top of the frame.
5. If the frame is divided into three free bodies, the result will be as shown below.

62 Recalling that the fixed end moment is $M^F = 6EI\frac{\Delta}{L^2} = 6EK_m\Delta$, where $K_m = \frac{I}{L^2} = \frac{K}{L}$ we can write

$$\Delta = \frac{M^F_{AB}}{6EK_m} = \frac{M^F_{DC}}{6EK_m} \quad (12.72\text{-a})$$

$$\Rightarrow \frac{M^F_{AB}}{M^F_{DC}} = \frac{K^{AB}_m}{K^{DC}_m} = \frac{10}{15} \quad (12.72\text{-b})$$

63 These fixed-end moments could, for example, have the values of −10 and −15 k.ft or −20 and −30, or −30 and −45, or any other combination so long as the above ratio is maintained. The proper procedure is to choose values for these fixed-end moments of approximately the same order of magnitude as the original fixed-end moments due to the real loads. This will result in the same accuracy for the results of the balance for the side-sway correction that was realized in the first balance for the real loads. Accordingly, it will be assumed that P, and the resulting Δ, are of such magnitudes as to result in the fixed-end moments shown below



8. Obviously, $\Sigma M = 0$ is not satisfied for joints B and C in this deflected frame. Therefore these joints must rotate until equilibrium is reached. The effect of this rotation is determined in the distribution below

Joint	*A*	*B*		*C*		*D*	Balance	CO
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM	-30.0	-30.0			-45.0	-45.0		
			+12.9	+25.8	+19.2	+9.6	C	BC; DC
	+2.8	+5.7	+11.4	+5.7			B	AB; CB
			-1.6	-3.3	-2.4	-1.2	C	BC; DC
	+0.2	+0.5	+1.1	+0.5			B	AB; CB
				-0.3	-0.2	-0.1	C	
Total	**-27.0**	**-23.8**	**+23.8**	**+28.4**	**-28.4**	**-36.7**		

9. During the rotation of joints B and C, as represented by the above distribution, *the value of* Δ *has remained constant, with* P *varying in magnitude as required to maintain* Δ.

10. It is now possible to determine the final value of P simply by adding the shears in the columns. The shear in any member, without external loads applied along its length, is obtained by adding the end moments algebraically and dividing by the length of the member. The final

Chapter 13

DIRECT STIFFNESS METHOD

13.1 Introduction

13.1.1 Structural Idealization

1 Prior to analysis, a structure must be idealized for a suitable mathematical representation. Since it is practically impossible (and most often unnecessary) to model every single detail, assumptions must be made. Hence, structural idealization is as much an art as a science. *Some* of the questions confronting the analyst include:

1. Two dimensional versus three dimensional; Should we model a single bay of a building, or the entire structure?
2. Frame or truss, can we neglect flexural stiffness?
3. Rigid or semi-rigid connections (most important in steel structures)
4. Rigid supports or elastic foundations (are the foundations over solid rock, or over clay which may consolidate over time)
5. Include or not secondary members (such as diagonal braces in a three dimensional analysis).
6. Include or not axial deformation (can we neglect the axial stiffness of a beam in a building?)
7. Cross sectional properties (what is the moment of inertia of a reinforced concrete beam?)
8. Neglect or not haunches (those are usually present in zones of high negative moments)
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations (In the analysis of a high rise building subjected to wind load, the moments should be amplified by the product of the axial load times the lateral deformation, $P-\Delta$ effects).
11. Time dependent effects (such as creep, which is extremely important in prestressed concrete, or cable stayed concrete bridges).

(such as beam element), while element 2 has a code 2 (such as a truss element). Material group 1 would have different elastic/geometric properties than material group 2.

Group No.	Element Type	Material Group
1	1	1
2	2	1
3	1	2

Table 13.3: Example of Group Number

6 From the analysis, we first obtain the nodal displacements, and then the element internal forces. Those internal forces vary according to the element type. For a two dimensional frame, those are the axial and shear forces, and moment at each node.

7 Hence, the need to define two coordinate systems (one for the entire structure, and one for each element), and a sign convention become apparent.

13.1.3 Coordinate Systems

8 We should differentiate between 2 coordinate systems:

Global: to describe the structure nodal coordinates. This system can be arbitrarily selected provided it is a Right Hand Side (RHS) one, and we will associate with it upper case axis labels, X, Y, Z, Fig. 13.1 or 1,2,3 (running indeces within a computer program).

Local: system is associated with each element and is used to describe the element internal forces. We will associate with it lower case axis labels, x, y, z (or 1,2,3), Fig. 13.2.

9 The x-axis is assumed to be along the member, and the direction is chosen such that it points from the 1st node to the 2nd node, Fig. 13.2.

10 Two dimensional structures will be defined in the X-Y plane.

13.1.4 Sign Convention

11 The sign convention in structural analysis is completely different than the one previously adopted in structural analysis/design, Fig. 13.3 (where we focused mostly on flexure and defined a positive moment as one causing "tension below". This would be awkward to program!).

12 In matrix structural analysis the sign convention adopted is consistent with the prevailing coordinate system. Hence, we define a positive moment as one which is counter-clockwise, Fig. 13.3

13 Fig. 13.4 illustrates the sign convention associated with each type of element.

14 Fig. 13.4 also shows the geometric (upper left) and elastic material (upper right) properties associated with each type of element.

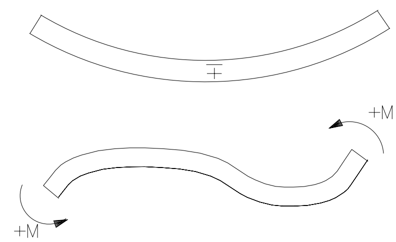


Figure 13.3: Sign Convention, Design and Analysis

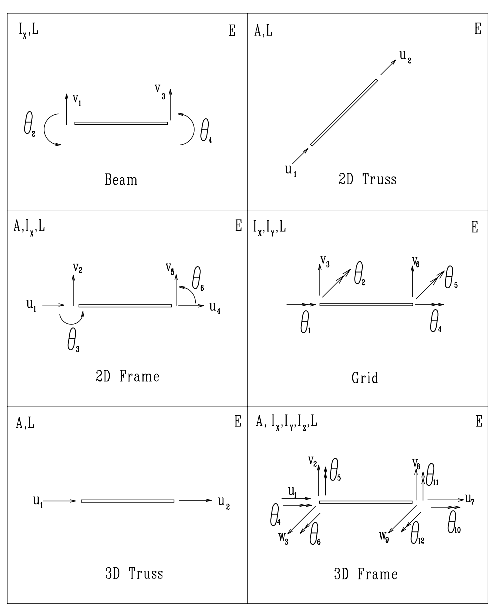


Figure 13.4: Total Degrees of Freedom for various Type of Elements

Type		Node 1	Node 2	[**k**]	[**K**]
				(Local)	(Global)
1 Dimensional					
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\boldsymbol{\delta}\}$	u_1	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathbf{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}	2×2	6×6
	$\{\boldsymbol{\delta}\}$	u_1,	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}	12×12	12×12
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}		

Table 13.4: Degrees of Freedom of Different Structure Types Systems

13.2 Stiffness Matrices

13.2.1 Truss Element

22 From strength of materials, the force/displacement relation in axial members is

$$\begin{aligned} \sigma &= E\epsilon \\ \underbrace{A\sigma}_{P} &= \frac{AE}{L}\underbrace{\Delta}_{1} \end{aligned} \tag{13.1}$$

Hence, for a unit displacement, the applied force should be equal to $\frac{AE}{L}$. From statics, the force at the other end must be equal and opposite.

23 The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node. Hence, from Eq. 13.1, we have

$$[\mathbf{k}^t] = \frac{AE}{L} \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{bmatrix} u_1 & u_2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{13.2}$$

13.2.2 Beam Element

24 Using Equations 12.10, 12.10, 12.12 and 12.12 we can determine the forces associated with each unit displacement.

$$[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ \text{Eq. } 12.12(v_1=1) & \text{Eq. } 12.12(\theta_1=1) & \text{Eq. } 12.12(v_2=1) & \text{Eq. } 12.12(\theta_2=1) \\ \text{Eq. } 12.10(v_1=1) & \text{Eq. } 12.10(\theta_1=1) & \text{Eq. } 12.10(v_2=1) & \text{Eq. } 12.10(\theta_2=1) \\ \text{Eq. } 12.12(v_1=1) & \text{Eq. } 12.12(\theta_1=1) & \text{Eq. } 12.12(v_2=1) & \text{Eq. } 12.12(\theta_2=1) \\ \text{Eq. } 12.10(v_1=1) & \text{Eq. } 12.10(\theta_1=1) & \text{Eq. } 12.10(v_2=1) & \text{Eq. } 12.10(\theta_2=1) \end{bmatrix} \tag{13.3}$$

25 The stiffness matrix of the beam element (neglecting shear and axial deformation) will thus be

$$[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \tag{13.4}$$

26 We note that this is identical to Eq.12.14

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \underbrace{\begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}}_{\mathbf{k}^{(e)}} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{13.5}$$

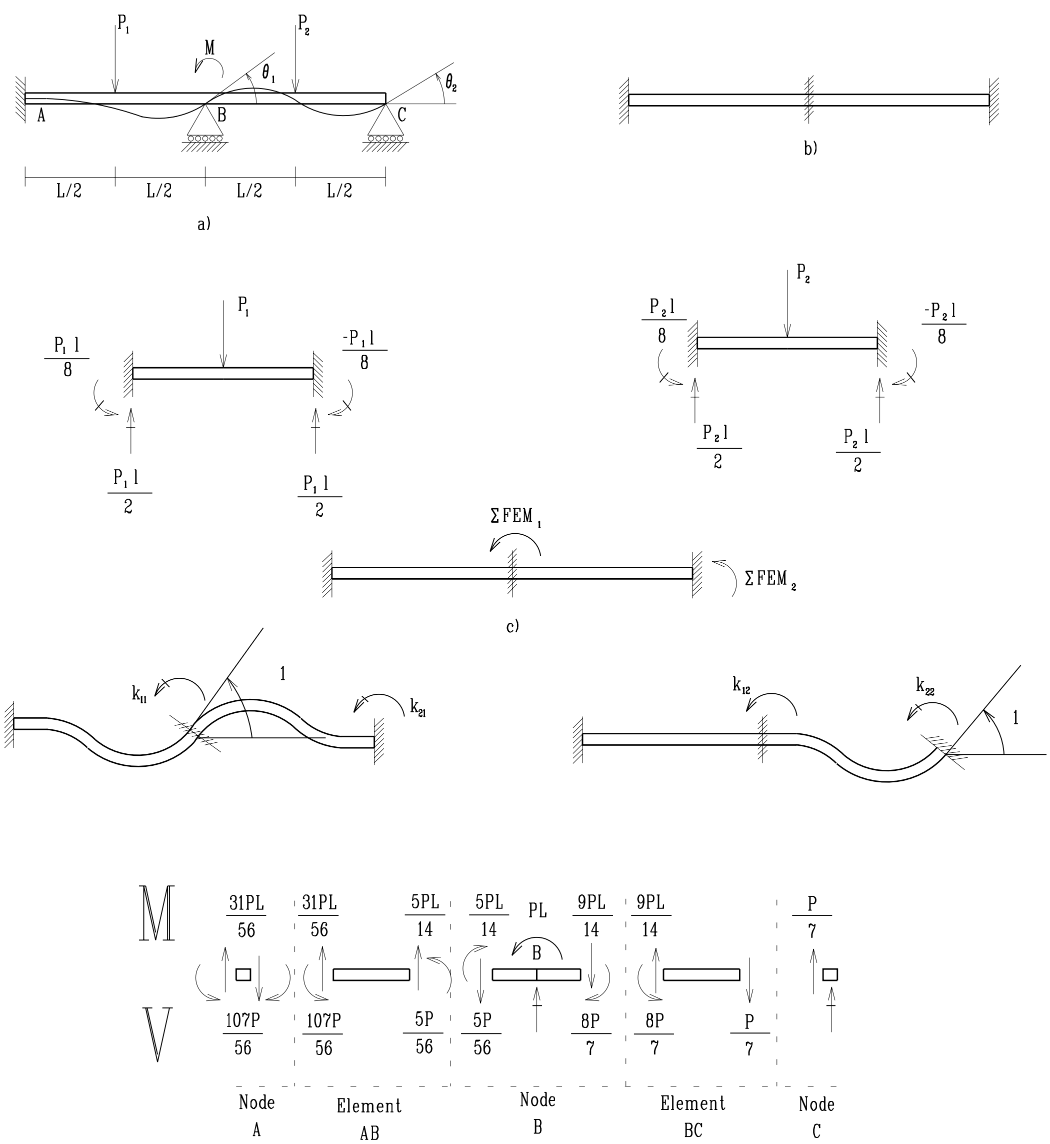


Figure 13.7: Problem with 2 Global d.o.f. θ_1 and θ_2

```
% Solve for the Displacements in meters and radians
Displacements=inv(Ktt)*P'
% Extract Kut
Kut=Kaug(4:9,1:3);
% Compute the Reactions and do not forget to add fixed end actions
Reactions=Kut*Displacements+FEA_Rest'
% Solve for the internal forces and do not forget to include the fixed end actions
dis_global(:,:,1)=[0,0,0,Displacements(1:3)'];
dis_global(:,:,2)=[Displacements(1:3)',0,0,0];
for elem=1:2
   dis_local=Gamma(:,:,elem)*dis_global(:,:,elem)';
   int_forces=k(:,:,elem)*dis_local+fea(1:6,elem)
end

function [k,K,Gamma]=stiff(EE,II,A,i,j)
% Determine the length
L=sqrt((j(2)-i(2))^2+(j(1)-i(1))^2);
% Compute the angle theta (carefull with vertical members!)
if(j(1)-i(1))~=0
   alpha=atan((j(2)-i(2))/(j(1)-i(1)));
else
   alpha=-pi/2;
end
% form rotation matrix Gamma
Gamma=[
cos(alpha)  sin(alpha)  0  0          0          0;
-sin(alpha) cos(alpha)  0  0          0          0;
0           0           1  0          0          0;
0           0           0  cos(alpha) sin(alpha) 0;
0           0           0 -sin(alpha) cos(alpha) 0;
0           0           0  0          0          1];
% form element stiffness matrix in local coordinate system
EI=EE*II;
EA=EE*A;
k=[EA/L,         0,         0, -EA/L,          0,         0;
   0,    12*EI/L^3,  6*EI/L^2,     0, -12*EI/L^3,  6*EI/L^2;
   0,     6*EI/L^2,    4*EI/L,     0,  -6*EI/L^2,    2*EI/L;
 -EA/L,          0,         0,  EA/L,          0,         0;
     0, -12*EI/L^3, -6*EI/L^2,     0,  12*EI/L^3, -6*EI/L^2;
     0,   6*EI/L^2,    2*EI/L,     0,  -6*EI/L^2,    4*EI/L];
% Element stiffness matrix in global coordinate system
K=Gamma'*k*Gamma;
```

This simple proigram will produce the following results:

```
Displacements =

    0.0010
   -0.0050
```

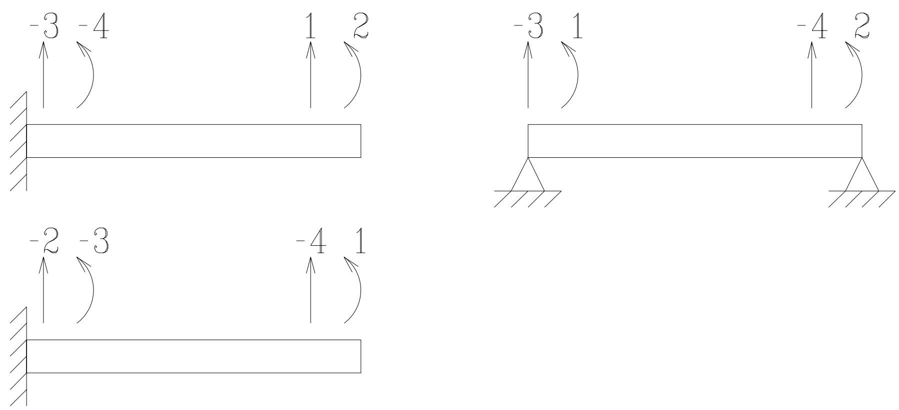


Figure 13.14: ID Values for Simple Beam

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c|cccc} & 1 & 2 & -3 & -4 \\ \hline 1 & 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ 2 & -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ -3 & -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -4 & -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array}$$

3. The global matrix can be rewritten as

$$\left\{ \begin{array}{c} -P\surd \\ 0\surd \\ \hline R_3? \\ R_4? \end{array} \right\} = \left[\begin{array}{cc|cc} 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right] \left\{ \begin{array}{c} \Delta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \theta_4\surd \end{array} \right\}$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$\left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\ \boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right]$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\begin{aligned} \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u &= \left\{ \begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array} \right\} - \left[\begin{array}{cc|cc} L^3/3EI & L^2/2EI & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} \\ L^2/2EI & L/EI & \boxed{6EI/L^2} & \boxed{2EI/L} \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right] \left\{ \begin{array}{c} -P \\ 0 \\ \hline \boxed{0} \\ \boxed{0} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array} \right\} \end{aligned}$$

5. We compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u = \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\} - \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ -L/6EI & L/3EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} 0 \\ M \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\}$$

6. Solve for the displacements, $\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\boldsymbol{\Delta}_t$

$$\left\{\begin{array}{c} \boxed{\theta_1} \\ \boxed{\theta_2} \\ \hline 0 \\ 0 \end{array}\right\} = \left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2 \\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{-ML/6EI} \\ \boxed{ML/3EI} \\ \hline 0 \\ 0 \end{array}\right\}$$

7. Solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\boldsymbol{\Delta}_{tt} + \mathbf{K}_{uu}\boldsymbol{\Delta}_u$, and overwrite $\boldsymbol{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} -ML/6EI \\ ML/3EI \\ \hline \boxed{R_1} \\ \boxed{R_2} \end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & 6EI/L^2 & -6EI/L^2 \\ -L/6EI & L/3EI & 6EI/L^2 & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{-12EI/L^3} \\ \boxed{-6EI/L^2} & \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{12EI/L^3} \end{array}\right] \left\{\begin{array}{c} \boxed{-ML/6EI} \\ \boxed{ML/3EI} \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -ML/6EI \\ ML/3EI \\ \hline \boxed{M/L} \\ \boxed{-M/L} \end{array}\right\}$$

Cantilivered Beam/Initial Displacement and Concentrated Moment

1. The *element* stiffness matrix is

$$\mathbf{k} = \begin{array}{c} \\ -2 \\ -3 \\ -4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} -2 & -3 & -4 & 1 \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & -2 & -3 & -4 \end{array} \\ \left[\begin{array}{cccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \end{array}$$

3. The global matrix can be rewritten as

$$\left\{\begin{array}{c} M\surd \\ \hline R_2? \\ R_3? \\ R_4? \end{array}\right\} = \left[\begin{array}{c|ccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1? \\ \hline \Delta_2\surd \\ \theta_3\surd \\ \Delta_4\surd \end{array}\right\}$$

Chapter 14

DESIGN PHILOSOPHIES of ACI and AISC CODES

14.1 Safety Provisions

1 Structures and structural members must always be designed to carry some reserve load above what is expected under normal use. This is to account for

Variability in Resistance: The **actual** strengths (resistance) of structural elements will differ from those **assumed** by the designer due to:

1. Variability in the strength of the material (greater variability in concrete strength than in steel strength).
2. Differences between the actual dimensions and those specified (mostly in placement of steel rebars in R/C).
3. Effect of simplifying assumptions made in the derivation of certain formulas.

Variability in Loadings: All loadings are variable. There is a greater variation in the live loads than in the dead loads. Some types of loadings are very difficult to quantify (wind, earthquakes).

Consequences of Failure: The consequence of a structural component failure must be carefully assessed. The collapse of a beam is likely to cause a localized failure. Alternatively the failure of a column is likely to trigger the failure of the whole structure. Alternatively, the failure of certain components can be preceded by warnings (such as excessive deformation), whereas other are sudden and catastrophic. Finally, if no redistribution of load is possible (as would be the case in a statically determinate structure), a higher safety factor must be adopted.

2 The purpose of safety provisions is to limit the **probability of failure** and yet permit economical structures.

3 The following items must be considered in determining safety provisions:

1. Seriousness of a failure, either to humans or goods.

$$\boxed{\sigma < \sigma_{all} = \frac{\sigma_{yld}}{F.S.}} \tag{14.1}$$

where $F.S.$ is the factor of safety.

10 Major limitations of this method

1. An elastic analysis can not easily account for creep and shrinkage of concrete.
2. For concrete structures, stresses are not linearly proportional to strain beyond $0.45f'_c$.
3. Safety factors are not rigorously determined from a probabilistic approach, but are the result of experience and judgment.

11 Allowable strengths are given in Table 14.1.

Steel, AISC/ASD	
Tension, Gross Area	$F_t = 0.6F_y$
Tension, Effective Net Area*	$F_t = 0.5F_u$
Bending	$F_b = 0.66F_y$
Shear	$F_v = 0.40F_y$
Concrete, ACI/WSD	
Tension	0
Compression	$0.45f'_c$

* Effective net area will be defined in section 17.2.1.2.

Table 14.1: Allowable Stresses for Steel and Concrete

14.3 Ultimate Strength Method

14.3.1 The Normal Distribution

12 The normal distribution has been found to be an excellent approximation to a large class of distributions, and has some very desirable mathematical properties:

1. $f(x)$ is symmetric with respect to the mean μ.
2. $f(x)$ is a "bell curve" with inflection points at $x = \mu \pm \sigma$.
3. $f(x)$ is a valid *probability distribution function* as:

$$\int_{-\infty}^{\infty} f(x) = 1 \tag{14.2}$$

4. The *probability* that $x_{min} < x < x_{max}$ is given by:

$$P(x_{min} < x < x_{max}) = \int_{x_{min}}^{x_{max}} f(x)dx \tag{14.3}$$

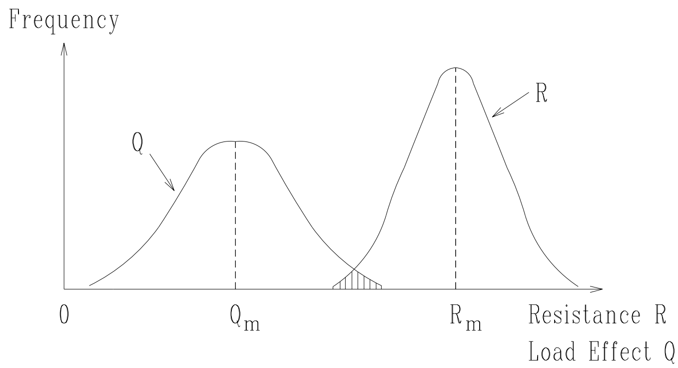


Figure 14.3: Frequency Distributions of Load Q and Resistance R

Failure would occur for negative values of X

19 The **probability of failure** P_f is equal to the ratio of the shaded area to the total area under the curve in Fig. 14.4.

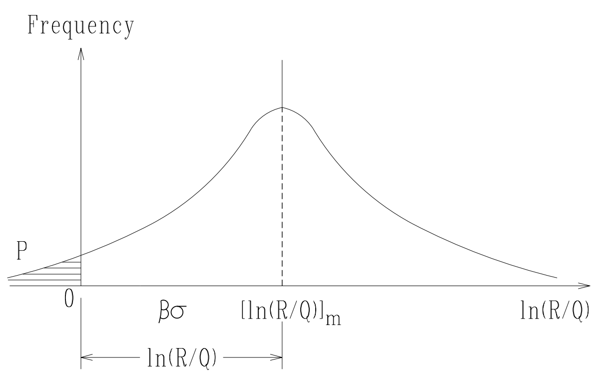


Figure 14.4: Definition of Reliability Index

20 If X is assumed to follow a **Normal Distribution** than it has a mean value $\overline{X} = \left(\ln \frac{R}{Q}\right)_m$ and a standard deviation σ.

21 We define the **safety index** (or **reliability index**) as $\beta = \frac{\overline{X}}{\sigma}$

22 For standard distributions and for $\beta = 3.5$, it can be shown that the probability of failure is $P_f = \frac{1}{9{,}091}$ or 1.1×10^{-4}. That is 1 in every 10,000 structural members designed with $\beta = 3.5$ will fail because of either excessive load or understrength sometime in its lifetime.

23 Reliability indices are a relative measure of the current condition and provide a qualitative estimate of the structural performance.

24 Structures with relatively high reliable indices will be expected to perform well. If the value is too low, then the structure may be classified as a hazard.

25 Target values for β are shown in Table 28.2, and in Fig. 28.3

Chapter 15

LOADS

15.1 Introduction

1 The main purpose of a structure is to transfer load from one point to another: bridge deck to pier; slab to beam; beam to girder; girder to column; column to foundation; foundation to soil.

2 There can also be secondary loads such as thermal (in restrained structures), differential settlement of foundations, P-Delta effects (additional moment caused by the product of the vertical force and the lateral displacement caused by lateral load in a high rise building).

3 Loads are generally subdivided into two categories

Vertical Loads or gravity load

1. **dead load** (DL)
2. **live load** (LL)

also included are snow loads.

Lateral Loads which act horizontally on the structure

1. **Wind load** (WL)
2. **Earthquake load** (EL)

this also includes hydrostatic and earth loads.

4 This distinction is helpful not only to compute a structure's load, but also to assign different factor of safety to each one.

5 For a detailed coverage of loads, refer to the Universal Building Code (UBC), (UBC 1995).

15.2 Vertical Loads

6 For closely spaced identical loads (such as joist loads), it is customary to treat them as a uniformly distributed load rather than as discrete loads, Fig. 15.1

Material	lb/ft^2
Ceilings	
Channel suspended system	1
Acoustical fiber tile	1
Floors	
Steel deck	2-10
Concrete-plain 1 in.	12
Linoleum 1/4 in.	1
Hardwood	4
Roofs	
Copper or tin	1-5
5 ply felt and gravel	6
Shingles asphalt	3
Clay tiles	9-14
Sheathing wood	3
Insulation 1 in. poured in place	2
Partitions	
Clay tile 3 in.	17
Clay tile 10 in.	40
Gypsum Block 5 in.	14
Wood studs 2x4 (12-16 in. o.c.)	2
Plaster 1 in. cement	10
Plaster 1 in. gypsum	5
Walls	
Bricks 4 in.	40
Bricks 12 in.	120
Hollow concrete block (heavy aggregate)	
4 in.	30
8 in.	55
12 in.	80
Hollow concrete block (light aggregate)	
4 in.	21
8 in.	38
12 in.	55

Table 15.2: Weights of Building Materials

Material	lb/ft^2
Timber	40-50
Steel	50-80
Reinforced concrete	100-150

Table 15.3: Average Gross Dead Load in Buildings

Floor	Roof	10	9	8	7	6	5	4	3	2	Total
Cumulative R (%)	8.48	16.96	25.44	33.92	42.4	51.32	59.8	60	60	60	
Cumulative LL	20	80	80	80	80	80	80	80	80	80	740
Cumulative R× LL	18.3	66.4	59.6	52.9	46.08	38.9	32.2	32	32	32	410

The resulting design live load for the bottom column has been reduced from 740 Kips to 410 Kips.

5. The total dead load is $DL = (10)(60) = 600$ Kips, thus the total reduction in load is $\frac{740-410}{740+600} \times 100 = 25\%$.

■

15.2.3 Snow

19 Roof snow load vary greatly depending on **geographic location and elevation**. They range from 20 to 45 psf, Fig. 15.2.

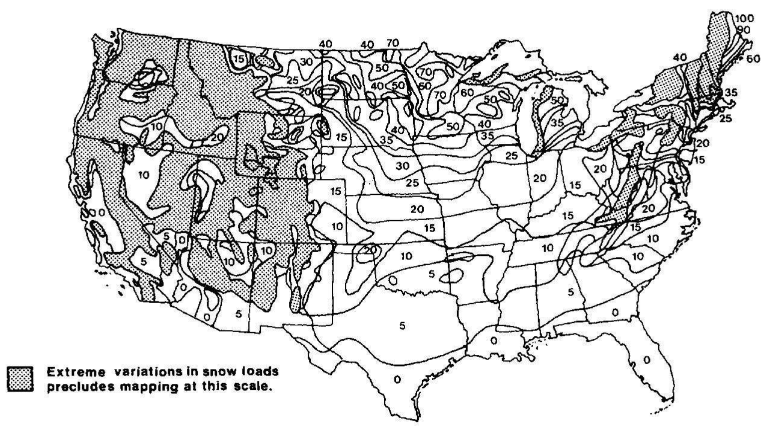


Figure 15.2: Snow Map of the United States, ubc

20 Snow loads are always given on the projected length or area on a slope, Fig. 15.3.

21 The steeper the roof, the lower the snow retention. For snow loads greater than 20 psf and roof pitches α more than 20° the snow load p may be reduced by

$$R = (\alpha - 20)\left(\frac{p}{40} - 0.5\right) \quad \text{(psf)} \tag{15.2}$$

22 Other examples of loads acting on inclined surfaces are shown in Fig. 15.4.

15.3 Lateral Loads

15.3.1 Wind

23 Wind load depend on: **velocity of the wind, shape of the building, height, geographical location, texture of the building surface and stiffness of the structure**.

Chapter 16

STRUCTURAL MATERIALS

1 Proper understanding of structural materials is essential to both structural analysis and to structural design.

2 Characteristics of the most commonly used structural materials will be highlighted.

16.1 Steel

16.1.1 Structural Steel

3 Steel is an **alloy** of iron and carbon. Its properties can be greatly varied by altering the carbon content (always less than 0.5%) or by adding other elements such as silicon, nickel, manganese and copper.

4 Practically all grades of steel have a Young Modulus equal to **29,000 ksi**, density of 490 lb/cu ft, and a coefficient of thermal expansion equal to 0.65×10^{-5} /deg F.

5 The yield stress of steel can vary from 40 ksi to 250 ksi. Most commonly used structural steel are **A36** ($\sigma_{\rm yld} = 36$ ksi) and A572 ($\sigma_{\rm yld} = 50$ ksi), Fig. 16.6

6 Structural steel can be rolled into a wide variety of shapes and sizes. Usually the most desirable members are those which have a large section moduli (S) in proportion to their area (A), Fig. 16.2.

7 Steel can be bolted, riveted or welded.

8 Sections are designated by the shape of their cross section, their depth and their weight. For example W 27× 114 is a W section, 27 in. deep weighing 114 lb/ft.

9 Common sections are:

S sections were the first ones rolled in America and have a slope on their inside flange surfaces of 1 to 6.

W or wide flange sections have a much smaller inner slope which facilitates connections and rivetting. W sections constitute about 50% of the tonnage of rolled structural steel.

C are channel sections

MC Miscellaneous channel which can not be classified as a C shape by dimensions.

HP is a bearing pile section.

M is a miscellaneous section.

L are angle sections which may have equal or unequal sides.

WT is a T section cut from a W section in two.

Properties of structural steel are tabulated in Table 16.1.

ASTM Desig.	**Shapes Available**	**Use**	σ_y **(ksi)**	σ_u **(ksi)**
A36	Shapes and bars	Riveted, bolted, welded; Buildings and bridges	36 up through 8 in. (32 above 8.)	
A500	Cold formed welded and seamless sections;	General structural purpose Riveted, welded or bolted;	Grade A: 33; Grade B: 42; Grade C: 46	
A501	Hot formed welded and seamless sections;	Bolted and welded	36	
A529	Plates and bars $\frac{1}{2}$ in and less thick;	Building frames and trusses; Bolted and welded	42	
A606	Hot and cold rolled sheets;	Atmospheric corrosion resistant	45-50	
A611	Cold rolled sheet in cut lengths	Cold formed sections	Grade C 33; Grade D 40; Grade E 80	
A 709	Structural shapes, plates and bars	Bridges	Grade 36: 36 (to 4 in.); Grade 50: 50; Grade 100: 100 (to 2.5in.) and 90 (over 2.5 to 4 in.)	

Table 16.1: Properties of Major Structural Steels

10 Rolled sections, Fig. 16.3 and welded ones, Fig 16.4 have **residual stresses**. Those originate during the rolling or fabrication of a member. The member is hot just after rolling or welding, it cools unevenly because of varying exposure. The area that cool first become stiffer, resist contraction, and develop compressive stresses. The remaining regions continue to cool and contract in the plastic condition and develop tensile stresses.

11 Due to those residual stresses, the stress-strain curve of a rolled section exhibits a non-linear segment prior to the theoretical yielding, Fig. 16.5. This would have important implications on the flexural and axial strength of beams and columns.

16.1.2 Reinforcing Steel

12 Steel is also used as reinforcing bars in concrete, Table 16.2. Those bars have a deformation on their surface to increase the bond with concrete, and usually have a yield stress of 60 ksi[1].

[1]Stirrups which are used as vertical reinforcement to resist shear usually have a yield stress of only 40 ksi.

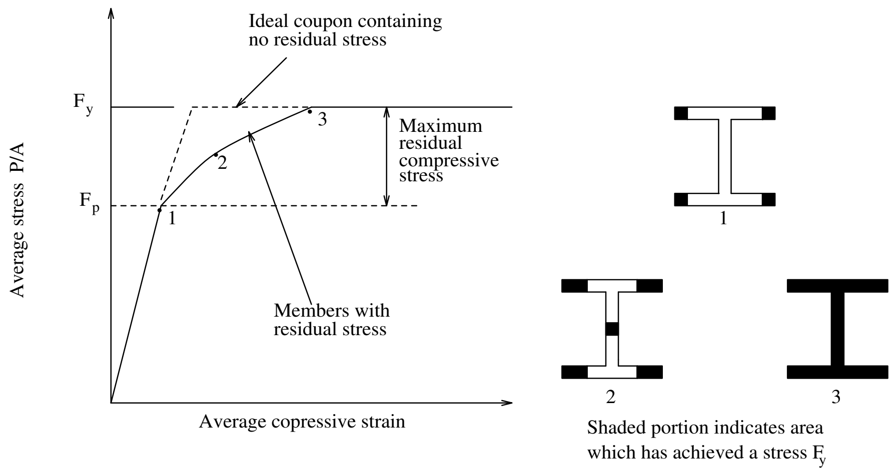


Figure 16.5: Influence of Residual Stress on Average Stress-Strain Curve of a Rolled Section

Bar Designation	Diameter (in.)	Area (in^2)	Perimeter in	Weight lb/ft
No. 2	2/8=0.250	0.05	0.79	0.167
No. 3	3/8=0.375	0.11	1.18	0.376
No. 4	4/8=0.500	0.20	1.57	0.668
No. 5	5/8=0.625	0.31	1.96	1.043
No. 6	6/8=0.750	0.44	2.36	1.5202
No. 7	7/8=0.875	0.60	2.75	2.044
No. 8	8/8=1.000	0.79	3.14	2.670
No. 9	9/8=1.128	1.00	3.54	3.400
No. 10	10/8=1.270	1.27	3.99	4.303
No. 11	11/8=1.410	1.56	4.43	5.313
No. 14	14/8=1.693	2.25	5.32	7.650
No. 18	18/8=2.257	4.00	7.09	13.60

Table 16.2: Properties of Reinforcing Bars

or

$$E = 33\gamma^{1.5}\sqrt{f'_c} \tag{16.2}$$

where both f'_c and E are in psi and γ is in lbs/ft^3.

24 Typical concrete (compressive) strengths range from 3,000 to 6,000 psi; However **high strength concrete** can go up to 14,000 psi.

25 All concrete fail at an ultimate strain of 0.003, Fig. 16.6.

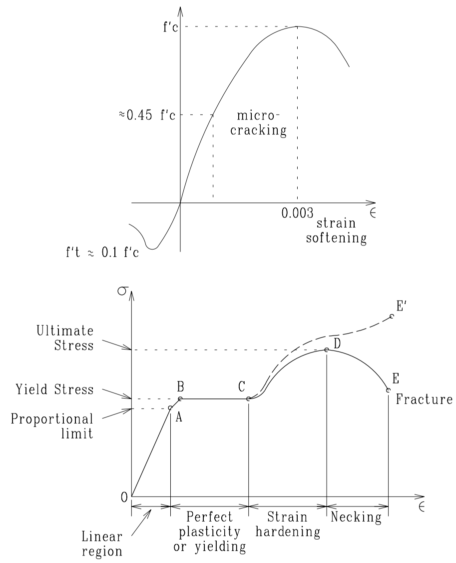


Figure 16.6: Concrete Stress-Strain curve

26 Pre-peak nonlinearity is caused by micro-cracking Fig. 16.7.

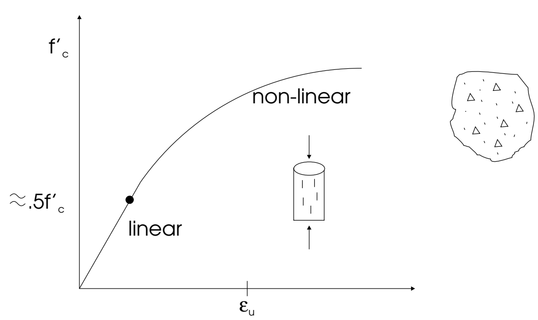


Figure 16.7: Concrete microcracking

27 The tensile strength of concrete f'_t is about 10% of the compressive strength.

28 Density of normal weight concrete is 145 lbs/ft^3 and 100 lbs/ft^3 for lightweight concrete.

Designation	A	d	t_w	b_f	t_f	$\frac{b_f}{2t_f}$	$\frac{h_c}{t_w}$	I_x	S_x	r_x	I_y	S_y	r_y	Z_x	Z_y	J
	in^2	in	in	in	in			in^4	in^3	in	in^4	in^3	in	in^3	in^3	in^4
WT 8.x 50.	14.7	8.48	0.585	10.425	0.985	0	12.1	76.8	11.4	2.28	93.10	17.90	2.51	20.70	27.40	3.85
WT 8.x 45.	13.1	8.38	0.525	10.365	0.875	0	13.5	67.2	10.1	2.27	81.30	15.70	2.49	18.10	24.00	2.72
WT 8.x 39.	11.3	8.26	0.455	10.295	0.760	0	15.6	56.9	8.6	2.24	69.20	13.40	2.47	15.30	20.50	1.78
WT 8.x 34.	9.8	8.16	0.395	10.235	0.665	0	18.0	48.6	7.4	2.22	59.50	11.60	2.46	13.00	17.70	1.19
WT 8.x 29.	8.4	8.22	0.430	7.120	0.715	0	16.5	48.7	7.8	2.41	21.60	6.06	1.60	13.80	9.43	1.10
WT 8.x 25.	7.4	8.13	0.380	7.070	0.630	0	18.7	42.3	6.8	2.40	18.60	5.26	1.59	12.00	8.16	0.76
WT 8.x 23.	6.6	8.06	0.345	7.035	0.565	0	20.6	37.8	6.1	2.39	16.40	4.67	1.57	10.80	7.23	0.65
WT 8.x 20.	5.9	8.01	0.305	6.995	0.505	0	23.3	33.1	5.3	2.37	14.40	4.12	1.57	9.43	6.37	0.40
WT 8.x 18.	5.3	7.93	0.295	6.985	0.430	0	24.1	30.6	5.1	2.41	12.20	3.50	1.52	8.93	5.42	0.27
WT 8.x 16.	4.6	7.94	0.275	5.525	0.440	0	25.8	27.4	4.6	2.45	6.20	2.24	1.17	8.27	3.52	0.23
WT 8.x 13.	3.8	7.84	0.250	5.500	0.345	0	28.4	23.5	4.1	2.47	4.80	1.74	1.12	8.12	2.74	0.13
WT 7.x365.	107.0	11.21	3.070	17.890	4.910	0	1.9	739.0	95.4	2.62	2360.00	264.00	4.69	211.00	408.00	714.00
WT 7.x333.	97.8	10.82	2.830	17.650	4.520	0	2.0	622.0	82.1	2.52	2080.00	236.00	4.62	182.00	365.00	555.00
WT 7.x303.	88.9	10.46	2.595	17.415	4.160	0	2.2	524.0	70.6	2.43	1840.00	211.00	4.55	157.00	326.00	430.00
WT 7.x275.	80.9	10.12	2.380	17.200	3.820	0	2.4	442.0	60.9	2.34	1630.00	189.00	4.49	136.00	292.00	331.00
WT 7.x250.	73.5	9.80	2.190	17.010	3.500	0	2.6	375.0	52.7	2.26	1440.00	169.00	4.43	117.00	261.00	255.00
WT 7.x228.	66.9	9.51	2.015	16.835	3.210	0	2.8	321.0	45.9	2.19	1280.00	152.00	4.38	102.00	234.00	196.00
WT 7.x213.	62.6	9.34	1.875	16.695	3.035	0	3.0	287.0	41.4	2.14	1180.00	141.00	4.34	91.70	217.00	164.00
WT 7.x199.	58.5	9.15	1.770	16.590	2.845	0	3.2	257.0	37.6	2.10	1090.00	131.00	4.31	82.90	201.00	135.00
WT 7.x185.	54.4	8.96	1.655	16.475	2.660	0	3.4	229.0	33.9	2.05	994.00	121.00	4.27	74.40	185.00	110.00
WT 7.x171.	50.3	8.77	1.540	16.360	2.470	0	3.7	203.0	30.4	2.01	903.00	110.00	4.24	66.20	169.00	88.30
WT 7.x156.	45.7	8.56	1.410	16.230	2.260	0	4.0	176.0	26.7	1.96	807.00	99.40	4.20	57.70	152.00	67.50
WT 7.x142.	41.6	8.37	1.290	16.110	2.070	0	4.4	153.0	23.5	1.92	722.00	89.70	4.17	50.40	137.00	51.80
WT 7.x129.	37.8	8.19	1.175	15.995	1.890	0	4.9	133.0	20.7	1.88	645.00	80.70	4.13	43.90	123.00	39.30
WT 7.x117.	34.2	8.02	1.070	15.890	1.720	0	5.3	116.0	18.2	1.84	576.00	72.50	4.10	38.20	110.00	29.60
WT 7.x106.	31.0	7.86	0.980	15.800	1.560	0	5.8	102.0	16.2	1.81	513.00	65.00	4.07	33.40	99.00	22.20
WT 7.x 97.	28.4	7.74	0.890	15.710	1.440	0	6.4	89.8	14.4	1.78	466.00	59.30	4.05	29.40	90.20	17.30
WT 7.x 88.	25.9	7.61	0.830	15.650	1.310	0	6.9	80.5	13.0	1.76	419.00	53.50	4.02	26.30	81.40	13.20
WT 7.x 80.	23.4	7.49	0.745	15.565	1.190	0	7.7	70.2	11.4	1.73	374.00	48.10	4.00	22.80	73.00	9.84
WT 7.x 73.	21.3	7.39	0.680	15.500	1.090	0	8.4	62.5	10.2	1.71	338.00	43.70	3.98	20.20	66.30	7.56
WT 7.x 66.	19.4	7.33	0.645	14.725	1.030	0	8.8	57.8	9.6	1.73	274.00	37.20	3.76	18.60	56.60	6.13
WT 7.x 60.	17.7	7.24	0.590	14.670	0.940	0	9.7	51.7	8.6	1.71	247.00	33.70	3.74	16.50	51.20	4.67
WT 7.x 55.	16.0	7.16	0.525	14.605	0.860	0	10.9	45.3	7.6	1.68	223.00	30.60	3.73	14.40	46.40	3.55
WT 7.x 50.	14.6	7.08	0.485	14.565	0.780	0	11.8	40.9	6.9	1.67	201.00	27.60	3.71	12.90	41.80	2.68
WT 7.x 45.	13.2	7.01	0.440	14.520	0.710	0	13.0	36.4	6.2	1.66	181.00	25.00	3.70	11.50	37.80	2.03
WT 7.x 41.	12.0	7.16	0.510	10.130	0.855	0	11.2	41.2	7.1	1.85	74.20	14.60	2.48	13.20	22.40	2.53
WT 7.x 37.	10.9	7.09	0.450	10.070	0.785	0	12.7	36.0	6.3	1.82	66.90	13.30	2.48	11.50	20.30	1.94
WT 7.x 34.	10.0	7.02	0.415	10.035	0.720	0	13.7	32.6	5.7	1.81	60.70	12.10	2.46	10.40	18.50	1.51
WT 7.x 31.	9.0	6.95	0.375	9.995	0.645	0	15.2	28.9	5.1	1.80	53.70	10.70	2.45	9.16	16.40	1.10
WT 7.x 27.	7.8	6.96	0.370	8.060	0.660	0	15.4	27.6	4.9	1.88	28.80	7.16	1.92	8.87	11.00	0.97
WT 7.x 24.	7.1	6.89	0.340	8.030	0.595	0	16.8	24.9	4.5	1.87	25.70	6.40	1.91	8.00	9.82	0.73
WT 7.x 22.	6.3	6.83	0.305	7.995	0.530	0	18.7	21.9	4.0	1.86	22.60	5.65	1.89	7.05	8.66	0.52
WT 7.x 19.	5.6	7.05	0.310	6.770	0.515	0	19.8	23.3	4.2	2.04	13.30	3.94	1.55	7.45	6.07	0.40
WT 7.x 17.	5.0	6.99	0.285	6.745	0.455	0	21.5	20.9	3.8	2.04	11.70	3.45	1.53	6.74	5.32	0.28
WT 7.x 15.	4.4	6.92	0.270	6.730	0.385	0	22.7	19.0	3.5	2.07	9.79	2.91	1.49	6.25	4.49	0.19
WT 7.x 13.	3.8	6.95	0.255	5.025	0.420	0	24.1	17.3	3.3	2.12	4.45	1.77	1.08	5.89	2.77	0.18
WT 7.x 11.	3.3	6.87	0.230	5.000	0.335	0	26.7	14.8	2.9	2.14	3.50	1.40	1.04	5.20	2.19	0.10
WT 6.x168.	49.4	8.41	1.775	13.385	2.955	0	2.7	190.0	31.2	1.96	593.00	88.60	3.47	68.40	137.00	120.00
WT 6.x153.	44.8	8.16	1.625	13.235	2.705	0	3.0	162.0	27.0	1.90	525.00	79.30	3.42	59.10	122.00	92.00
WT 6.x140.	41.0	7.93	1.530	13.140	2.470	0	3.2	141.0	24.1	1.86	469.00	71.30	3.38	51.90	110.00	70.90
WT 6.x126.	37.0	7.70	1.395	13.005	2.250	0	3.5	121.0	20.9	1.81	414.00	63.60	3.34	44.80	97.90	53.50
WT 6.x115.	33.9	7.53	1.285	12.895	2.070	0	3.8	106.0	18.5	1.77	371.00	57.50	3.31	39.40	88.40	41.60
WT 6.x105.	30.9	7.36	1.180	12.790	1.900	0	4.1	92.1	16.4	1.73	332.00	51.90	3.28	34.50	79.70	32.20
WT 6.x 95.	27.9	7.19	1.060	12.670	1.735	0	4.6	79.0	14.2	1.68	295.00	46.50	3.25	29.80	71.30	24.40
WT 6.x 85.	25.0	7.01	0.960	12.570	1.560	0	5.1	67.8	12.3	1.65	259.00	41.20	3.22	25.60	63.00	17.70
WT 6.x 76.	22.4	6.86	0.870	12.480	1.400	0	5.6	58.5	10.8	1.62	227.00	36.40	3.19	22.00	55.60	12.80
WT 6.x 68.	20.0	6.70	0.790	12.400	1.250	0	6.1	50.6	9.5	1.59	199.00	32.10	3.16	19.00	49.00	9.22
WT 6.x 60.	17.6	6.56	0.710	12.320	1.105	0	6.8	43.4	8.2	1.57	172.00	28.00	3.13	16.20	42.70	6.43
WT 6.x 53.	15.6	6.45	0.610	12.220	0.990	0	8.0	36.3	6.9	1.53	151.00	24.70	3.11	13.60	37.50	4.55
WT 6.x 48.	14.1	6.36	0.550	12.160	0.900	0	8.8	32.0	6.1	1.51	135.00	22.20	3.09	11.90	33.70	3.42
WT 6.x 44.	12.8	6.26	0.515	12.125	0.810	0	9.4	28.9	5.6	1.50	120.00	19.90	3.07	10.70	30.20	2.54
WT 6.x 40.	11.6	6.19	0.470	12.080	0.735	0	10.3	25.8	5.0	1.49	108.00	17.90	3.05	9.49	27.20	1.92
WT 6.x 36.	10.6	6.13	0.430	12.040	0.670	0	11.3	23.2	4.5	1.48	97.50	16.20	3.04	8.48	24.60	1.46
WT 6.x 33.	9.5	6.06	0.390	12.000	0.605	0	12.4	20.6	4.1	1.47	87.20	14.50	3.02	7.50	22.00	1.09
WT 6.x 29.	8.5	6.09	0.360	10.010	0.640	0	13.5	19.1	3.8	1.50	53.50	10.70	2.51	6.97	16.30	1.05
WT 6.x 27.	7.8	6.03	0.345	9.995	0.575	0	14.1	17.7	3.5	1.51	47.90	9.58	2.48	6.46	14.60	0.79
WT 6.x 25.	7.3	6.09	0.370	8.080	0.640	0	13.1	18.7	3.8	1.60	28.20	6.97	1.96	6.90	10.70	0.89
WT 6.x 23.	6.6	6.03	0.335	8.045	0.575	0	14.5	16.6	3.4	1.58	25.00	6.21	1.94	6.12	9.50	0.66
WT 6.x 20.	5.9	5.97	0.295	8.005	0.515	0	16.5	14.4	3.0	1.57	22.00	5.51	1.93	5.30	8.41	0.48
WT 6.x 18.	5.2	6.25	0.300	6.560	0.520	0	18.1	16.0	3.2	1.76	12.20	3.73	1.54	5.71	5.73	0.37
WT 6.x 15.	4.4	6.17	0.260	6.520	0.440	0	20.9	13.5	2.8	1.75	10.20	3.12	1.52	4.83	4.78	0.23
WT 6.x 13.	3.8	6.11	0.230	6.490	0.380	0	23.6	11.7	2.4	1.75	8.66	2.67	1.51	4.20	4.08	0.15
WT 6.x 11.	3.2	6.16	0.260	4.030	0.425	0	20.9	11.7	2.6	1.90	2.33	1.16	0.85	4.63	1.83	0.15
WT 6.x 10.	2.8	6.08	0.235	4.005	0.350	0	23.1	10.1	2.3	1.90	1.88	0.94	0.82	4.11	1.49	0.09
WT 6.x 8.	2.4	5.99	0.220	3.990	0.265	0	24.7	8.7	2.0	1.92	1.41	0.71	0.77	3.72	1.13	0.05
WT 6.x 7.	2.1	5.95	0.200	3.970	0.225	0	27.2	7.7	1.8	1.92	1.18	0.59	0.75	3.32	0.95	0.04
WT 5.x 56.	16.5	5.68	0.755	10.415	1.250	0	5.2	28.6	6.4	1.32	118.00	22.60	2.68	13.40	34.60	7.50
WT 5.x 50.	14.7	5.55	0.680	10.340	1.120	0	5.8	24.5	5.6	1.29	103.00	20.00	2.65	11.40	30.50	5.41
WT 5.x 44.	12.9	5.42	0.605	10.265	0.990	0	6.5	20.8	4.8	1.27	89.30	17.40	2.63	9.65	26.50	3.75
WT 5.x 39.	11.3	5.30	0.530	10.190	0.870	0	7.4	17.4	4.0	1.24	76.80	15.10	2.60	8.06	22.90	2.55
WT 5.x 34.	10.0	5.20	0.470	10.130	0.770	0	8.4	14.9	3.5	1.22	66.80	13.20	2.59	6.85	20.00	1.78
WT 5.x 30.	8.8	5.11	0.420	10.080	0.680	0	9.4	12.9	3.0	1.21	58.10	11.50	2.57	5.87	17.50	1.23
WT 5.x 27.	7.9	5.05	0.370	10.030	0.615	0	10.6	11.1	2.6	1.19	51.70	10.30	2.56	5.05	15.70	0.91
WT 5.x 25.	7.2	4.99	0.340	10.000	0.560	0	11.6	10.0	2.4	1.18	46.70	9.34	2.54	4.52	14.20	0.69
WT 5.x 23.	6.6	5.05	0.350	8.020	0.620	0	11.2	10.2	2.5	1.24	26.70	6.65	2.01	4.65	10.10	0.75
WT 5.x 20.	5.7	4.96	0.315	7.985	0.530	0	12.5	8.8	2.2	1.24	22.50	5.64	1.98	3.99	8.59	0.49
WT 5.x 17.	4.8	4.86	0.290	7.960	0.435	0	13.6	7.7	1.9	1.26	18.30	4.60	1.94	3.48	7.01	0.29
WT 5.x 15.	4.4	5.24	0.300	5.810	0.510	0	14.8	9.3	2.2	1.45	8.35	2.87	1.37	4.01	4.42	0.31
WT 5.x 13.	3.8	5.16	0.260	5.770	0.440	0	17.0	7.9	1.9	1.44	7.05	2.44	1.36	3.39	3.75	0.20
WT 5.x 11.	3.2	5.09	0.240	5.750	0.360	0	18.4	6.9	1.7	1.46	5.71	1.99	1.33	3.02	3.05	0.12
WT 5.x 10.	2.8	5.12	0.250	4.020	0.395	0	17.7	6.7	1.7	1.54	2.15	1.07	0.87	3.10	1.68	0.12
WT 5.x 9.	2.5	5.05	0.240	4.010	0.330	0	18.4	6.1	1.6	1.56	1.78	0.89	0.84	2.90	1.40	0.08
WT 5.x 8.	2.2	4.99	0.230	4.000	0.270	0	19.2	5.4	1.5	1.57	1.45	0.72	0.81	3.03	1.15	0.05
WT 5.x 6.	1.8	4.93	0.190	3.960	0.210	0	23.3	4.3	1.2	1.57	1.09	0.55	0.79	2.50	0.87	0.03

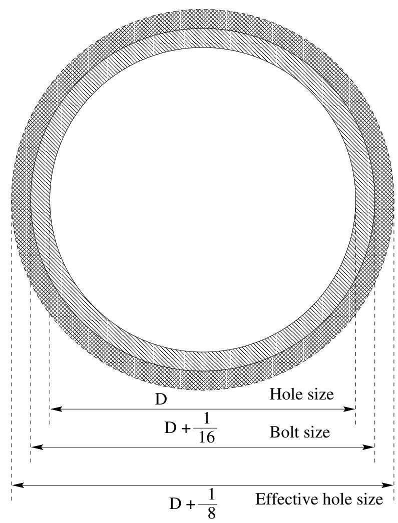


Figure 17.2: Hole Sizes

17 Accordingly, for a plate of width w and thickness t, and with a single hole which accomodates a bolt of diameter D, the net area would be

$$\boxed{A_n = wt - \left(D + \frac{1}{8}\right) t} \tag{17.2}$$

18 Whenever there is a need for more than two rivets or bolts, the holes are not aligned but rather **staggered**. Thus there is more than one potential failure line which can be used to determine the net area.

19 For staggered holes, we define the net are in terms of, Fig. 17.3,

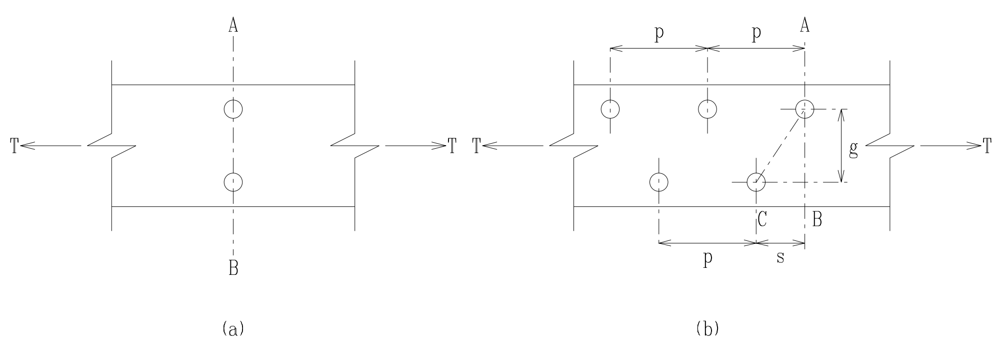
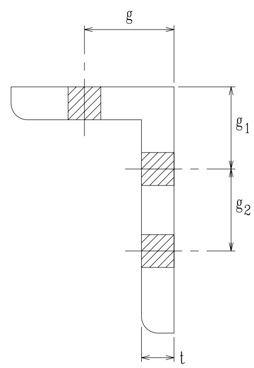


Figure 17.3: Effect of Staggered Holes on Net Area

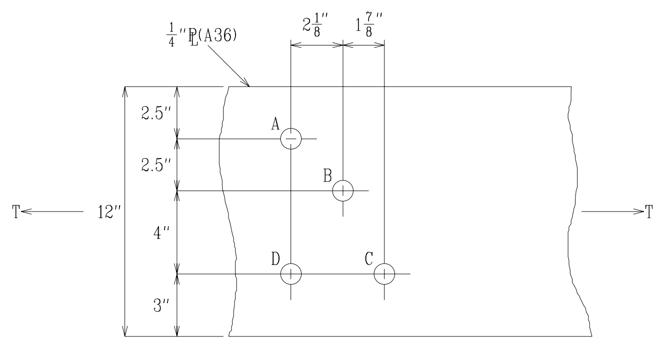
pitch s, longitudinal center to center spacing between two consecutive holes.

gage g, transverse center to center spacing between two adjacent holes.



Leg	8	7	6	5	4	$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$1\frac{1}{4}$	1
g	$4\frac{1}{2}$	4	$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	$1\frac{3}{4}$	$1\frac{3}{8}$	$1\frac{1}{8}$	1	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$
g_1	3	$2\frac{1}{2}$	$2\frac{1}{4}$	2										
g_2	3	3	$2\frac{1}{2}$	$1\frac{3}{4}$										

Table 17.1: Usual Gage Lengths g, g_1, g_2



Solution:
We consider various paths:

$$A-D: \quad \left[12-2\left(\frac{15}{16}+\frac{1}{16}\right)\right](0.25) = 2.50 \text{ in}^2 \tag{17.5-a}$$

$$A-B-D: \quad \left[12-3\left(\frac{15}{16}+\frac{1}{16}\right)+\frac{(2.125)^2}{4(2.5)}+\frac{(2.125)^2}{4(4)}\right](0.25) = 2.43 \text{ in}^2 \tag{17.5-b}$$

$$A-B-C: \quad \left[12-3\left(\frac{15}{16}+\frac{1}{16}\right)+\frac{(2.125)^2}{4(2.5)}+\frac{(1.875)^2}{4(4)}\right](0.25) = \boxed{2.4 \text{ in}^2} \tag{17.5-c}$$

Thus path A-B-C controls. ■

■ Example 17-2: Net Area, Angle

Determine the net area A_n for the angle shown below if $\frac{15}{16}$ in. diameter holes are used.

Type of members	Minimum Number of Fasteners/line	Special Requirements	A_e
Fastened Rolled Sections			
Members having all cross sectional elements connected to transmit tensile force	1 (or welds)		A_n
W, M, or S shapes with flange widths not less than 2/3 the depth, and structural tees cut from these shapes, provided the connection is to the flanges. Bolted or riveted connections shall have no fewer than three fasteners per line in the direction of stress	3 (or welds)	$\frac{b}{d} \geq 0.67$	$0.9A_n$
W, M, or S not meeting above conditions, structural tees cut from these shapes and all other shapes, including built-up cross-sections. Bolted or riveted connections shall have no fewer than three fasteners per line in the direction of stress	3 (or welds)		$0.85A_n$
Structural tees cut from W, M, or S connected at flanges only	3 (or welds)	$\frac{b}{d} \geq 0.67$	$0.9A_n$
All members with bolted or riveted connections having only two fasteners per line in the direction of stress	2		$0.75A_n$
Welded Plates			
	Welds	$l > 2w$	A_g
	Welds	$2w > l > 1.5w$	$0.87A_g$
	Welds	$l/w.1$	$0.75A_g$

b Flange width; d section depth; l Weld length; w Plate width;

Table 17.2: Effective Net Area A_e for Bolted and Welded Connections

where Φ_t = resistance factor relating to tensile strength
T_n = nominal strength of a tension member
T_u = factored load on a tension member

30 From Table 14.3, Φ_t is 0.75 and 0.9 for fracture and yielding failure respectively.

17.3.1 Tension Failure

31 The design strength $\Phi_t T_n$ is the smaller of that based on, Fig. 17.5

1. **Yielding in the gross section:** We can not allow yielding of the gross section, because this will result in unacceptable elongation of the entire member.

$$\boxed{\Phi_t T_n = \Phi_t F_y A_g = 0.90 F_y A_g} \tag{17.12}$$

or

2. **Fracture in the net section:** Yielding is locally allowed, because A_e is applicable only on a small portion of the element. Local excessive elongation is allowed, however fracture must be prevented. This mode of failure usually occurs if there is insufficient distance behind the pin.

$$\boxed{\Phi_t T_n = \Phi_t F_u A_e = 0.75 F_u A_e} \tag{17.13}$$

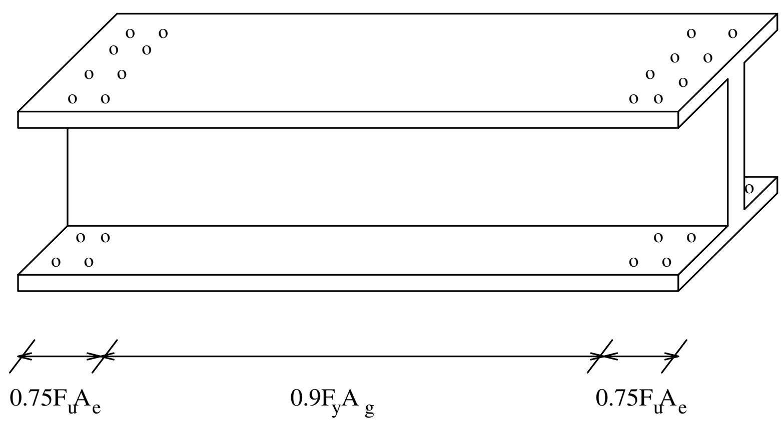


Figure 17.5: Net and Gross Areas

17.3.2 Block Shear Failure

32 For bolted connections, **tearing failure** may occur and control the strength of the tension member.

33 For instance, with reference to Fig. 17.6 the angle tension member attached to the **gusset**

Chapter 18

COLUMN STABILITY

18.1 Introduction; Discrete Rigid Bars

18.1.1 Single Bar System

1 Let us begin by considering a rigid bar connected to the support by a spring and axially loaded at the other end, Fig. 18.1. Taking moments about point A:

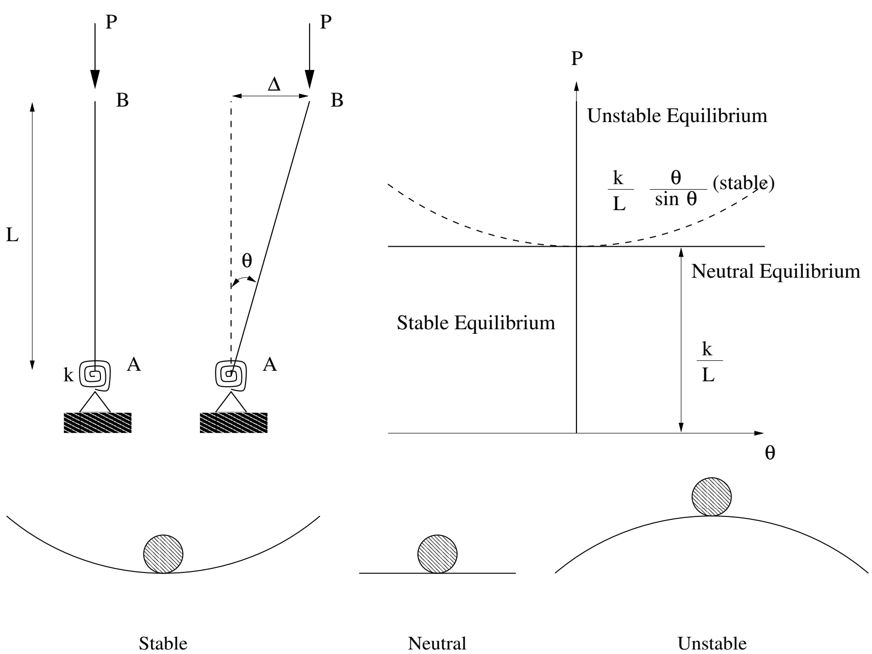


Figure 18.1: Stability of a Rigid Bar

$$\Sigma M_A = P\Delta - k\theta = 0 \quad (18.1\text{-a})$$

$$\Delta = L\theta \text{ for small rotation} \quad (18.1\text{-b})$$

$$P\theta L - k\theta = 0 \quad (18.1\text{-c})$$

$$(P - \frac{k}{L}) = 0 \quad (18.1\text{-d})$$

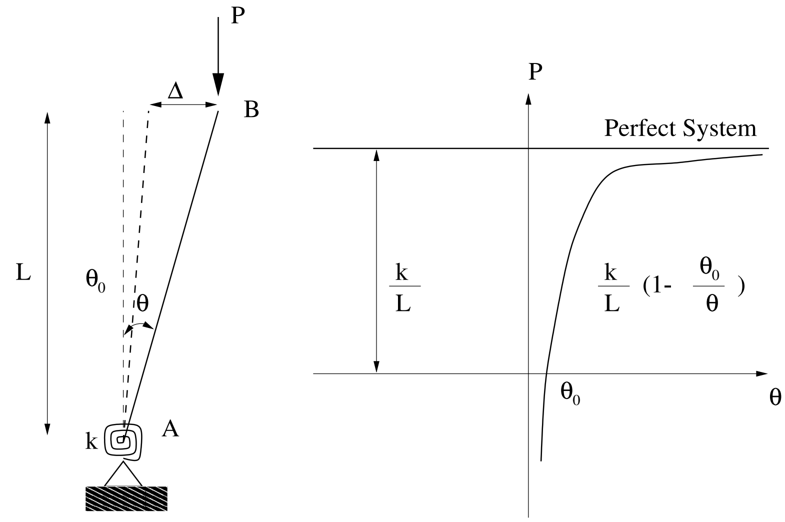


Figure 18.2: Stability of a Rigid Bar with Initial Imperfection

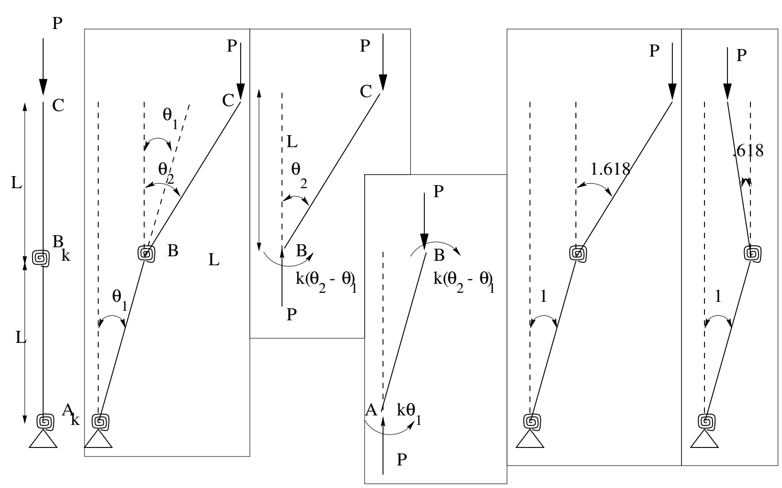


Figure 18.3: Stability of a Two Rigid Bars System

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & 1-\frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18.12-c}$$

$$\begin{bmatrix} -0.618 & -1 \\ -1 & -1.618 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18.12-d}$$

we now arbitrarily set $\theta_1 = 1$, then $\theta_2 = -1/1.618 = -0.618$, thus the second eigenmode is

$$\boxed{\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}} \tag{18.13}$$

18.1.3 ‡Analogy with Free Vibration

12 The problem just considered bears great ressemblence with the vibration of a two degree of freedom mass spring system, Fig. 18.4.

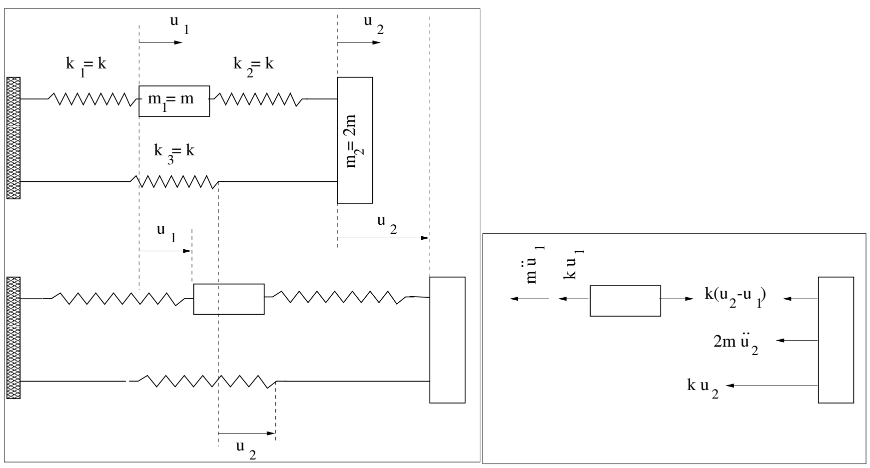


Figure 18.4: Two DOF Dynamic System

13 Each mass is subjected to an inertial force equals to the mass times the acceleration, and the spring force:

$$2m\ddot{\mathbf{u}}_2 + k\mathbf{u}_2 + k(\mathbf{u}_2 - \mathbf{u}_1) = 0 \tag{18.14-a}$$

$$m\ddot{\mathbf{u}}_1 + k\mathbf{u}_1 + k9\mathbf{u}_2 - \mathbf{u}_1 0 = 0 \tag{18.14-b}$$

or in matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{Bmatrix} \ddot{\mathbf{u}}_1 \\ \ddot{\mathbf{u}}_2 \end{Bmatrix}}_{\ddot{\mathbf{U}}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix}}_{\mathbf{U}} \tag{18.15}$$

14 The characteristic equation is $|\mathbf{K} - \lambda\mathbf{M}|$ where $\lambda = \omega^2$, and ω is the natural frequency.

$$\begin{vmatrix} 2h-\lambda & -h \\ -h & 2h-2\lambda \end{vmatrix} = 0 \tag{18.16}$$

Chapter 19

STEEL COMPRESSION MEMBERS

1 This chapter will cover the elementary analysis and design of steel column according to the LRFD provisions.

19.1 AISC Equations

2 By introducing a **slenderness parameter** λ_c (not to be confused with the slenderness ratio) defined as

$$\lambda_c^2 = \frac{F_y}{F_{cr}^{Euler}} = \frac{F_y}{\frac{\pi^2 E}{\left(\frac{KL}{r_{min}}\right)^2}} \tag{19.1-a}$$

$$\lambda_c = \frac{KL}{r_{min}}\sqrt{\frac{F_y}{\pi^2 E}} \tag{19.1-b}$$

Hence, this parameter accounts for both the slenderness ratio as well as steel material properties. This new parameter is a more suitable one than the slenderness ratio (which was a delimeter for elastic buckling) for the inelastic buckling.

3 Equation 18.53 becomes

$$\frac{F_{cr}}{F_y} = 1 - \frac{\lambda_c^2}{4} \qquad \text{for } \lambda_c \le \sqrt{2} \tag{19.2}$$

4 When $\lambda_c > \sqrt{2}$, then Euler equation applies

$$\frac{F_{cr}}{F_y} = \frac{1}{\lambda_c^2} \qquad \text{for } \lambda_c \ge \sqrt{2} \tag{19.3}$$

Hence the first equation is based on inelastic buckling (with gross yielding as a limiting case), and the second one on elastic buckling. The two curves are tangent to each other.

5 The above equations are valid for concentrically straight members.

$\frac{Kl}{r_{min}}$	λ_c	$\phi_c F_{cr}$	$\frac{Kl}{r_{min}}$	λ_c	$\phi_c F_{cr}$	$\frac{Kl}{r_{min}}$	λ_c	$\phi_c F_{cr}$	$\frac{Kl}{r_{min}}$	λ_c	$\phi_c F_{cr}$	$\frac{Kl}{r_{min}}$	λ_c	$\phi_c F_{cr}$
1	0.01	30.60	41	0.46	28.01	81	0.91	21.66	121	1.36	14.16	161	1.81	8.23
2	0.02	30.59	42	0.47	27.89	82	0.92	21.48	122	1.37	13.98	162	1.82	8.13
3	0.03	30.59	43	0.48	27.76	83	0.93	21.29	123	1.38	13.80	163	1.83	8.03
4	0.04	30.57	44	0.49	27.63	84	0.94	21.11	124	1.39	13.62	164	1.84	7.93
5	0.06	30.56	45	0.50	27.51	85	0.95	20.92	125	1.40	13.44	165	1.85	7.84
6	0.07	30.54	46	0.52	27.37	86	0.96	20.73	126	1.41	13.27	166	1.86	7.74
7	0.08	30.52	47	0.53	27.24	87	0.98	20.54	127	1.42	13.09	167	1.87	7.65
8	0.09	30.50	48	0.54	27.10	88	0.99	20.35	128	1.44	12.92	168	1.88	7.56
9	0.10	30.47	49	0.55	26.97	89	1.00	20.17	129	1.45	12.74	169	1.90	7.47
10	0.11	30.44	50	0.56	26.83	90	1.01	19.98	130	1.46	12.57	170	1.91	7.38
11	0.12	30.41	51	0.57	26.68	91	1.02	19.79	131	1.47	12.40	171	1.92	7.30
12	0.13	30.37	52	0.58	26.54	92	1.03	19.60	132	1.48	12.23	172	1.93	7.21
13	0.15	30.33	53	0.59	26.39	93	1.04	19.41	133	1.49	12.06	173	1.94	7.13
14	0.16	30.29	54	0.61	26.25	94	1.05	19.22	134	1.50	11.88	174	1.95	7.05
15	0.17	30.24	55	0.62	26.10	95	1.07	19.03	135	1.51	11.71	175	1.96	6.97
16	0.18	30.19	56	0.63	25.94	96	1.08	18.84	136	1.53	11.54	176	1.97	6.89
17	0.19	30.14	57	0.64	25.79	97	1.09	18.65	137	1.54	11.37	177	1.99	6.81
18	0.20	30.08	58	0.65	25.63	98	1.10	18.46	138	1.55	11.20	178	2.00	6.73
19	0.21	30.02	59	0.66	25.48	99	1.11	18.27	139	1.56	11.04	179	2.01	6.66
20	0.22	29.96	60	0.67	25.32	100	1.12	18.08	140	1.57	10.89	180	2.02	6.59
21	0.24	29.90	61	0.68	25.16	101	1.13	17.89	141	1.58	10.73	181	2.03	6.51
22	0.25	29.83	62	0.70	24.99	102	1.14	17.70	142	1.59	10.58	182	2.04	6.44
23	0.26	29.76	63	0.71	24.83	103	1.16	17.51	143	1.60	10.43	183	2.05	6.37
24	0.27	29.69	64	0.72	24.66	104	1.17	17.32	144	1.61	10.29	184	2.06	6.30
25	0.28	29.61	65	0.73	24.50	105	1.18	17.13	145	1.63	10.15	185	2.07	6.23
26	0.29	29.53	66	0.74	24.33	106	1.19	16.94	146	1.64	10.01	186	2.09	6.17
27	0.30	29.45	67	0.75	24.16	107	1.20	16.75	147	1.65	9.87	187	2.10	6.10
28	0.31	29.36	68	0.76	23.99	108	1.21	16.56	148	1.66	9.74	188	2.11	6.04
29	0.33	29.27	69	0.77	23.82	109	1.22	16.37	149	1.67	9.61	189	2.12	5.97
30	0.34	29.18	70	0.79	23.64	110	1.23	16.18	150	1.68	9.48	190	2.13	5.91
31	0.35	29.09	71	0.80	23.47	111	1.24	16.00	151	1.69	9.36	191	2.14	5.85
32	0.36	28.99	72	0.81	23.29	112	1.26	15.81	152	1.70	9.23	192	2.15	5.79
33	0.37	28.90	73	0.82	23.11	113	1.27	15.62	153	1.72	9.11	193	2.16	5.73
34	0.38	28.79	74	0.83	22.94	114	1.28	15.44	154	1.73	9.00	194	2.18	5.67
35	0.39	28.69	75	0.84	22.76	115	1.29	15.25	155	1.74	8.88	195	2.19	5.61
36	0.40	28.58	76	0.85	22.58	116	1.30	15.07	156	1.75	8.77	196	2.20	5.55
37	0.41	28.47	77	0.86	22.40	117	1.31	14.88	157	1.76	8.66	197	2.21	5.50
38	0.43	28.36	78	0.87	22.21	118	1.32	14.70	158	1.77	8.55	198	2.22	5.44
39	0.44	28.25	79	0.89	22.03	119	1.33	14.52	159	1.78	8.44	199	2.23	5.39
40	0.45	28.13	80	0.90	21.85	120	1.35	14.34	160	1.79	8.33	200	2.24	5.33

Table 19.1: Design Stress ϕF_{cr} for F_y=36 ksi in Terms of $\frac{KL}{r_{min}}$

Chapter 20

BRACED ROLLED STEEL BEAMS

1 This chapter deals with the behavior and design of **laterally supported** steel beams according to the LRFD provisions.

2 If a beam is not laterally supported, we will have a failure mode governed by lateral torsional buckling.

3 By the end of this lecture you should be able to select the most efficient section (light weight with adequate strength) for a given bending moment and also be able to determine the flexural strength of a given beam.

20.1 Review from Strength of Materials

20.1.1 Flexure

4 Fig.20.1 shows portion of an originally straight beam which has been bent to the radius ρ by end couples M, thus the segment is subjected to pure bending. It is assumed that plane cross-sections normal to the length of the unbent beam remain plane after the beam is bent. Therefore, considering the cross-sections AB and CD a unit distance apart, the similar sectors Oab and bcd give

$$\varepsilon = \frac{y}{\rho} \tag{20.1}$$

where y is measured from the axis of rotation (neutral axis). Thus strains are proportional to the distance from the neutral axis.

5 The corresponding variation in stress over the cross-section is given by the stress-strain diagram of the material rotated 90^o from the conventional orientation, provided the strain axis ε is scaled with the distance y (Fig.20.1). The bending moment M is given by

$$M = \int y\sigma dA \tag{20.2}$$

where dA is an differential area a distance y (Fig.20.1) from the neutral axis. Thus the moment M can be determined if the stress-strain relation is known.

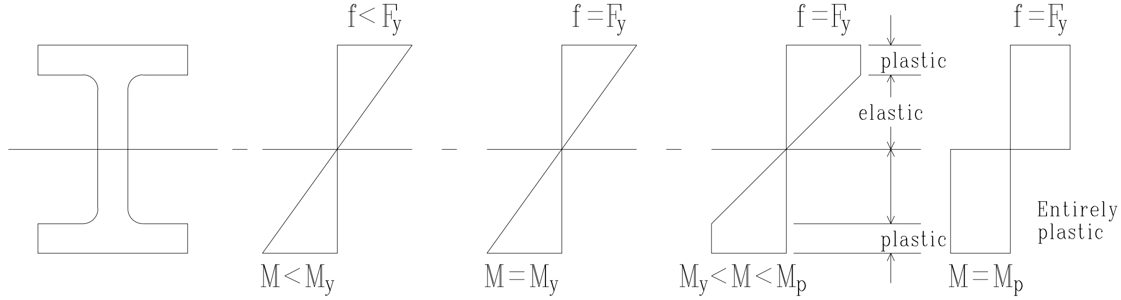


Figure 20.2: Stress distribution at different stages of loading

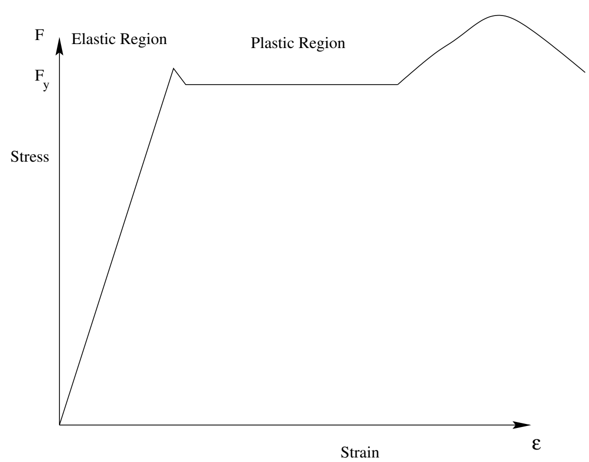


Figure 20.3: Stress-strain diagram for most structural steels

$$F_2 = \int_A \frac{(M+dM)y}{I} dA \tag{20.8-c}$$

$$\tau b dx = F_2 - F_1 \tag{20.8-d}$$

$$= \int_A \frac{dMy}{I} dA \tag{20.8-e}$$

$$\tau = \frac{dM}{dx}\left(\frac{1}{Ib}\right)\int_A y dA \tag{20.8-f}$$

(20.8-g)

substituting $V = dM/dx$ we now obtain

$$\boxed{\begin{array}{rcl}\tau &=& \frac{VQ}{Ib}\\ Q &=& \int_A y dA\end{array}} \tag{20.9}$$

this is the shear formula, and Q is the first moment of the area.

12 For a rectangular beam, we will have a parabolic shear stress distribution.

13 For a W section, it can be easily shown that about 95% of the shear force is carried by the web, and that the average shear stress is within 10% of the actual maximum shear stress.

20.2 Nominal Strength

The strength requirement for beams in load and resistance factor design is stated as

$$\boxed{\phi_b M_n \geq M_u} \tag{20.10}$$

where:

ϕ_b	strength reduction factor; for flexure 0.90
M_n	nominal moment strength
M_u	factored service load moment.

14 The equations given in this chapter are valid for flexural members with the following kinds of cross section and loading:

1. Doubly symmetric (such as W sections) and loaded in Plane of symmetry

2. Singly symmetric (channels and angles) loaded in plane of symmetry or through the shear center parallel to the web[1].

20.3 Flexural Design

20.3.1 Failure Modes and Classification of Steel Beams

15 The strength of flexural members is limited by:

Plastic Hinge: at a particular cross section.

[1]More about shear centers in *Mechanics of Materials II.*

19 The nominal strength M_n for laterally stable compact sections according to LRFD is

$$\boxed{M_n = M_p} \tag{20.12}$$

where:

- M_p plastic moment strength $= ZF_y$
- Z plastic section modulus
- F_y specified minimum yield strength

20 Note that section properties, including Z values are tabulated in Section 16.6.

20.3.1.2 Partially Compact Section

21 If the width to thickness ratios of the compression elements exceed the λ_p values mentioned in Eq. 20.11 but do not exceed the following λ_r, the section is partially compact and we can have local buckling.

$$\boxed{\begin{array}{rlll} \text{Flange:} & \lambda_p < \frac{b_f}{2t_f} \leq \lambda_r & \lambda_p = \frac{65}{\sqrt{F_y}} & \lambda_r = \frac{141}{\sqrt{F_y - F_r}} \\ \text{Web:} & \lambda_p < \frac{h_c}{t_w} \leq \lambda_r & \lambda_p = \frac{640}{\sqrt{F_y}} & \lambda_r = \frac{970}{\sqrt{F_y}} \end{array}} \tag{20.13}$$

where, Fig. 20.6:

- F_y specified minimum yield stress in ksi
- b_f width of the flange
- t_f thickness of the flange
- h_c unsupported height of the web which is twice the distance from the neutral axis to the inside face of the compression flange less the fillet or corner radius.
- t_w thickness of the web.
- F_r residual stress = 10.0 ksi for rolled sections and 16.5 ksi for welded sections.

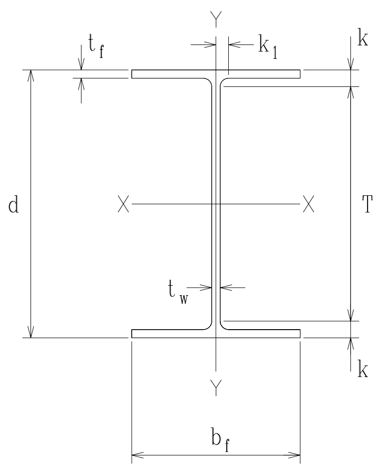


Figure 20.6: W Section

22 The nominal strength of partially compact sections according to LRFD is, Fig. 20.7

Chapter 21

UNBRACED ROLLED STEEL BEAMS

21.1 Introduction

1 In a previous chapter we have examined the behavior of laterally supported beams. Under those conditions, the potential modes of failures were either the formation of a plastic hinge (if the section is compact), or local buckling of the flange or the web (partially compact section).

2 Rarely are the compression flange of beams entirely free of all restraint, and in general there are two types of lateral supports:

1. Continuous lateral support by embedment of the compression flange in a concrete slab.
2. Lateral support at intervals through cross beams, cross frames, ties, or struts.

3 Now that the beam is not laterally supported, we ought to consider a third potential mode of failure, lateral torsional buckling.

21.2 Background

4 Whereas it is beyond the scope of this course to derive the governing differential equation for flexural torsional buckling (which is covered in either *Mechanics of Materials II* or in *Steel Structures*), we shall review some related topics in order to understand the AISC equations later on

5 There are two types of torsion:

Saint-Venant's torsion: or pure torsion (torsion is constant throughout the length) where it is assumed that the cross-sectional plane prior to the application of torsion remains plane, and only rotation occurs.

Warping torsion: out of plane effects arise when the flanges are laterally displaced during twisting. Compression flange will bend in one direction laterally while its tension flange will bend in another. In this case part of the torque is resisted by bending and the rest by Saint-Venant's torsion.

$$X_1 = \frac{\pi}{S_x}\sqrt{\frac{EGJA}{2}} \quad \text{(21.4-a)}$$

$$X_2 = 4\frac{C_w}{I_y}\left(\frac{S_x}{GJ}\right)^2 \quad \text{(21.4-b)}$$

are not really physical properties but a combination of cross-sectional ones which simplifies the writing of Eq. 21.3. Those values are tabulated in the AISC manual.

9 The flexural efficiency of the member increases when X_1 decreases and/or X_2 increases.

21.3.2 Governing Moments

1. $L_b < L_p$: "very short" Plastic hinge

$$\boxed{M_n = M_p = Z_x F_y} \quad (21.5)$$

2. $L_p < L_b < L_r$: "short" inelastic lateral torsional buckling

$$\boxed{M_n = C_b\left[M_p - (M_p - M_r)\left(\frac{L_b - L_p}{L_r - L_p}\right)\right] \leq M_p} \quad (21.6)$$

and

$$\boxed{C_b = 1.75 + 1.05\left(\frac{M_1}{M_2}\right) + 0.3\left(\frac{M_1}{M_2}\right)^2 \leq 2.3} \quad (21.7)$$

where M_1 is the smaller and M_2 is the larger end moment *in the unbraced segment* $\frac{M_1}{M_2}$ is negative when the moments cause single curvature (i.e. one of them is clockwise, the other counterclockwise), hence the most severe loading case with constant M gives $C_b = 1.75 - 1.05 + 0.3 = 1.0$.

M_r is the moment strength available for service load when extreme fiber reach the yield stress F_y;

$$M_r = (F_y - F_r)S_x \quad (21.8)$$

3. $L_r < L_b$ "long" elastic lateral torsional buckling, and the critical moment is the same as in Eq.21.2

$$\boxed{M_{cr} = C_b\frac{\pi}{L_b}\sqrt{\left(\frac{\pi E}{L_b}\right)^2 C_w I_y + EI_y GJ} \leq C_b M_r \text{ and } M_p} \quad (21.9)$$

or if expressed in terms of X_1 and X_2

$$\boxed{M_{cr} = \frac{C_b S_x X_1\sqrt{2}}{\frac{L_b}{r_y}}\sqrt{1 + \frac{X_1^2 X_2}{2\left(\frac{L_b}{r_y}\right)^2}} \leq C_b M_r} \quad (21.10)$$

10 Fig. 21.2 summarizes the governing equations.

Chapter 22

Beam Columns, (Unedited)

UNEDITED

`Examples taken from` *Structural Steel Design; LRFD*`, by T. Burns, Delmar Publishers, 1995`

22.1 Potential Modes of Failures

22.2 AISC Specifications

$$\boxed{\begin{array}{rcl} \dfrac{P_u}{\phi_c P_n} + \dfrac{8}{9}\left(\dfrac{M_{ux}}{\phi_b M_{nx}} + \dfrac{M_{uy}}{\phi_b M_{ny}}\right) & \leq & 1.0 \text{ if } \frac{P_u}{\phi_c P_n} \geq .20 \\ \dfrac{P_u}{2\phi_c P_n} + \dfrac{M_{ux}}{\phi_b M_{nx}} & \leq & 1.0 \text{ if } \frac{P_u}{\phi_c P_n} \leq .20 \end{array}} \tag{22.1}$$

22.3 Examples

22.3.1 Verification

■ **Example 22-1: Verification, (?)**

A W 12 × 120 is used to support the loads and moments as shown below, and is not subjected to sidesway. Determine if the member is adequate and if the factored bending moment occurs about the weak axis. The column is assumed to be perfectly pinned ($K = 1.0$) in both the strong and weak directions and no bracing is supplied. Steel is A36 and assumed $C_b = 1.0$.

be developed or whether buckling behavior controls the bending behavior. The values for L_p and L_r can be calculated using Eq. 6-1 and 6-2 respectively or they can be found in the beam section of the LRFD manual. In either case these values for a W 12 × 120 are:

$$\begin{aligned} L_p &= 13 \text{ ft} \\ L_r &= 75.5 \text{ ft} \end{aligned}$$

5. Since our unbraced length falls between these two values, the beam will be controlled by inelastic buckling, and the nominal moment capacity M_n can be calculated from Eq. 6-5. Using this equation we must first calculate the plastic and elastic moment capacity values, M_p and M_r.

$$\begin{aligned} M_p &= F_y Z_x = (36) \text{ ksi}(186) \text{ in}^3 \frac{\text{ft}}{(12) \text{ in}} = 558 \text{ k.ft} \\ M_r &= (F_{yw} - 10) \text{ ksi} S_x \\ & (36 - 10) \text{ ksi}(163) \text{ in}^3 = 353.2 \text{ k.ft} \end{aligned}$$

6. Using Eq. 6-5 (assuming C_b is equal to 1.0) we find:

$$\begin{aligned} M_n &= C_b[M_p - (M_p - M_r)] \frac{l_b - L_p}{L_r - L_p} \\ M_n &= 1.0[558 - (558 - 453.2)] \text{ k.ft} \frac{(15-13) \text{ ft}}{(75.5-15) \text{ ft}} \\ &= 551.4 \text{ k.ft} \end{aligned}$$

7. Therefore the design moment capacity is as follows:

$$\phi_b M_n = 0.90(551.4) \text{ k.ft} = 496.3 \text{ k.ft}$$

8. Now consider the effects of moment magnification on this section. Based on the alternative method and since the member is not subjected to sidesway ($M_{lt} = 0$)

$$\begin{aligned} M_u &= B_1 M_{nt} \\ B_1 &= \frac{C_m}{1 - \frac{P_u}{P_e}} \\ C_m &= .60 - .4\frac{M_1}{M_2} \\ C_m &= .60 - .4\frac{-50}{50} = 1.0 \\ P_u &= 400 \text{ k} \\ P_e &= \frac{\pi^2 E A_g}{\frac{Kl}{r}^2} \quad \text{(Euler's Buckling Load Equation)} \\ P_e &= \frac{\pi^2(29{,}000 \text{ ksi})(35.3 \text{ in}^2)}{32.67^2} = 9,456 \text{ k} \end{aligned}$$

9. Therefore, calculating the B_1 magnifier we find:

$$B_1 = \frac{1}{1 - \frac{(400) \text{ k}}{(9{,}456) \text{ k}}} = 1.044$$

Calculating the amplified moment as follows:

$$\begin{aligned} M_u &= B_1 M_{nt} \\ M_u &= 1.044(50) \text{ k.ft} = 52.2 \text{ k.ft} \end{aligned}$$

Therefore the adequacy of the section is calculated from Eq. as follows:

$$\begin{aligned} \frac{P_u}{\phi_c P_n} + \frac{8}{9}\frac{M_u x}{\phi_b M_n x} &\leq 1.0 \\ \frac{94000 \text{ k}}{(907.6) \text{ k}} + \frac{8}{9}\frac{(52.2) \text{ k.ft}}{(496.3) \text{ k.ft}} &= \boxed{.53} < 1.0\checkmark \end{aligned}$$

■

Chapter 23

STEEL CONNECTIONS

23.1 Bolted Connections

1 Bolted connections, Fig. 23.1 are increasingly used instead of rivets (too labor intensive) and more often than welds (may cause secondary cracks if not properly performed).

23.1.1 Types of Bolts

2 Most common high strength bolts are the A325 and A490.

3 A325 is made from heat-treated medium carbon steel, min. $F_y^b \approx 81 - 92$ ksi, $F_u^b = 120$ ksi

4 A490 is a higher strength manufactured from an alloy of steel, min. $F_y^b \approx 115 - 130$ ksi, and $F_u^b = 150$ ksi

5 Most common diameters are 3/4", 7/8" for building constructions; 7/8" and 1" for bridges, Table 23.1.

Bolt Diameter (in.)	Nominal Area (in^2)
5/8	0.3068
3/4	0.4418
7/8	0.6013
1	0.7854
1 1/8	0.9940
1 1/4	1.2272

Table 23.1: Nominal Areas of Standard Bolts

23.1.2 Types of Bolted Connections

6 There are two types of bolted connections:

Bearing type which transmits the load by a combination of shear and bearing on the bolt, Fig. 23.2.

Slip-critical transmits load by friction, Fig. 23.3. In addition of providing adequate at ultimate load, it must not slip during service loads.

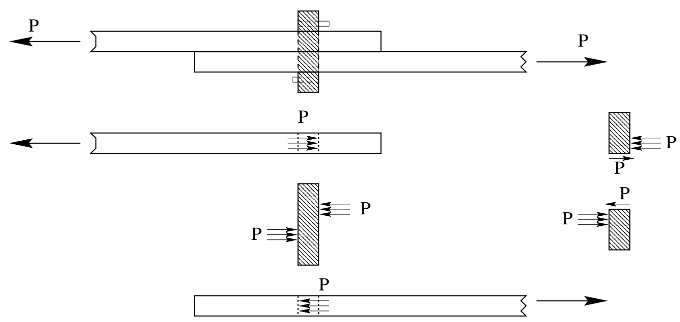


Figure 23.2: Stress Transfer by Shear and Bearing in a Bolted Connection

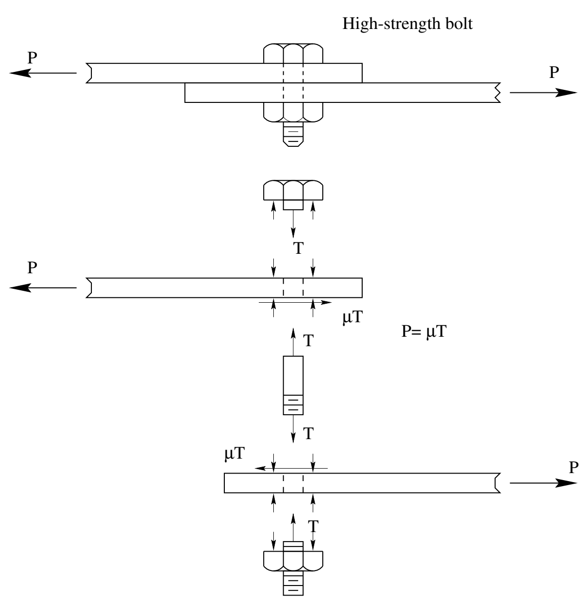


Figure 23.3: Stress Transfer in a Friction Type Bolted Connection

7 Possible failure modes (or "limit states") which may control the strength of a bolted connection are shown in Fig. 23.4.

23.1.3 Nominal Strength of Individual Bolts

Tensile Strength The nominal strength R_n of one fastener in tension is

$$R_n = F_u^b A_n \tag{23.1}$$

where F_u^b is the tensile strength of the bolt, and A_n the area through the threaded portion, also known as "tensile stress area". The ratio of the tensile stress area to the gross area A_g ranges from 0.75 to 0.79.

Chapter 24

REINFORCED CONCRETE BEAMS; Part I

24.1 Introduction

1 Recalling that concrete has a tensile strength (f_t') about one tenth its compressive strength (f_c'), concrete by itself is a very poor material for flexural members.

2 To provide tensile resistance to concrete beams, a reinforcement must be added. Steel is almost universally used as reinforcement (longitudinal or as fibers), but in poorer countries other indigenous materials have been used (such as bamboos).

3 The following lectures will focus exclusively on the flexural design and analysis of reinforced concrete rectangular sections. Other concerns, such as shear, torsion, cracking, and deflections are left for subsequent ones.

4 Design of reinforced concrete structures is governed in most cases by the *Building Code Requirements for Reinforced Concrete*, of the American Concrete Institute (ACI-318). Some of the most relevant provisions of this code are enclosed in this set of notes.

5 We will focus on determining the amount of flexural (that is longitudinal) reinforcement required at a **given section**. For that section, the moment which should be considered for design is the one obtained from the **moment envelope** at that particular point.

24.1.1 Notation

6 In R/C design, it is customary to use the following notation

24.1.4 Basic Relations and Assumptions

13 In developing a design/analysis method for reinforced concrete, the following **basic relations** will be used:

1. Equilibrium: of forces and moment at the cross section. 1) $\Sigma F_x = 0$ or Tension in the reinforcement = Compression in concrete; and 2) $\Sigma M = 0$ or external moment (that is the one obtained from the moment envelope) equal and opposite to the internal one (tension in steel and compression of the concrete).

2. Material Stress Strain: We recall that all normal strength concrete have a failure strain $\epsilon_u = .003$ in compression irrespective of f'_c.

14 Basic **assumptions** used:

Compatibility of Displacements: Perfect bond between steel and concrete (no slip). Note that those two materials do also have very close coefficients of thermal expansion under normal temperature.

Plane section remain plane $\Rightarrow$ strain is proportional to distance from neutral axis.

24.1.5 ACI Code

15 The ACI code is based on limit strength, or $\Phi M_n \geq M_u$ thus a similar design philosophy is used as the one adopted by the LRFD method of the AISC code, **ACI-318: 8.1.1; 9.3.1; 9.3.2**

16 The required strength is based on (**ACI-318: 9.2**)

$$\boxed{\begin{aligned} U &= 1.4D + 1.7L \\ &= 0.75(1.4D + 1.7L + 1.7W) \end{aligned}} \tag{24.1}$$

17 We should consider the behaviors of a reinforced concrete section under increasing load:

1. Section uncracked

2. Section cracked, elastic

3. Section cracked, limit state

The second analysis gives rise to the Working Stress Design (WSD) method (to be covered in Structural Engineering II), and the third one to the Ultimate Strength Design (USD) method.

24.2 Cracked Section, Ultimate Strength Design Method

24.2.1 Equivalent Stress Block

18 In determining the limit state moment of a cross section, we consider Fig. 24.1. Whereas the strain distribution is linear (**ACI-318 10.2.2**), the stress distribution is non-linear because the stress-strain curve of concrete is itself non-linear beyond $0.5f'_c$.

19 Thus we have two alternatives to investigate the moment carrying capacity of the section, **ACI-318: 10.2.6**

24 We have two equations and three unknowns (α, β_1, and β). Thus we need to use test data to solve this problem[1]. From **experimental tests**, the following relations are obtained

f'_c (ppsi)	< 4,000	5,000	6,000	7,000	8,000
α	.72	.68	.64	.60	.56
β	.425	.400	.375	.350	.325
β_1	.85	.80	.75	.70	.65

Thus we have a general equation for β_1 (**ACI-318 10.2.7.3**):

$$\boxed{\begin{array}{rcll}\beta_1 & = & .85 & \text{if } f'_c \leq 4,000 \\ & = & .85 - (.05)(f'_c - 4,000)\frac{1}{1,000} & \text{if } 4,000 < f'_c < 8,000\end{array}} \tag{24.2}$$

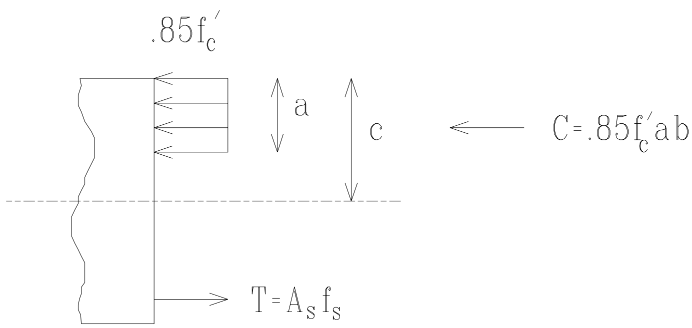


Figure 24.2: Whitney Stress Block

24.2.2 Balanced Steel Ratio

25 Next we seek to determine the **balanced steel ratio** ρ_b such that failure occurs by simultaneous yielding of the steel $f_s = f_y$ and crushing of the concrete $\varepsilon_c = 0.003$, **ACI-318: 10.3.2** We will separately consider the two failure possibilities:

Tension Failure: we stipulate that the steel stress is equal to f_y:

$$\left.\begin{array}{rcl}\rho & = & \frac{A_s}{bd} \\ A_s f_y & = & .85 f'_c ab = .85 f'_c b \beta_1 c\end{array}\right\} \Rightarrow c = \frac{\rho f_y}{0.85 f'_c \beta_1} d \tag{24.3}$$

Compression Failure: where the concrete strain is equal to the ultimate strain; From the strain diagram

$$\left.\begin{array}{rcl}\varepsilon_c & = & 0.003 \\ \frac{c}{d} & = & \frac{.003}{.003+\varepsilon_s}\end{array}\right\} \Rightarrow c = \frac{.003}{\frac{f_s}{E_s} + .003} d \tag{24.4}$$

[1]This approach is often used in Structural Engineering. Perform an analytical derivation, if the number of unknowns then exceeds the number of equations, use experimental data.

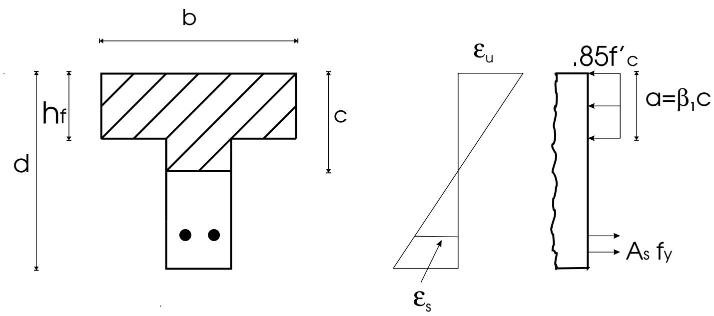


Figure 25.3: T Beam Strain and Stress Diagram

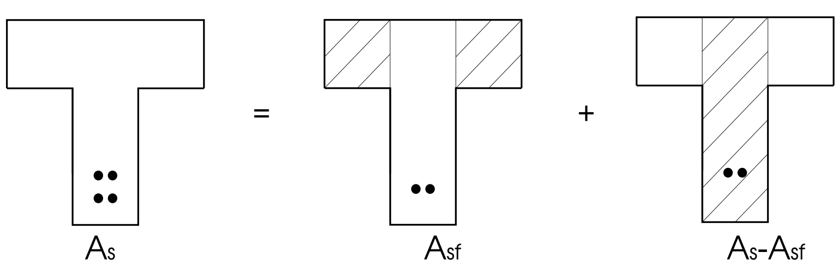


Figure 25.4: Decomposition of Steel Reinforcement for T Beams

Solution:

1. Check requirements for isolated T sections
   (a) $b_e = 30$ in should not exceed $4b_w = 4(14) = 56$ in$\surd$
   (b) $h_f \geq \frac{b_u}{2} \Rightarrow 7 \geq \frac{14}{2}\surd$
2. Assume Rectangular section

$$a = \frac{(12.48)(50)}{(0.85)(3)(30)} = 8.16 \text{ in} > h_f$$

3. For a T section

$$\begin{aligned}
A_{sf} &= \tfrac{.85f'_c h_f(b-b_w)}{f_y} \\
&= \tfrac{(.85)(3)(7)(30-14)}{50} = 5.71 \text{ in}^2 \\
\rho_f &= \tfrac{5.71}{(14)(36)} = .0113 \\
A_{sw} &= 12.48 - 5.71 = 6.77 \text{ in}^2 \\
\rho_w &= \tfrac{12.48}{(14)(36)} = .025 \\
\rho_b &= .85\beta_1 \tfrac{f'_c}{f_y} \tfrac{87}{87+f_y} \\
&= (.85)(.85)\tfrac{3}{50}\tfrac{87}{87+50} = .0275
\end{aligned}$$

4. Maximum permissible ratio

$$\begin{aligned}
\rho_{max} &= .75(\rho_b + \rho_f) \\
&= .75(.0275 + .0113) = .029 > \rho_w \surd
\end{aligned}$$

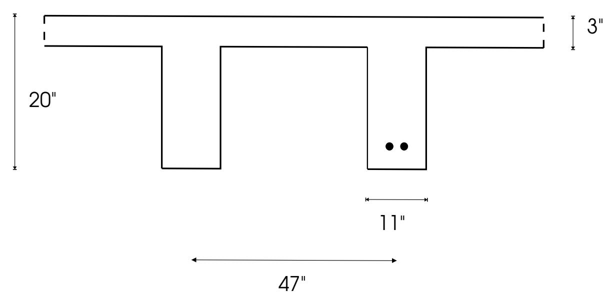
5. The design moment is then obtained from

$$\begin{aligned}
M_{n1} &= (5.71)(50)(36 - \tfrac{7}{2}) = 9,280 \text{ k.in} \\
a &= \tfrac{(6.77)(50)}{(.85)(3)(14)} = 9.48 \text{ in} \\
M_{n2} &= (6.77)(50)(36 - \tfrac{9.48}{2}) = 10,580 \text{ k.in} \\
M_d &= (.9)(9,280 + 10,580) = 17,890 \text{ k.in} \rightarrow \boxed{17,900 \text{ k.in}}
\end{aligned}$$

■

■ Example 25-2: T Beam; Moment Capacity II

Determine the moment capacity of the following section, assume flange dimensions to satisfy ACI requirements; $A_s = 6\#10 = 7.59$ in^2; $f'_c = 3$ ksi; f_y=60 ksi.



Solution:

1. Determine effective flange width:

$$\left.\begin{array}{lcl} \frac{1}{2}(b-b_w) \leq 8h_f & & \\ 16h_f + b_w = (16)(3) + 11 & = & 59 \text{ in} \\ \frac{L}{4} = \frac{24}{4}12 & = & 72 \text{ in} \\ \text{Center Line spacing} & = & 47 \text{ in} \end{array}\right\} b = 47 \text{ in}$$

2. Assume $a = 3$ in

$$\begin{array}{rcl} A_s & = & \frac{M_d}{\phi f_y (d-\frac{a}{2})} = \frac{6{,}400}{0.9)(60)(20-\frac{3}{2})} = 6.40 \text{ in}^2 \\ a & = & \frac{A_s f_y}{(.85)f'_c b} = \frac{(6.4)(60)}{(.85)(3)(47)} = 3.20 \text{ in} > h_f \end{array}$$

3. Thus a T beam analysis is required.

$$\begin{array}{rcl} A_{sf} & = & \frac{.85 f'_c (b-b_w) h_f}{f_y} = \frac{(.85)(3)(47-11)(3)}{60} = 4.58 \text{ in}^2 \\ M_{d1} & = & \phi A_{sf} f_y (d - \frac{h_f}{2}) = (.90)(4.58)(60)(20 - \frac{3}{2}) = 4,570 \text{ k.in} \\ M_{d2} & = & M_d - M_{d1} = 6,400 - 4,570 = 1,830 \text{ k.in} \end{array}$$

4. Now, this is similar to the design of a rectangular section. Assume $a = \frac{d}{5} = \frac{20}{5} = 4.$ in

$$A_s - A_{sf} = \frac{1,830}{(.90)(60)\left(20 - \frac{4}{2}\right)} = 1.88 \text{ in}^2$$

5. check

$$\begin{array}{rcl} a & = & \frac{1.88)(60)}{(.85)(3)(11)} = 4.02 \text{ in} \approx 4.00 \\ A_s & = & 4.58 + 1.88 = \boxed{6.46 \text{ in}^2} \\ \rho_w & = & \frac{6.46}{(11)(20)} = .0294 \\ \rho_f & = & \frac{4.58}{(11)(20)} = .0208 \\ \rho_b & = & (.85)(.85)\left(\frac{3}{60}\right)\left(\frac{87}{87+60}\right) = .0214 \\ \rho_{max} & = & .75(.0214 + .0208) = .0316 > \rho_w \surd \end{array}$$

6. Note that 6.46 in^2 (T beam) is close to $A_s = 6.40$ in^2 if rectangular section was assumed.

■

Chapter 26

PRESTRESSED CONCRETE

26.1 Introduction

1 Beams with longer spans are architecturally more appealing than those with short ones. However, for a reinforced concrete beam to span long distances, it would have to have to be relatively deep (and at some point the self weight may become too large relative to the live load), or higher grade steel and concrete must be used.

2 However, if we were to use a steel with f_y much higher than $\approx$ 60 ksi in reinforced concrete (R/C), then to take full advantage of this higher yield stress while maintaining full bond between concrete and steel, will result in unacceptably wide crack widths. Large crack widths will in turn result in corrosion of the rebars and poor protection against fire.

3 One way to control the concrete cracking and reduce the tensile stresses in a beam is to prestress the beam by applying an initial state of stress which is opposite to the one which will be induced by the load.

4 For a simply supported beam, we would then seek to apply an initial tensile stress at the top and compressive stress at the bottom. In prestressed concrete (P/C) this can be achieved through prestressing of a tendon placed below the elastic neutral axis.

5 Main advantages of P/C: Economy, deflection & crack control, durability, fatigue strength, longer spans.

6 There two type of Prestressed Concrete beams:

Pretensioning: Steel is first stressed, concrete is then poured around the stressed bars. When enough concrete strength has been reached the steel restraints are released, Fig. 26.1.

Postensioning: Concrete is first poured, then when enough strength has been reached a steel cable is passed thru a hollow core inside and stressed, Fig. 26.2.

26.1.1 Materials

7 P/C beams usually have higher compressive strength than R/C. Prestressed beams can have f'_c as high as 8,000 psi.

8 The importance of high yield stress for the steel is illustrated by the following simple example.

If we consider the following:

1. An unstressed steel cable of length l_s
2. A concrete beam of length l_c
3. Prestress the beam with the cable, resulting in a stressed length of concrete and steel equal to $l'_s = l'_c$.
4. Due to shrinkage and creep, there will be a change in length

$$\Delta l_c = (\varepsilon_{sh} + \varepsilon_{cr}) l_c \tag{26.1}$$

   we want to make sure that this amout of deformation is substantially smaller than the stretch of the steel (for prestressing to be effective).

5. Assuming ordinary steel: $f_s = 30$ ksi, $E_s = 29,000$ ksi, $\varepsilon_s = \frac{30}{29,000} = 1.03 \times 10^{-3}$ in/ in
6. The total steel elongation is $\varepsilon_s l_s = 1.03 \times 10^{-3} l_s$
7. The creep and shrinkage strains are about $\varepsilon_{cr} + \varepsilon_{sh} \simeq .9 \times 10^{-3}$
8. The residual stres which is left in the steel after creep and shrinkage took place is thus

$$(1.03 - .90) \times 10^{-3} (29 \times 10^3) = 4 \text{ ksi} \tag{26.2}$$

   Thus the total loss is $\frac{30-4}{30} = 87\%$ which is unacceptably too high.

9. Alternatively if initial stress was 150 ksi after losses we would be left with 124 ksi or a 17% loss.
10. Note that the actual loss is $(.90 \times 10^{-3})(29 \times 10^3) = 26$ ksiin each case

9 Having shown that losses would be too high for low strength steel, we will use

Strands usually composed of 7 wires. Grade 250 or 270 ksi, Fig. 26.3.

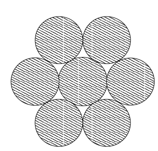


Figure 26.3: 7 Wire Prestressing Tendon

Tendon have diameters ranging from 1/2 to 1 3/8 of an inch. Grade 145 or 160 ksi.

Wires come in bundles of 8 to 52.

Note that yield stress is not well defined for steel used in prestressed concrete, usually we take 1% strain as effective yield.

10 Steel relaxation is the reduction in stress at constant strain (as opposed to creep which is reduction of strain at constant stress) occrs. Relaxation occurs indefinitely and produces

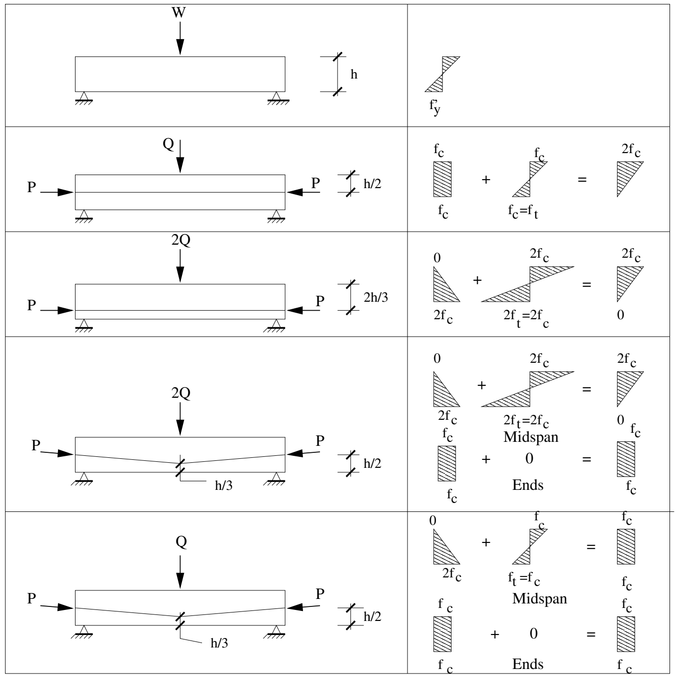


Figure 26.4: Alternative Schemes for Prestressing a Rectangular Concrete Beam, (Nilson 1978)

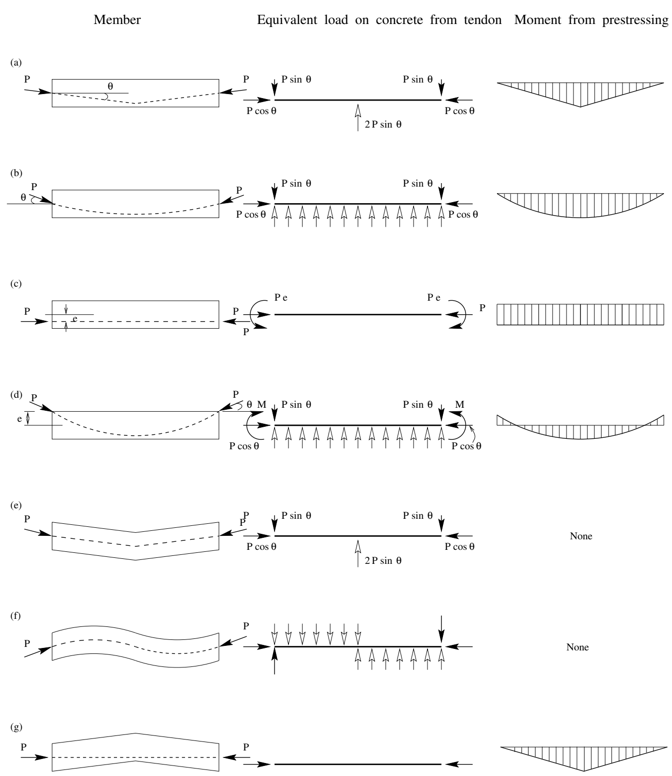


Figure 26.5: Determination of Equivalent Loads

4. P_e and $M_0 + M_{DL} + M_{LL}$

$$\boxed{\begin{array}{rcl} f_1 & = & -\frac{P_e}{A_c}\left(1-\frac{ec_1}{r^2}\right) - \frac{M_0+M_{DL}+M_{LL}}{S_1} \\ f_2 & = & -\frac{P_e}{A_c}\left(1+\frac{ec_2}{r^2}\right) + \frac{M_0+M_{DL}+M_{LL}}{S_2} \end{array}} \qquad (26.7)$$

The internal stress distribution at each one of those four stages is illustrated by Fig. 26.7.

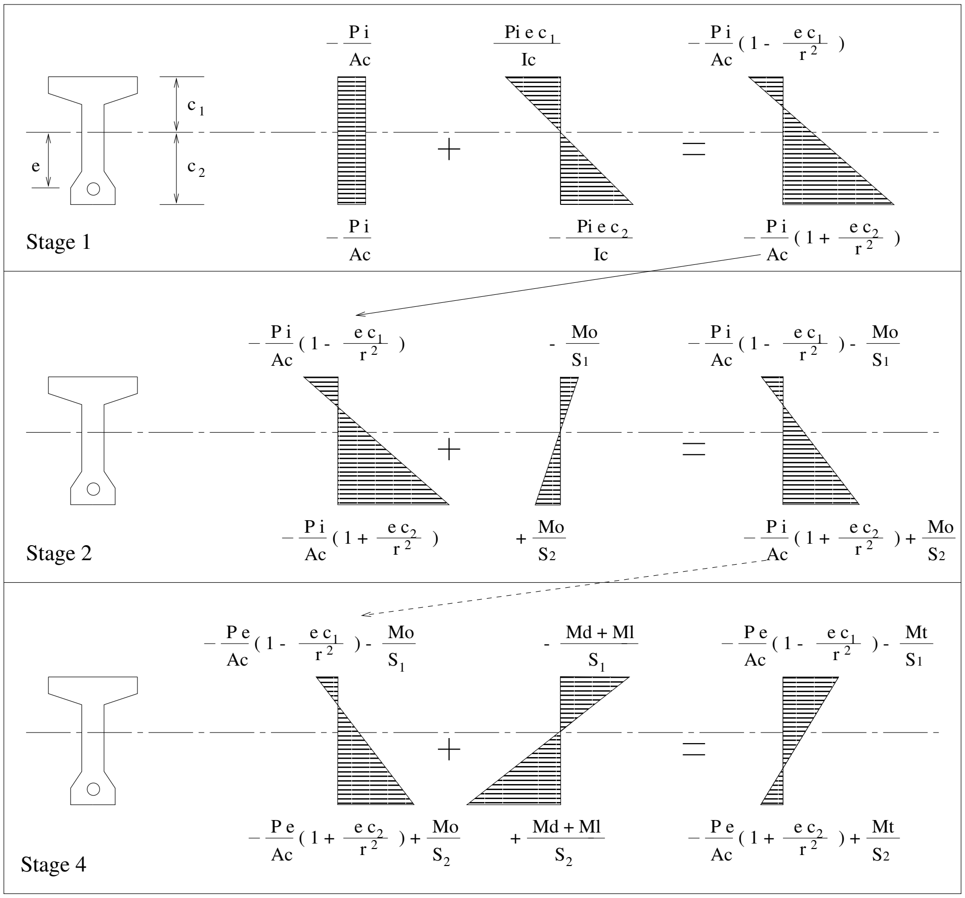


Figure 26.7: Flexural Stress Distribution for a Beam with Variable Eccentricity; Maximum Moment Section and Support Section, (Nilson 1978)

17 Those (service) flexural stresses must be below those specified by the ACI code (where the subscripts c, t, i and s refer to compression, tension, initial and service respectively):

f_{ci}	permitted concrete compression stress at initial stage	$.60f'_{ci}$
f_{ti}	permitted concrete tensile stress at initial stage	$< 3\sqrt{f'_{ci}}$
f_{cs}	permitted concrete compressive stress at service stage	$.45f'_c$
f_{ts}	permitted concrete tensile stress at initial stage	$6\sqrt{f'_c}$ or $12\sqrt{f'_c}$

Note that f_{ts} can reach $12\sqrt{f'_c}$ only if appropriate deflection analysis is done, because section would be cracked.

18 Based on the above, we identify two types of prestressing:

$$M_0 = \frac{(.183)(40)^2}{8} = 36.6 \text{ k.ft} \tag{26.9-b}$$

The flexural stresses will thus be equal to:

$$f_{1,2}^{w_0} = \mp\frac{M_0}{S_{1,2}} = \mp\frac{(36.6)(12,000)}{1,000} = \mp 439 \text{ psi} \tag{26.10}$$

$$f_1 = -\frac{P_i}{A_c}\left(1 - \frac{ec_1}{r^2}\right) - \frac{M_0}{S_1} \tag{26.11-a}$$

$$= -83 - 439 = \boxed{-522 \text{ psi}} \tag{26.11-b}$$

$$f_{ti} = 3\sqrt{f'_c} = +190\surd \tag{26.11-c}$$

$$f_2 = -\frac{P_i}{A_c}\left(1 + \frac{ec_2}{r^2}\right) + \frac{M_0}{S_2} \tag{26.11-d}$$

$$= -1,837 + 439 = \boxed{-1,398 \text{ psi}} \tag{26.11-e}$$

$$f_{ci} = .6f'_c = -2,400\surd \tag{26.11-f}$$

3. P_e and M_0. If we have 15% losses, then the effective force P_e is equal to $(1-0.15)169 = 144$ k

$$f_1 = -\frac{P_e}{A_c}\left(1 - \frac{ec_1}{r^2}\right) - \frac{M_0}{S_1} \tag{26.12-a}$$

$$= -\frac{144,000}{176}\left(1 - \frac{(5.19)(12)}{68.2}\right) - 439 \tag{26.12-b}$$

$$= -71 - 439 = \boxed{-510 \text{ psi}} \tag{26.12-c}$$

$$f_2 = -\frac{P_e}{A_c}\left(1 + \frac{ec_2}{r^2}\right) + \frac{M_0}{S_2} \tag{26.12-d}$$

$$= -\frac{144,000}{176}\left(1 + \frac{(5.19)(12)}{68.2}\right) + 439 \tag{26.12-e}$$

$$= -1,561 + 439 = \boxed{-1,122 \text{ psi}} \tag{26.12-f}$$

note that -71 and $-1,561$ are respectively equal to $(0.85)(-83)$ and $(0.85)(-1,837)$ respectively.

4. P_e and $M_0 + M_{DL} + M_{LL}$

$$M_{DL} + M_{LL} = \frac{(0.55)(40)^2}{8} = 110 \text{ k.ft} \tag{26.13}$$

and corresponding stresses

$$f_{1,2} = \mp\frac{(110)(12,000)}{1,000} = \mp 1,320 \text{ psi} \tag{26.14}$$

Thus,

$$f_1 = -\frac{P_e}{A_c}\left(1 - \frac{ec_1}{r^2}\right) - \frac{M_0 + M_{DL} + M_{LL}}{S_1} \tag{26.15-a}$$

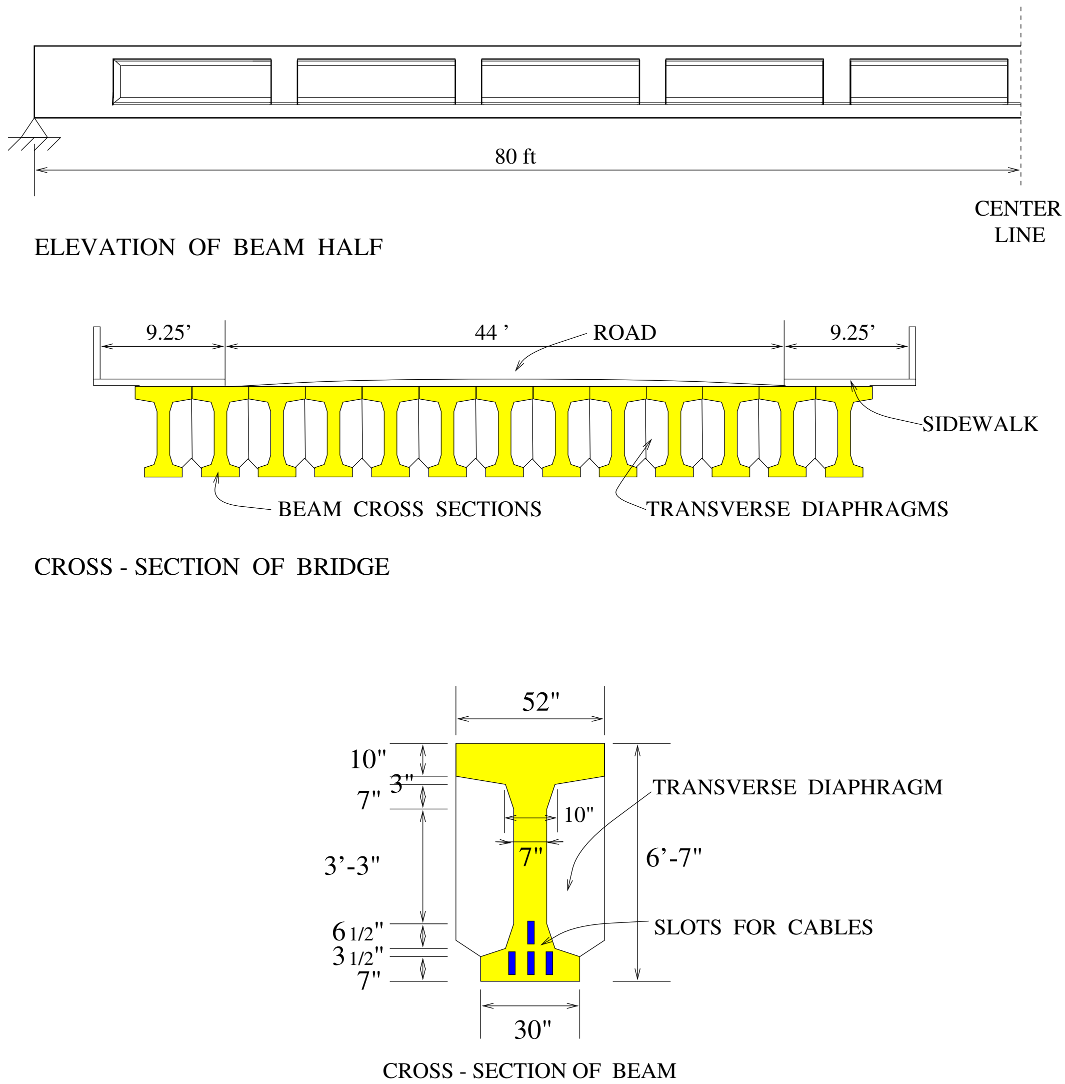


Figure 26.8: Walnut Lane Bridge, Plan View

Chapter 27

COLUMNS

27.1 Introduction

1 Columns resist a combination of axial P and flexural load M, (or $M = Pe$ for eccentrically applied load).

27.1.1 Types of Columns

Types of columns, Fig. 27.1

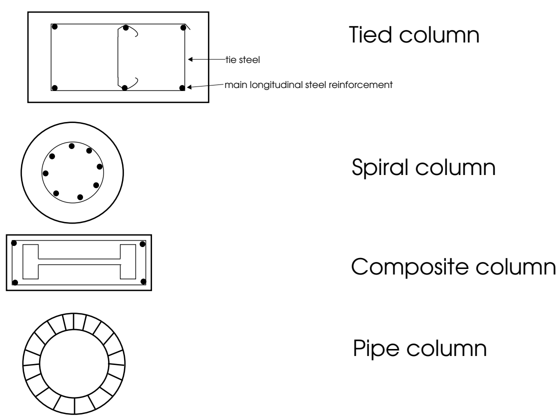


Figure 27.1: Types of columns

2 Lateral reinforcement, Fig. 27.2

1. Restrains longitudinal steel from outward buckling
2. Restrains Poisson's expansion of concrete
3. Acts as shear reinforcement for horizontal (wind & earthquake) load
4. Provide ductility

very important to resist earthquake load.

5 note:

1. 0.85 is obtained also from test data
2. Matches with beam theory using rect. stress block
3. Provides an adequate factor of safety

27.2.2 Eccentric Columns

5 Sources of flexure, Fig. 27.4

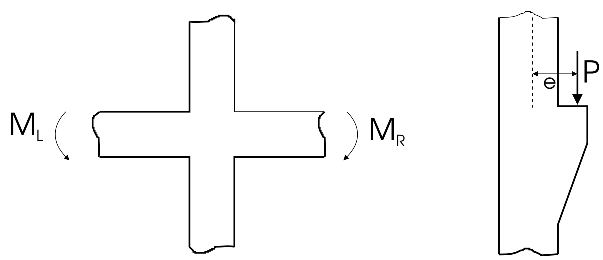


Figure 27.4: Sources of Bending

1. Unsymmetric moments $M^L \neq M^R$
2. uncertainty of loads (must assume a minimum eccentricity)
3. unsymmetrical reinforcement

6 Types of Failure, Fig. 27.5

1. large eccentricity of load $\Rightarrow$ failure by yielding of steel
2. small eccentricity of load $\Rightarrow$ failure by crushing of concrete
3. balanced condition

7 Assumptions $A'_s = A_s; \rho = \frac{A_s}{bd} = \frac{A'_s}{bd} = f'_s = f_y$

27.2.2.1 Balanced Condition

8 There is one specific eccentricity $e_b = \frac{M}{P}$ such that failure will be triggered by simultaneous

1. steel yielding
2. concrete crushing

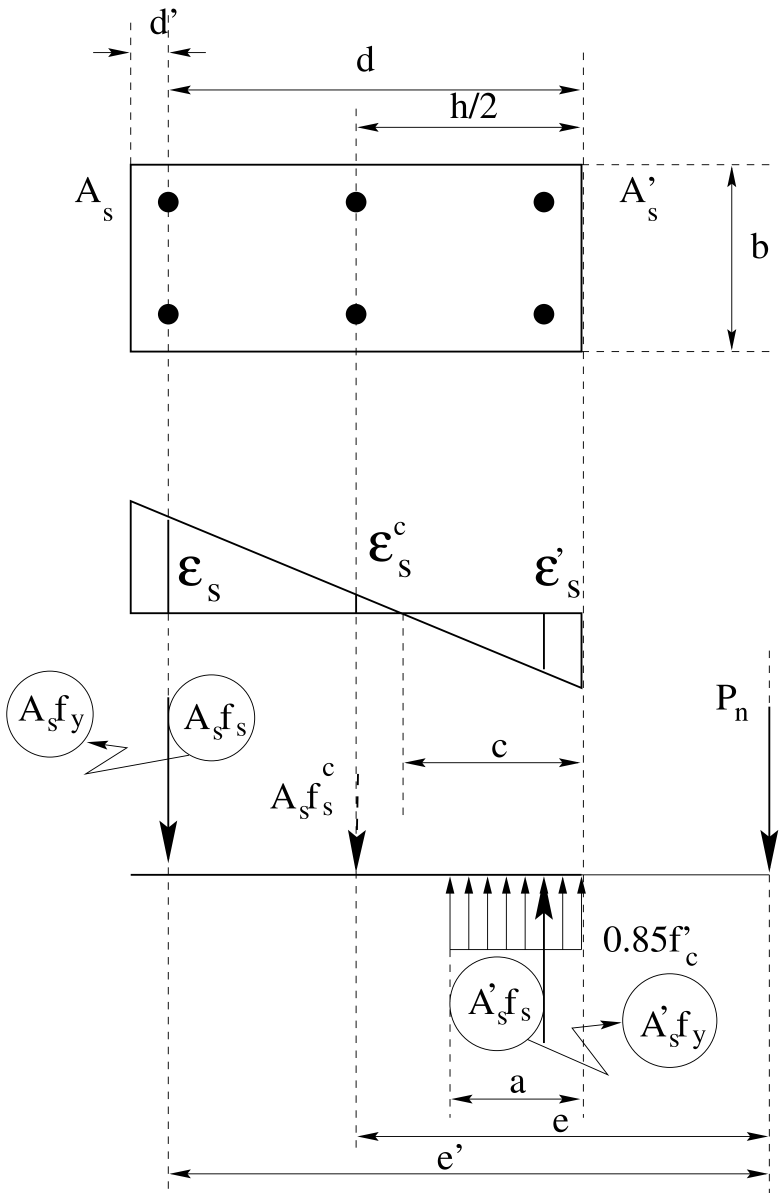


Figure 27.6: Strain and Stress Diagram of a R/C Column

```
error=0.1805
a= 20.7445
iter= 2
c=24.4053
epsi_sc=0.0015
f_sc=44.2224
f_sc=40
M_n=1.6860e+04
P_n=1.4789e+03
epsi_s=-4.1859e-04
f_s=12.1392

Failure by Tension
e=24
error=1
a=10
iter=0
iter=1
c=11.7647
epsi_sc=-6.0000e-
f_sc=-1.7400
P_n_1=504.5712
P_n_2=381.9376
a_new=7.5954
error=-0.3166
a=7.5954
iter=2
c=8.9358
epsi_sc=-0.0010
f_sc=-29.8336
P_n_1=294.2857
P_n_2=312.6867
a_new=7.9562
error=0.0453
a=7.9562
iter=2
P_n_1=294.2857
```

■

■ Example 27-3: R/C Column, Using Design Charts

Design the reinforcement for a column with $h = 20$ in, $b = 12$ in, $d' = 2.5$ in, $f'_c = 4,000$ psi, $f_y = 60,000$ psi, to support $P_{DL} = 56$ k, $P_{LL} = 72$ k, $M_{DL} = 88$ k.ft, $M_{LL} = 75$ k.ft,
Solution:

1. Ultimate loads

$$\begin{array}{lllll} P_u & = & (1.4)(56) + (1.7)(72) = 201 \text{ k} & \Rightarrow P_n & = \frac{201}{0.7} = 287 \text{ k} \\ M_u & = & (1.4)(88) + (1.7)(75) = 251 \text{ k.ft} & \Rightarrow M_n & = \frac{251}{0.7} = 358 \text{ k.ft} \end{array} \quad (27.26)$$

Chapter 28

ELEMENTS of STRUCTURAL RELIABILITY

28.1 Introduction

1 Traditionally, evaluations of structural adequacy have been expressed by safety factors $SF = \frac{C}{D}$, where C is the *capacity* (i.e. strength) and D is the demand (i.e. load). Whereas this evaluation is quite simple to understand, it suffers from many limitations: it 1) treats all loads equally; 2) does not differentiate between capacity and demands respective uncertainties; 3) is restricted to service loads; and last but not least 4) does not allow comparison of relative reliabilities among different structures for different performance modes. Another major deficiency is that all parameters are assigned a single value in an analysis which is then *deterministic*.

2 Another approach, a *probabilistic* one, extends the factor of safety concept to explicitly incorporate uncertainties in the parameters. The uncertainties are quantified through statistical analysis of existing data or judgmentally assigned.

3 This chapter will thus develop a procedure which will enable the Engineer to perform a *reliability* based analysis of a structure, which will ultimately yield a *reliability index*. This is turn is a "universal" indicator on the adequacy of a structure, and can be used as a metric to 1) assess the health of a structure, and 2) compare different structures targeted for possible remediation.

28.2 Elements of Statistics

4 Elementary statistics formulaes will be reviewed, as they are needed to properly understand structural reliability.

5 When a set of N values x_i is clustered around a particular one, then it may be useful to characterize the set by a few numbers that are related to its *moments* (the sums of integer powers of the values):

Mean: estimates the value around which the data clusters.

$$\boxed{\mu = \frac{1}{N}\sum_{i=1}^{N} x_i} \tag{28.1}$$

Skewness: characterizes the degree of asymmetry of a distribution around its mean. It is defined in a non-dimensional value. A positive one signifies a distribution with an asymmetric tail extending out toward more positive x

$$\text{Skew} = \frac{1}{N}\Sigma_{i=1}^{N}\left[\frac{x_i-\mu}{\sigma}\right]^3 \tag{28.8}$$

Kurtosis: is a nondimensional quantity which measures the "flatness" or "peakedness" of a distribution. It is normalized with respect to the curvature of a normal distribution. Hence a negative value would result from a distribution resembling a loaf of bread, while a positive one would be induced by a sharp peak:

$$\text{Kurt} = \frac{1}{N}\Sigma_{i=1}^{N}\left[\frac{x_i-\mu}{\sigma}\right]^4 - 3 \tag{28.9}$$

the -3 term makes the value zero for a normal distribution.

6 The expected value (or mean), standard deviation and coefficient of variation are interdependent: knowing any two, we can determine the third.

28.3 Distributions of Random Variables

7 Distribution of variables can be mathematically represented.

28.3.1 Uniform Distribution

8 Uniform distribution implies that any value between x_{min} and x_{max} is equaly likely to occur.

28.3.2 Normal Distribution

9 The general normal (or Gauss) distribution is given by, Fig. 28.1:

$$\boxed{\phi(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}} \tag{28.10}$$

10 A normal distribution $N(\mu,\sigma^2)$ can be normalized by defining

$$\boxed{y = \frac{x-\mu}{\sigma}} \tag{28.11}$$

and y would have a distribution $N(0,1)$:

$$\boxed{\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}} \tag{28.12}$$

11 The normal distribution has been found to be an excellent approximation to a large class of distributions, and has some very desirable mathematical properties:

28.3.3 Lognormal Distribution

12 A random variable is lognormally distributed if the natural logarithm of the variable is normally distributed.

13 It provides a reasonable shape when the coefficient of variation is large.

28.3.4 Beta Distribution

14 Beta distributions are very flexible and can assume a variety of shapes including the normal and uniform distributions as special cases.

15 On the other hand, the beta distribution requires four parameters.

16 Beta distributions are selected when a particular shape for the probability density function is desired.

28.3.5 BiNormal distribution

28.4 Reliability Index

28.4.1 Performance Function Identification

17 Designating F the capacity to demand ratio C/D (or *performance function*), in general F is a function of one or more variables x_i which describe the geometry, material, loads, and boundary conditions

$$F = F(x_i) \tag{28.17}$$

and thus F is in turn a random variable with its own probability distribution function, Fig. 28.2.

18 A performance function evaluation typically require a structural analysis, this may range from a simple calculation to a detailled finite element study.

28.4.2 Definitions

19 Reliability indices, β are used as a relative measure of the reliability or confidence in the ability of a structure to perform its function in a satisfactory manner. In other words they are a measure of the performance function.

20 Probabilistic methods are used to systematically evaluate uncertainties in parameters that affect structural performance, and there is a relation between the reliability index and risk.

21 Reliability index is defined in terms of the performance function capacity C, and the applied load or demand D. It is assumed that both C and D are **random variables**.

22 The **safety margin** is defined as $Y = C - D$. Failure would occur if $Y < 0$ Next, C and D can be combined and the result expressed logarithmically, Fig. 28.2.

$$X = \ln \frac{C}{D} \tag{28.18}$$

30 The objective is to determine the mean and standard deviation of the performance function defined in terms of C/D.

31 Those two parameters, in turn, will later be required to compute the reliability index.

28.4.3.1 Direct Integration

32 Given a function random variable x, the mean value of the function is obtained by integrating the function over the probability distribution function of the random variable

$$\mu[F(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx \tag{28.21}$$

33 For more than one variable,

$$\mu[F(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(x_1, x_2, \cdots, x_n)F(x_1, x_2, \cdots, x_n)dx_1dx_2\cdots dx_n \tag{28.22}$$

34 Note that in practice, the function $F(x)$ is very rarely available for practical problems, and hence this method is seldom used.

28.4.3.2 Monte Carlo Simulation

35 The performance function is evaluated for many possible values of the random variables.

36 Assuming that all variables have a normal distribution, then this is done through the following algorithm

1. initialize random number generators
2. Perform n analysis, for each one:
   (a) For each variable, determine a random number for the given distribution
   (b) Transform the random number
   (c) Analyse
   (d) Determine the performance function, and store the results
3. From all the analyses, determine the mean and the standard deviation, compute the reliability index.
4. Count the number of analyses, n_f which performance function indicate failure, the likelihood of structural failure will be $p(f) = n_f/n$.

37 A sample program (all subroutines are taken from (Press, Flannery, Teukolvsky and Vetterling 1988) which generates n normally distributed data points, and then analyze the results, determines mean and standard deviation, and sort them (for histogram plotting), is shown below:

```
program nice
parameter(ns=100000)
real x(ns), mean, sd
write(*,*)'enter mean, standard-deviation and n '
read(*,*)mean,sd,n
```

```
      ix3=mod(ia3*ix3+ic3,m3)
      j=1+(97*ix3)/m3
      if(j.gt.97.or.j.lt.1)pause
      ran1=r(j)
      r(j)=(float(ix1)+float(ix2)*rm2)*rm1
      return
      end
c-------------------------------------------------------------------------------
      subroutine moment(data,n,ave,adev,sdev,var,skew,curt)
c given array of data of length N, returns the mean AVE, average
c      deviation ADEV, standard deviation SDEV, variance VAR, skewness SKEW,
c      and kurtosis CURT
      dimension data(n)
      if(n.le.1)pause 'n must be at least 2'
      s=0.
      do 11 j=1,n
        s=s+data(j)
11    continue
      ave=s/n
      adev=0.
      var=0.
      skew=0.
      curt=0.
      do 12 j=1,n
        s=data(j)-ave
        adev=adev+abs(s)
        p=s*s
        var=var+p
        p=p*s
        skew=skew+p
        p=p*s
        curt=curt+p
12    continue
      adev=adev/n
      var=var/(n-1)
      sdev=sqrt(var)
      if(var.ne.0.)then
        skew=skew/(n*sdev**3)
        curt=curt/(n*var**2)-3.
      else
        pause 'no skew or kurtosis when zero variance'
      endif
      return
      end
c--------------------------------------------------------------------------------
      subroutine sort(n,ra)
      dimension ra(n)
      l=n/2+1
      ir=n
10    continue
        if(l.gt.1)then
          l=l-1
          rra=ra(l)
        else
          rra=ra(ir)
          ra(ir)=ra(1)
          ir=ir-1
          if(ir.eq.1)then
            ra(1)=rra
```

42 This is accomplished by limiting ourselves to all possible combinations of $\mu_i \pm \sigma_i$.

43 For each analysis we determine SFF, as well as its logarithm.

44 Mean and standard deviation of the logarithmic values are then determined from the 2^n analyses, and then β is the ratio of the mean to the standard deviation.

28.4.3.4 Taylor's Series-Finite Difference Estimation

45 In the previous method, we have cut down the number of deterministic analyses to 2^n, in the following method, we reduce it even further to $2n+1$, (US Army Corps of Engineers 1992, US Army Corps of Engineers 1993, Bryant, Brokaw and Mlakar 1993).

46 This simplified approach starts with the first order Taylor series expansion of Eq. 28.17 about the mean and limited to linear terms, (Benjamin and Cornell 1970).

$$\boxed{\mu_F = F(\mu_i)} \tag{28.23}$$

where μ_i is the mean for all random variables.

47 For independent random variables, the variance can be approximated by

$$\begin{aligned} Var(F) = \sigma_F^2 &= \sum \left(\frac{\partial F}{\partial x_i} \sigma_i \right)^2 && \text{(28.24-a)} \\ \frac{\partial F}{\partial x_i} &\approx \frac{F_i^+ - F_i^-}{2\sigma_i} && \text{(28.24-b)} \\ F_i^+ &= F(\mu_1, \cdots, \mu_i + \sigma_i, \cdots, \mu_n) && \text{(28.24-c)} \\ F_i^- &= F(\mu_1, \cdots, \mu_i - \sigma_i, \cdots, \mu_n) && \text{(28.24-d)} \end{aligned}$$

where σ_i are the standard deviations of the variables. Hence,

$$\boxed{\sigma_F = \sum \left(\frac{F_i^+ - F_i^-}{2} \right)} \tag{28.25}$$

Finally, the reliability index is given by

$$\boxed{\beta = \frac{\ln \mu_F}{\sigma_F}} \tag{28.26}$$

48 The procedure can be summarized as follows:

1. Perform an initial analysis in which all variables are set equal to their mean value. This analysis provides the mean μ.

2. Perform $2n$ analysis, in which all variables are set equal to their mean values, except variable i, which assumes a value equal to $\mu_i + \sigma_i$, and then $\mu_i - \sigma_i$.

3. For each pair of analysis in which variable x_i is modified, determine

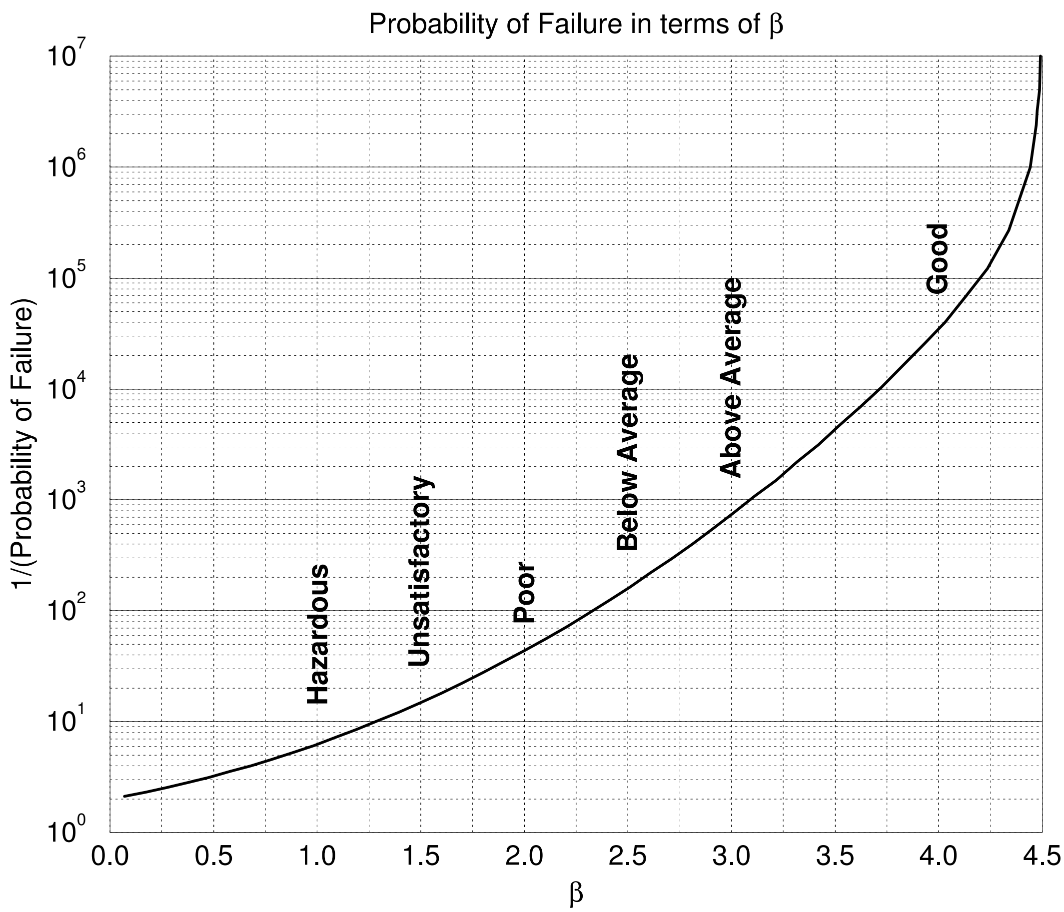


Figure 28.3: Probability of Failure in terms of β

Chapter 29

DESIGN II

29.1 Frames

29.1.1 Beam Column Connections

1 The connection between the beam and the column can be, Fig. 33.1:

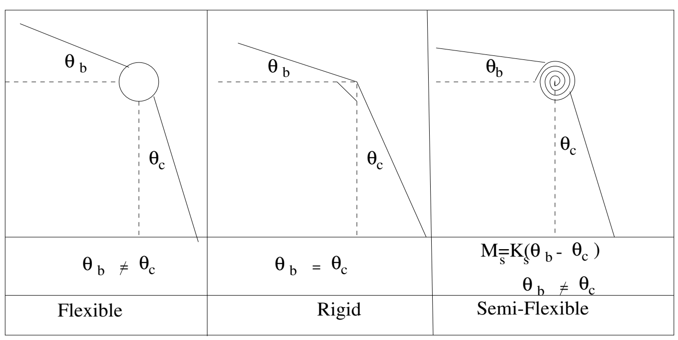


Figure 29.1: Flexible, Rigid, and Semi-Flexible Joints

Flexible that is a hinge which can transfer forces only. In this case we really have cantiliver action only. In a flexible connection the column and beam end moments are both equal to zero, $M_{\rm col} = M_{\rm beam} = 0$. The end rotation are not equal, $\theta_{\rm col} \neq \theta_{\rm beam}$.

Rigid: The connection is such that $\theta_{\rm beam} = \theta_{\rm col}$ and moment can be transmitted through the connection. In a rigid connection, the end moments and rotations are equal (unless there is an externally applied moment at the node), $M_{\rm col} = M_{\rm beam} \neq 0$, $\theta_{\rm col} = \theta_{\rm beam}$.

Semi-Rigid: The end moments are equal and not equal to zero, but the rotation are different. $\theta_{\rm beam} \neq \theta_{\rm col}$, $M_{\rm col} = M_{\rm beam} \neq 0$. Furthermore, the difference in rotation is resisted by the spring $M_{\rm spring} = K_{\rm spring}(\theta_{\rm col} - \theta_{\rm beam})$.

29.1.2 Behavior of Simple Frames

2 For vertical load across the beam rigid connection will reduce the maximum moment in the beam (at the expense of a negative moment at the ends which will in turn be transferred to

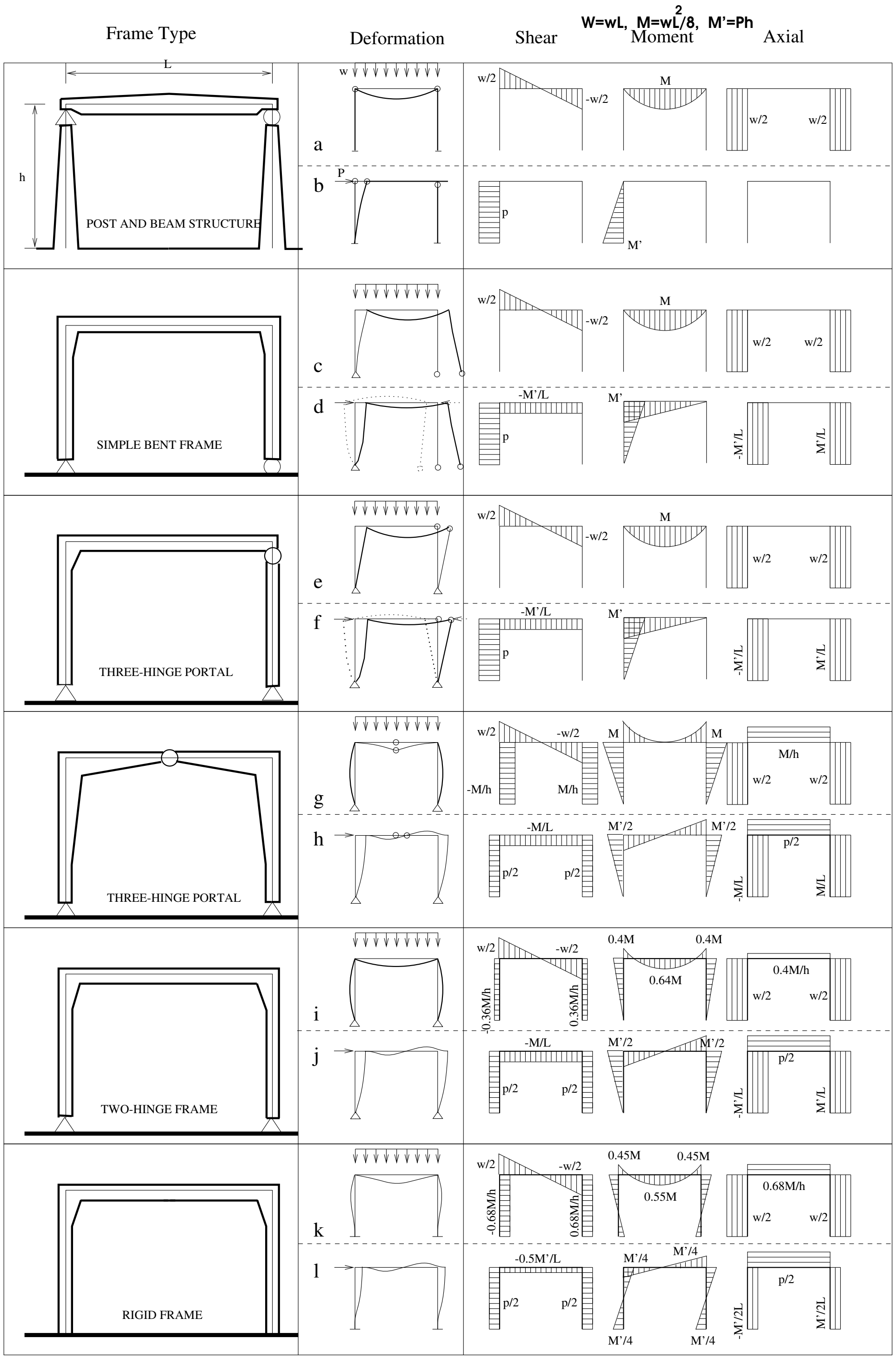


Figure 29.3: Deformation, Shear, Moment, and Axial Diagrams for Various Types of Portal Frames Subjected to Vertical and Horizontal Loads

29.1.4 Design of a Statically Indeterminate Arch

`Adapted from (Kinney 1957)`

Design a two-hinged, solid welded-steel arch rib for a hangar. The moment of inertia of the rib is to vary as necessary. The span, center to center of hinges, is to be 200 ft. Ribs are to be placed 35 ft center to center, with a rise of 35 ft. Roof deck, purlins and rib will be assumed to weight 25 lb/ft^2 on roof surface, and snow will be assumed at 40 lb/ft^2 of this surface. Twenty purlins will be equally spaced around the rib.

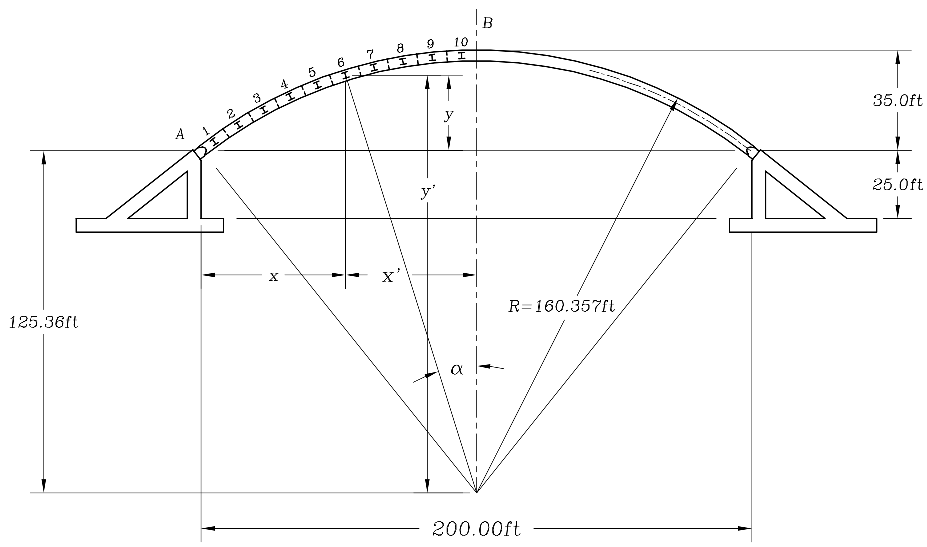


Figure 29.5: Design of a Statically Indeterminate Arch

1. The center line of the rib will be taken as the segment of a circle. By computation the radius of this circle is found to be 160.357 ft, and the length of the *arc AB* to be 107.984 ft.

2. For the analysis the arc AB will be considered to be divided into ten segments, each with a length of 10.798 ft. Thus a concentrated load is applied to the rib by the purlins framing at the center of each segment. (The numbered segments are indicated in Fig. **??**.

3. Since the total dead and snow load is 65 lb/ft^2 of roof surface, the value of each concentrated force will be

$$P = 10.798 \times 35 \times 65 = 24.565 \text{ k} \tag{29.8}$$

4. The computations necessary to evaluate the coodinates of the centers of the various segments, referred to the hinge at A, are shown in Table 29.1. Also shown are the values of Δx, the horizontal projection of the distance between the centers of the several segments.

5. If experience is lacking and the designing engineer is therefore at a loss as to the initial assumptions regarding the sectional variation along the rib, it is recommended that the

Segment	(2) x (ft)	(3) $y=\delta\overline{M}$ (ft · k)	(4) Shear to right (k)	(5) Δx (ft)	(6) M increment (ft · k)	(7) M simple beam (ft· k)	(8) $M\delta\overline{M}$	(9) $\delta\overline{M}\overline{M}$
A	0.0	0.0	245.650					
1	4.275	3.29	221.085	4.275	1,050	1,050	3,500	10.9
2	13.149	9.44	196.520	8.874	1,962	3,010	28,400	89.2
3	22.416	14.98	171.955	9.267	1,821	4,830	72,400	224.4
4	32.034	19.88	147.390	9.618	1,654	6,490	129,100	395.4
5	41.960	24.13	122.825	9.926	1,463	7,950	191,800	582.2
6	52.149	27.69	98.260	10.189	1,251	9,200	254,800	767.0
7	62.557	30.57	73.695	10.408	1,023	10,220	312,400	934.4
8	73.134	32.73	49.130	10.577	779	11,000	360,000	1,071.4
9	83.831	34.18	24.565	10.697	526	11,530	394,100	1,168.4
10	94.601	34.91	0.000	10.770	265	11,790	411,600	1,218.6
Crown	100.00	35.00	____	5.399	0	11,790		
$\sum$							2,158,100	6,461.9

Table 29.2: Calculation of Horizontal Force

Segment	y (ft)	M simple beam (ft · k)	H_Ay (ft · k)	Total M at segment (ft · k)
A	0			
1	3.29	1,050	–1,100	–50
2	9.44	3,010	–3,150	–140
3	14.98	4,830	–5,000	–170
4	19.88	6,490	–6,640	–150
5	24.13	7,950	–8,060	–110
6	27.69	9,200	–9,250	–50
7	30.57	10,220	–10,210	+10
8	32.73	11,000	–10,930	+70
9	34.18	11,530	–11,420	+110
10	34.19	11,790	–11,660	+130
Crown	35.00	11,790	–11,690	+100

Table 29.3: Moment at the Centers of the Ribs

Segment	V (k)	V cos α	$-H$ sin α	$=S$ (k)	V sin α	$+H$ cos α	$=N$ (k)
A	245.6	192	-208	= -16	153	+ 261	= 414
1	221.1	177	-199	= -22	132	+268	= 400
2	196.5	165	-181	= -16	106	+281	= 387
3	172.0	150	-162	= -12	83	+292	= 375
4	147.4	133	-142	= -9	62	+303	= 365
5	122.8	114	-121	= -7	44	+311	= 355
6	98.3	94	-100	= -6	29	+319	= 348
7	73.7	72	-78	= -6	17	+325	= 342
8	49.1	48	-56	= -8	8	+329	= 337
9	24.6	244	-34	= -10	2	+332	= 334
10	0.0	0	-11	= -11	0	+334	= 334
Crown						+334	= 334

Table 29.4: Values of Normal and Shear Forces

3 to the crown is made to vary linerarly. The adequacies of the sections thus determined for the centers of the several segments are checked in Table 29.5.

17. It is necessary to recompute the value of H_A because the rib now has a varying moment of inertia. Equation 29.11 must be altered to include the I of each segment and is now written as

$$H_A = -\frac{\sum M\delta\overline{M}/I}{\sum \delta\overline{M}\overline{M}/I} \tag{29.14}$$

18. The revised value for H_A is easily determined as shown in Table 29.6. Note that the values in column (2) are found by dividing the values of $M\delta\overline{M}$ for the corresponding segments in column (8) of Table 29.2 by the total I for each segment as shown in Table 29.5. The values in column (3) of Table 29.6 are found in a similar manner from the values in column (9) of Table 29.2. The simple beam moments in column (5) of Table 29.6 are taken directly from column (7) of Table 29.2.

19. Thus the revised H_A is

$$H_A = -\frac{2\sum M\delta\overline{M}/I}{2\sum \delta\overline{M}\overline{M}/I} = -frac2 \times 1,010.852 \times 3.0192 = -334.81k \tag{29.15}$$

20. The revised values for the axial thrust N at the centers of the various segments are computed in Table 29.7.

21. The sections previously designed at the centers of the segments are checked for adequacy in Table 29.8. From this table it appears that all sections of the rib are satisfactory. This cannot be definitely concluded, however, until the secondary stresses caused by the deflection of the rib are investigated.

29.1.5 Temperature Changes in an Arch

`Adapted from (Kinney 1957)`

Determine the effects of rib shortening and temperature changes in the arch rib of Fig. 29.5. Consider a temperature drop of 100°F.

Chapter 30

Case Study I: EIFFEL TOWER

`Adapted from (Billington and Mark 1983)`

30.1 Materials, & Geometry

1 The tower was built out of wrought iron, less expensive than steel,and Eiffel had more expereince with this material, Fig. 30.1

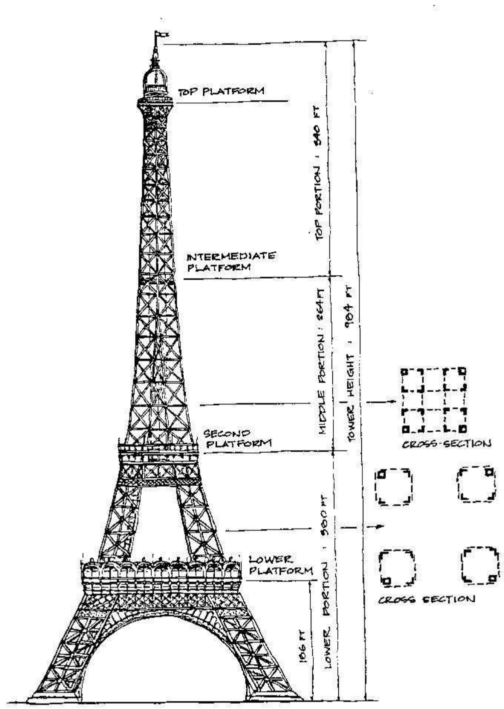


Figure 30.1: Eiffel Tower (Billington and Mark 1983)

Location	Height	Width/2	Width		$\frac{dy}{dx}$	β
			Estimated	Actual		
Support	0	164	328		.333	18.4^o
First platform	186	108	216	240	.270	15.1^o
second platform	380	62	123	110	.205	11.6^o
Intermediate platform	644	20	40		.115	6.6^o
Top platform	906	1	2		.0264	1.5^o
Top	984	0	0		0.000	0^o

4 The tower is supported by four inclined supports, each with a cross section of 800 in^2. An idealization of the tower is shown in Fig. 30.2.

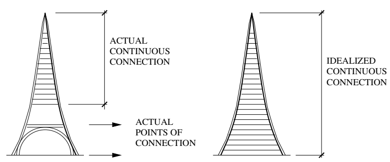


Figure 30.2: Eiffel Tower Idealization, (Billington and Mark 1983)

30.2 Loads

5 The total weight of the tower is 18,800 k.

6 The dead load is not uniformly distributed, and is approximated as follows, Fig. 30.3:

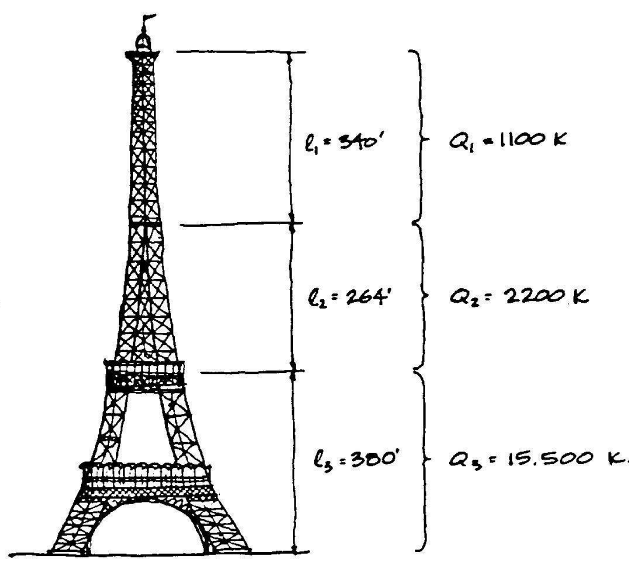


Figure 30.3: Eiffel Tower, Dead Load Idealization; (Billington and Mark 1983)

Draft

Chapter 31

Case Study II: GEORGE WASHINGTON BRIDGE

31.1 Theory

1 Whereas the forces in a cable can be determined from statics alone, its configuration must be derived from its deformation. Let us consider a cable with distributed load $p(x)$ **per unit horizontal projection** of the cable length (thus neglecting the weight of the cable). An infinitesimal portion of that cable can be assumed to be a straight line, Fig. 31.1 and in the absence of any horizontal load we have H =constant. Summation of the vertical forces yields

$$(+\downarrow)\ \Sigma F_y = 0 \Rightarrow -V + wdx + (V + dV) = 0 \quad (31.1\text{-a})$$

$$dV + wdx = 0 \quad (31.1\text{-b})$$

where V is the vertical component of the cable tension at x (Note that if the cable was subjected to its own weight then we would have wds instead of wdx). Because the cable must be tangent to T, we have

$$\tan\theta = \frac{V}{H} \quad (31.2)$$

Substituting into Eq. 31.1-b yields

$$d(H\tan\theta) + wdx = 0 \Rightarrow -\frac{d}{dx}(H\tan\theta) = w \quad (31.3)$$

2 But H is constant (no horizontal load is applied), thus, this last equation can be rewritten as

$$-H\frac{d}{dx}(\tan\theta) = w \quad (31.4)$$

3 Written in terms of the vertical displacement v, $\tan\theta = \frac{dv}{dx}$ which when substituted in Eq. 31.4 yields the **governing equation for cables**

$$-Hv'' = w \quad (31.5)$$

4 For a cable subjected to a uniform load w, we can determine its shape by double integration of Eq. 31.5

$$-Hv' = wx + C_1 \quad (31.6\text{-a})$$

$$-Hv = \frac{wx^2}{2} + C_1x + C_2 \tag{31.6-b}$$

and the constants of integrations C_1 and C_2 can be obtained from the boundary conditions: $v = 0$ at $x = 0$ and at $x = L \Rightarrow C_2 = 0$ and $C_1 = -\frac{wL}{2}$. Thus

$$v = \frac{w}{2H}x(L - x) \tag{31.7}$$

This equation gives the shape $v(x)$ in terms of the horizontal force H,

5 Since the maximum sag h occurs at midspan ($x = \frac{L}{2}$) we can solve for the horizontal force

$$\boxed{H = \frac{wL^2}{8h}} \tag{31.8}$$

we note the analogy with the maximum moment in a simply supported uniformly loaded beam $M = Hh = \frac{wL^2}{8}$. Furthermore, this relation clearly shows that the horizontal force is inversely proportional to the sag h, as $h \searrow H \nearrow$. Finally, we can rewrite this equation as

$$r \stackrel{\text{def}}{=} \frac{h}{L} \tag{31.9-a}$$

$$\frac{wL}{H} = 8r \tag{31.9-b}$$

6 Eliminating H from Eq. 31.7 and 31.8 we obtain

$$\boxed{v = 4h\left(-\frac{x^2}{L^2} + \frac{x}{L}\right)} \tag{31.10}$$

Thus the cable assumes a parabolic shape (as the moment diagram of the applied load).

7 Whereas the horizontal force H is constant throughout the cable, the tension T is not. The maximum tension occurs at the support where the vertical component is equal to $V = \frac{wL}{2}$ and the horizontal one to H, thus

$$T_{\max} = \sqrt{V^2 + H^2} = \sqrt{\left(\frac{wL}{2}\right)^2 + H^2} = H\sqrt{1 + \left(\frac{wL/2}{H}\right)^2} \tag{31.11}$$

Combining this with Eq. 31.8 we obtain[1].

$$\boxed{T_{\max} = H\sqrt{1 + 16r^2} \approx H(1 + 8r^2)} \tag{31.12}$$

8 Had we assumed a uniform load w **per length of cable** (rather than horizontal projection), the equation would have been one of a catenary[2].

$$\boxed{v = \frac{H}{w}\cosh\left[\frac{w}{H}\left(\frac{L}{2} - x\right)\right] + h} \tag{31.13}$$

The cable between transmission towers is a good example of a catenary.

[1] Recalling that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdot$ or $(1+b)^n = 1 + nb + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \cdots$; Thus for $b^2 << 1$, $\sqrt{1+b} = (1+b)^{\frac{1}{2}} \approx 1 + \frac{b}{2}$

[2] Derivation of this equation is beyond the scope of this course.

Chapter 32

Case Study III: MAGAZINI GENERALI

`Adapted from (Billington and Mark 1983)`

32.1 Geometry

1 This sotrage house, built by Maillart in Chiasso in 1924, provides a good example of the mariage between aesthetic and engineering.

2 The most strking feature of the Magazini Generali is not the structure itself, but rather the shape of its internal supporting frames, Fig. 32.1.

3 The frame can be idealized as a simply supported beam hung from two cantilever column supports. Whereas the beam itself is a simple structural idealization, the overhang is designed in such a way as to minimize the net moment to be transmitted to the supports (foundations), Fig. 32.2.

32.2 Loads

4 The load applied on the frame is from the weights of the roof slab, and the frame itself. Given the space between adjacent frames is 14.7 ft, and that the roof load is 98 psf, and that the total frame weight is 13.6 kips, the total uniform load becomes, Fig. 32.3:

$$q_{roof} = (98)\text{ psf}(14.7)\text{ ft} = 1.4\text{ k/ft} \quad (32.1\text{-a})$$

$$q_{frame} = \frac{(13.6)\text{ k}}{(63.6)\text{ ft}} = 0.2\text{ k/ft} \quad (32.1\text{-b})$$

$$q_{total} = 1.4 + 0.2 = \boxed{1.6\text{ k/ft}} \quad (32.1\text{-c})$$

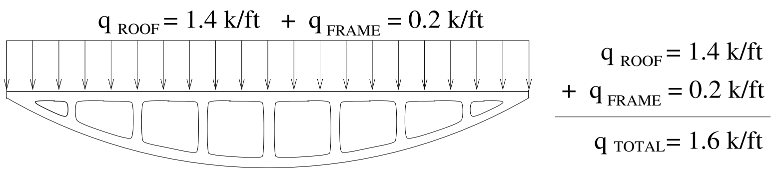


Figure 32.3: Magazzini Generali; Loads (Billington and Mark 1983)

32.3 Reactions

5 Reactions for the beam are determined first taking advantage of symmetry, Fig. 32.4:

$$W = (1.6) \text{ k/ft}(63.6) \text{ ft} = 102 \text{ k} \quad (32.2\text{-a})$$

$$R = \frac{W}{2} = \frac{102}{2} = \boxed{51 \text{ k}} \quad (32.2\text{-b})$$

We note that these reactions are provided by the internal shear forces.

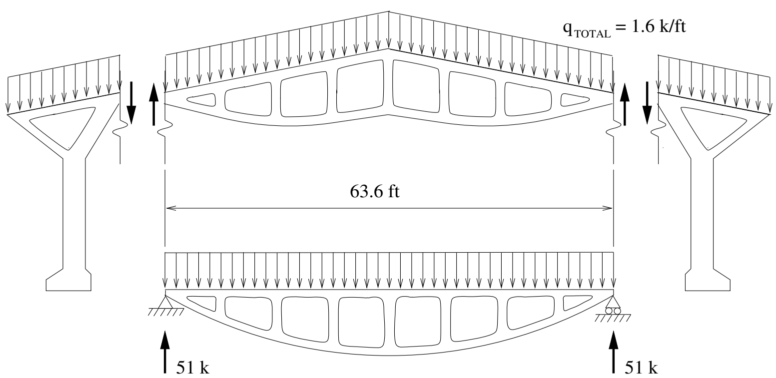


Figure 32.4: Magazzini Generali; Beam Reactions, (Billington and Mark 1983)

32.4 Forces

6 The internal forces are pimarily the shear and moments. Those can be easily determined for a simply supported uniformly loaded beam. The shear varies linearly from 51 kip to -51 kip with zero at the center, and the moment diagram is parabolic with the maximum moment at the center, Fig. 32.5, equal to:

$$M_{max} = \frac{qL^2}{8} = \frac{(1.6) \text{ k/ft}(63.6) \text{ ft}^2}{8} = \boxed{808 \text{ k.ft}} \quad (32.3)$$

7 The externally induced moment at midspan must be resisted by an equal and opposite internal moment. This can be achieved through a combination of compressive force on the upper fibers, and tensile ones on the lower. Thus the net axial force is zero, however there is a net internal couple, Fig. 32.6.

Chapter 33

BUILDING STRUCTURES

33.1 Introduction

33.1.1 Beam Column Connections

1 The connection between the beam and the column can be, Fig. 33.1:

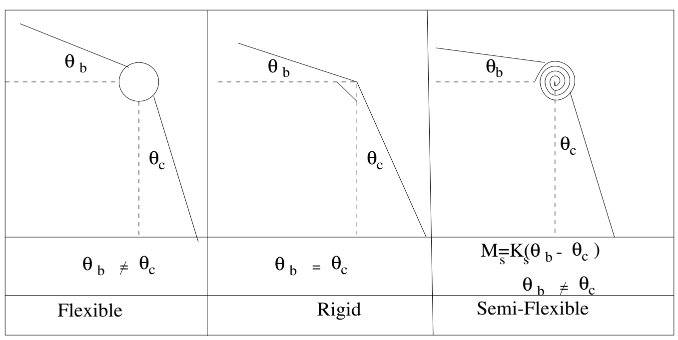


Figure 33.1: Flexible, Rigid, and Semi-Flexible Joints

Flexible that is a hinge which can transfer forces only. In this case we really have cantiliver action only. In a flexible connection the column and beam end moments are both equal to zero, $M_{\rm col} = M_{\rm beam} = 0$. The end rotation are not equal, $\theta_{\rm col} \neq \theta_{\rm beam}$.

Rigid: The connection is such that $\theta_{\rm beam} = \theta_{\rm col}$ and moment can be transmitted through the connection. In a rigid connection, the end moments and rotations are equal (unless there is an externally applied moment at the node), $M_{\rm col} = M_{\rm beam} \neq 0$, $\theta_{\rm col} = \theta_{\rm beam}$.

Semi-Rigid: The end moments are equal and not equal to zero, but the rotation are different. $\theta_{\rm beam} \neq \theta_{\rm col}$, $M_{\rm col} = M_{\rm beam} \neq 0$. Furthermore, the difference in rotation is resisted by the spring $M_{\rm spring} = K_{\rm spring}(\theta_{\rm col} - \theta_{\rm beam})$.

33.1.2 Behavior of Simple Frames

2 For vertical load across the beam rigid connection will reduce the maximum moment in the beam (at the expense of a negative moment at the ends which will in turn be transferred to

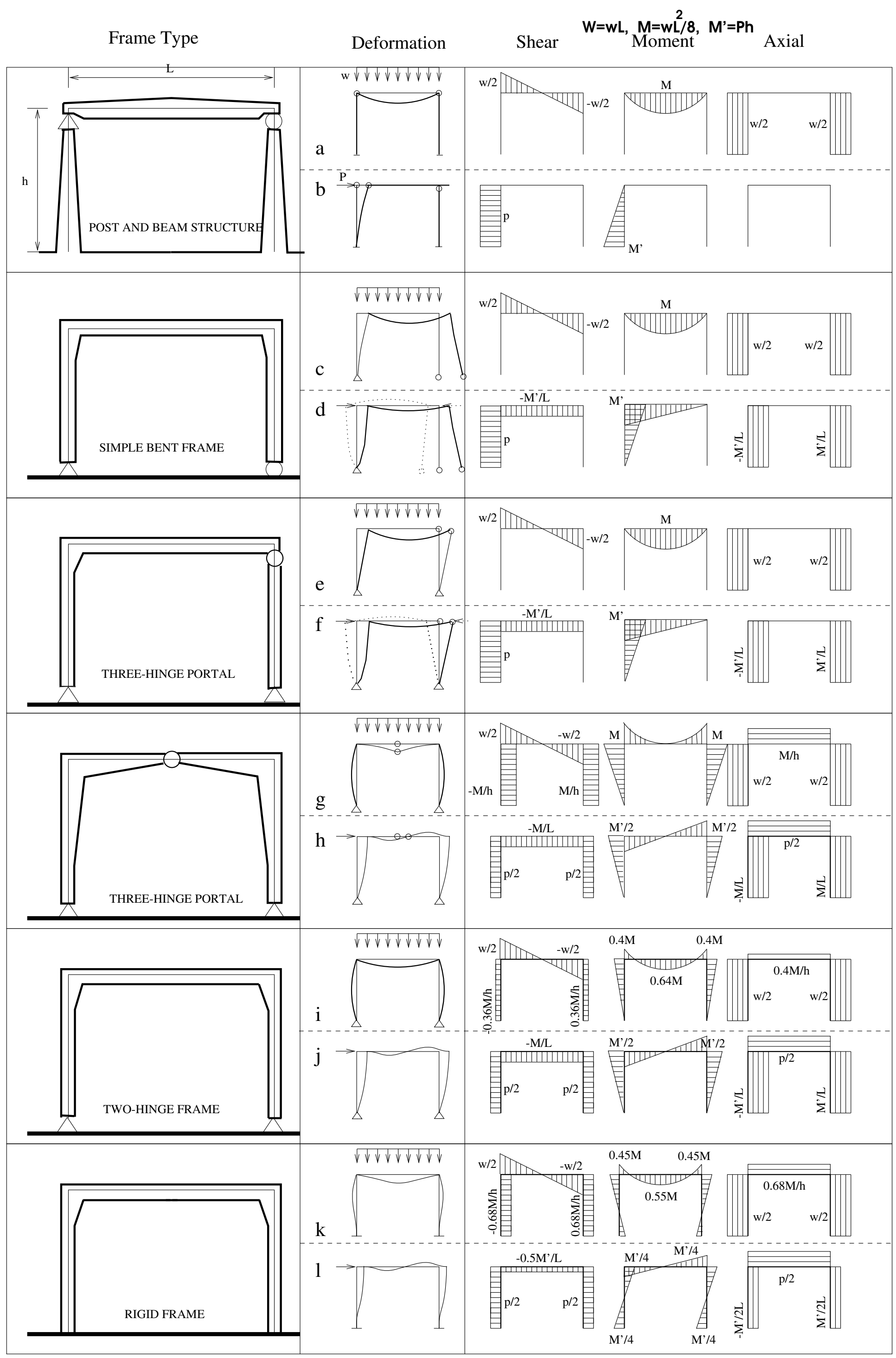


Figure 33.3: Deformation, Shear, Moment, and Axial Diagrams for Various Types of Portal Frames Subjected to Vertical and Horizontal Loads

33.2 Buildings Structures

11 There are three primary types of building systems:

Wall Subsytem: in which very rigid walls made up of solid masonry, paneled or braced timber, or steel trusses constitute a rigid subsystem. This is only adequate for small rise buildings.

Vertical Shafts: made up of four solid or trussed walls forming a tubular space structure. The tubular structure may be interior (housing elevators, staircases) and/or exterior. Most efficient for very high rise buildings.

Rigid Frame: which consists of linear vertical components (columns) rigidly connected to stiff horizontal ones (beams and girders). This is not a very efficient structural form to resist lateral (wind/earthquake) loads.

33.2.1 Wall Subsystems

12 Whereas exterior wall provide enclosure and interior ones separation, both of them can also have a structural role in trnsfering vertical and horizontal loads.

13 Walls are constructed out of masonry, timber concrete or steel.

14 If the wall is braced by floors, then it can provide an excellent resitance to horizontal load in the plane of the wall (but not orthogonal to it).

15 When shear-walls subsytems are used, it is best if the center of orthogonal shear resistance is close to the centroid of lateral loads as applied. If this is not the case, then there will be torsional design problems.

33.2.1.1 Example: Concrete Shear Wall

`From (Lin and Stotesbury 1981)`

16 We consider a reinforced concrete wall 20 ft wide, 1 ft thick, and 120 ft high with a vertical load of 400 k acting on it at the base. As a result of wind, we assume a uniform horizontal force of 0.8 kip/ft of vertical height acting on the wall. It is required to compute the flexural stresses and the shearing stresses in the wall to resist the wind load, Fig. 33.5.

1. Maximum shear force and bending moment at the base

$$V_{max} = wL = (0.8)\text{ k.ft}(120)\text{ ft} = 96\text{ k} \quad (33.8\text{-a})$$

$$M_{max} = \frac{wL^2}{2} = \frac{(0.8)\text{ k.ft}(120)^2\text{ ft}^2}{2} = 5,760\text{ k.ft} \quad (33.8\text{-b})$$

2. The resulting eccentricity is

$$e_{\text{Actual}} = \frac{M}{P} = \frac{(5,760)\text{ k.ft}}{(400)\text{ k}} = 14.4\text{ ft} \quad (33.9)$$

3. The critical eccentricity is

$$e_{cr} = \frac{L}{6} = \frac{(20)\text{ ft}}{6} = 3.3\text{ ft} < e_{\text{Actual}} N.G. \quad (33.10)$$

   thus there will be tension at the base.

9. The compressive stress of 740 psi can easily be sustained by concrete, as to the tensile stress of 460 psi, it would have to be resisted by some steel reinforcement.

10. Given that those stresses are *service* stresses and not factored ones, we adopt the WSD approach, and use an allowable stress of 20 ksi, which in turn will be increased by 4/3 for seismic and wind load,
$$\sigma_{all} = \frac{4}{3}(20) = 26.7 \text{ ksi} \tag{33.16}$$

11. The stress distribution is linear, compression at one end, and tension at the other. The length of the tension area is given by (similar triangles)
$$\frac{x}{460} = \frac{20}{460+740} \Rightarrow x = \frac{460}{460+740}(20) = 7.7 \text{ ft} \tag{33.17}$$

12. The total tensile force inside this triangular stress block is
$$T = \frac{1}{2}(460) \text{ ksi}(7.7 \times 12) \text{ in} \underbrace{(12) \text{ in}}_{\text{width}} = 250 \text{ k} \tag{33.18}$$

13. The total amount of steel reinforcement needed is
$$A_s = \frac{(250) \text{ k}}{(26.7) \text{ ksi}} = \boxed{9.4 \text{ in}^2} \tag{33.19}$$

    This amount of reinforcement should be provided at both ends of the wall since the wind or eartquake can act in any direction. In addition, the foundations should be designed to resist tensile uplift forces (possibly using piles).

33.2.1.2 Example: Trussed Shear Wall

`From (Lin and Stotesbury 1981)`

17 We consider the same problem previously analysed, but use a trussed shear wall instead of a concrete one, Fig. 33.6.

1. Using the maximum moment of 5,760 kip-ft (Eq. 33.8-b), we can compute the compression and tension in the columns for a lever arm of 20 ft.
$$F = \pm\frac{(5,760) \text{ k.ft}}{(20) \text{ ft}} = \pm 288 \text{ k} \tag{33.20}$$

2. If we now add the effect of the 400 kip vertical load, the forces would be
$$C = -\frac{(400) \text{ k}}{2} - 288 = \boxed{-488 \text{ k}} \tag{33.21-a}$$
$$T = -\frac{(400) \text{ k}}{2} + 288 = \boxed{88 \text{ k}} \tag{33.21-b}$$

3. The force in the diagonal which must resist a base shear of 96 kip is (similar triangles)
$$\frac{F}{96} = \frac{\sqrt{(20)^2 + (24)^2}}{20} \Rightarrow F = \frac{\sqrt{(20)^2 + (24)^2}}{20}(96) = \boxed{154 \text{ k}} \tag{33.22}$$

4. The design could be modified to have no tensile forces in the columns by increasing the width of the base (currently at 20 ft).