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12.1 Introduction

In this chapter we define a new type of algebra: the Clifford algebra. Clifford algebra was originally proposed by William Kingdon Clifford (Clifford) based on Grassmann's ideas. Clifford algebras have now found an important place as mathematical systems for describing some of the more fundamental theories of mathematical physics. Clifford algebra is based on a new kind of product called the *Clifford product*.

In this chapter we will show how the Clifford product can be defined straightforwardly in terms of the exterior and interior products that we have already developed, without introducing any new axioms. This approach has the dual advantage of ensuring consistency and enabling all the results we have developed previously to be applied to Clifford numbers.

In the previous chapter we have gone to some lengths to define and explore the generalized Grassmann product. In this chapter we will use the generalized Grassmann product to define the Clifford product in its most general form as a simple sum of generalized products. Computations in general Clifford algebras can be complicated, and this approach at least permits us to render the complexities in a straightforward manner.

From the general case, we then show how Clifford products of intersecting and orthogonal elements simplify. This is the normal case treated in introductory discussions of Clifford algebra.

Clifford algebras occur throughout mathematical physics, the most well known being the real numbers, complex numbers, quaternions, and complex quaternions. In this book we show how Clifford algebras can be firmly based on the Grassmann algebra as a sum of generalized Grassmann products.

Historical Note

The seminal work on Clifford algebra is by Clifford in his paper *Applications of Grassmann's Extensive Algebra* that he published in the *American Journal of Mathematics Pure and Applied* in 1878 (Clifford). Clifford became a great admirer of Grassmann and one of those rare contemporaries who appears to have understood his work. The first paragraph of this paper contains the following passage.

> Until recently I was unacquainted with the *Ausdehnungslehre*, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers*. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence on the future of mathematical science.

12.2 The Clifford Product

Definition of the Clifford product

The Clifford product of elements $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is denoted by $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ and defined to be

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.1$$

Here, $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}$ is the generalized Grassmann product of order λ.

The most surprising property of the Clifford product is its associativity, even though it is defined in terms of products which are non-associative. The associativity of the Clifford products is not directly evident from the definition.

Tabulating Clifford products

In order to see what this formula gives we can tabulate the Clifford products in terms of generalized products. Here we maintain the grade of the first factor general, and vary the grade of the second.

```
Table[
  {α_m ◇ β_k, Sum[CreateSignedGeneralizedProducts[λ][α_m, β_k], {λ, 0, k}]},
  {k, 0, 4}] // SimplifySigns // TableForm
```

$\underset{m}{\alpha} \diamond \underset{0}{\beta}$	$\underset{m}{\alpha} \underset{0}{\triangle} \underset{0}{\beta}$
$\underset{m}{\alpha} \diamond \beta$	$\underset{m}{\alpha} \underset{0}{\triangle} \beta - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \beta\right)$
$\underset{m}{\alpha} \diamond \underset{2}{\beta}$	$\underset{m}{\alpha} \underset{0}{\triangle} \underset{2}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}$
$\underset{m}{\alpha} \diamond \underset{3}{\beta}$	$\underset{m}{\alpha} \underset{0}{\triangle} \underset{3}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{3}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{3}{\beta} + (-1)^m \left(\underset{m}{\alpha} \underset{3}{\triangle} \underset{3}{\beta}\right)$
$\underset{m}{\alpha} \diamond \underset{4}{\beta}$	$\underset{m}{\alpha} \underset{0}{\triangle} \underset{4}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{4}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{4}{\beta} + (-1)^m \left(\underset{m}{\alpha} \underset{3}{\triangle} \underset{4}{\beta}\right) + \underset{m}{\alpha} \underset{4}{\triangle} \underset{4}{\beta}$

The grade of a Clifford product

It can be seen that, from its definition 12.1 in terms of generalized products, the Clifford product of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is a sum of elements with grades ranging from $m + k$ to $| m - k |$ in steps of 2. All the elements of a given Clifford product are either of even grade (if $m + k$ is even), or of odd grade (if $m + k$ is odd). To calculate the full range of grades in a space of unspecified dimension we can use the *GrassmannAlgebra* function `RawGrade`. For example:

```
RawGrade[α₃ ⋄ β₂]
```

```
{1, 3, 5}
```

Given a space of a certain dimension, some of these elements may necessarily be zero (and thus result in a grade of `Grade0`), because their grade is larger than the dimension of the space. For example, in a 3-space:

```
𝕍₃; Grade[α₃ ⋄ β₂]
```

```
{1, 3, Grade0}
```

In the general discussion of the Clifford product that follows, we will assume that the dimension of the space is high enough to avoid any terms of the product becoming zero because their grade exceeds the dimension of the space. In later more specific examples, however, the dimension of the space becomes an important factor in determining the structure of the particular Clifford algebra under consideration.

Clifford products in terms of generalized products

As can be seen from the definition, the Clifford product of a simple *m*-element and a simple *k*-element can be expressed as the sum of signed generalized products.

For example, $\underset{3}{\alpha} \diamond \underset{2}{\beta}$ may be expressed as the sum of three signed generalized products of grades 1, 3 and 5. In *GrassmannAlgebra* we can effect this conversion by applying the `ToGeneralizedProducts` function.

```
ToGeneralizedProducts[α₃ ⋄ β₂]
```

$$\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta} + \underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta} - \underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}$$

Multiple Clifford products can be expanded in the same way. For example:

```
ToGeneralizedProducts[α₃ ⋄ β₂ ⋄ γ]
```

$$\left(\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + -\left(\underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma + -\left(\underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma$$

`ToGeneralizedProducts` can also be used to expand Clifford products in any Grassmann expression, or list of Grassmann expressions. For example we can express the Clifford product of two general Grassmann numbers in 2-space in terms of generalized products.

$\mathbb{V}_2$; X = CreateGrassmannNumber[ξ] ⋄ CreateGrassmannNumber[ψ]

$$(\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1 \wedge e_2) \diamond (\psi_0 + e_1\,\psi_1 + e_2\,\psi_2 + \psi_3\,e_1 \wedge e_2)$$

X_1 = ToGeneralizedProducts[X]

$$\begin{aligned}
&\xi_0 \underset{0}{\triangle} \psi_0 + \xi_0 \underset{0}{\triangle} (e_1\,\psi_1) + \xi_0 \underset{0}{\triangle} (e_2\,\psi_2) + \xi_0 \underset{0}{\triangle} (\psi_3\,e_1 \wedge e_2) + (e_1\,\xi_1) \underset{0}{\triangle} \psi_0 + \\
&\quad (e_1\,\xi_1) \underset{0}{\triangle} (e_1\,\psi_1) + (e_1\,\xi_1) \underset{0}{\triangle} (e_2\,\psi_2) + (e_1\,\xi_1) \underset{0}{\triangle} (\psi_3\,e_1 \wedge e_2) + (e_2\,\xi_2) \underset{0}{\triangle} \psi_0 + \\
&\quad (e_2\,\xi_2) \underset{0}{\triangle} (e_1\,\psi_1) + (e_2\,\xi_2) \underset{0}{\triangle} (e_2\,\psi_2) + (e_2\,\xi_2) \underset{0}{\triangle} (\psi_3\,e_1 \wedge e_2) + \\
&\quad (\xi_3\,e_1 \wedge e_2) \underset{0}{\triangle} \psi_0 + (\xi_3\,e_1 \wedge e_2) \underset{0}{\triangle} (e_1\,\psi_1) + (\xi_3\,e_1 \wedge e_2) \underset{0}{\triangle} (e_2\,\psi_2) + \\
&\quad (\xi_3\,e_1 \wedge e_2) \underset{0}{\triangle} (\psi_3\,e_1 \wedge e_2) + (e_1\,\xi_1) \underset{1}{\triangle} (e_1\,\psi_1) + (e_1\,\xi_1) \underset{1}{\triangle} (e_2\,\psi_2) + \\
&\quad (e_1\,\xi_1) \underset{1}{\triangle} (\psi_3\,e_1 \wedge e_2) + (e_2\,\xi_2) \underset{1}{\triangle} (e_1\,\psi_1) + (e_2\,\xi_2) \underset{1}{\triangle} (e_2\,\psi_2) + \\
&\quad (e_2\,\xi_2) \underset{1}{\triangle} (\psi_3\,e_1 \wedge e_2) - (\xi_3\,e_1 \wedge e_2) \underset{1}{\triangle} (e_1\,\psi_1) - (\xi_3\,e_1 \wedge e_2) \underset{1}{\triangle} (e_2\,\psi_2) - \\
&\quad (\xi_3\,e_1 \wedge e_2) \underset{1}{\triangle} (\psi_3\,e_1 \wedge e_2) - (\xi_3\,e_1 \wedge e_2) \underset{2}{\triangle} (\psi_3\,e_1 \wedge e_2)
\end{aligned}$$

Clifford products in terms of interior products

The primary definitions of Clifford and generalized products are in terms of exterior and interior products. To transform a Clifford product to exterior and interior products, we can either apply the *GrassmannAlgebra* function `ToInteriorProducts` to the results obtained above, or directly to the expression involving the Clifford product.

ToInteriorProducts$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$

$$\begin{aligned}
&-(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2) + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \beta_1) \wedge \beta_2 - \\
&\quad (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \beta_2) \wedge \beta_1 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2
\end{aligned}$$

Note that `ToInteriorProducts` works with explicit elements, but also as above, automatically creates a simple exterior product from an underscripted element.

`ToInteriorProducts` expands the Clifford product by using the A form of the generalized product. A second function `ToInteriorProductsB` expands the Clifford product by using the `B` form of the expansion to give a different but equivalent expression.

ToInteriorProductsB$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$

$$\begin{aligned}
&-(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2) - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \ominus \beta_2 + \\
&\quad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_2 \ominus \beta_1 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2
\end{aligned}$$

In this example, the difference is evidenced in the second and third terms.

Clifford products in terms of inner products

Clifford products may also be expressed in terms of inner products, some of which may be scalar products. From this form it is easy to read the grade of the terms.

```
ToInnerProducts[α ⋄ β]
                3   2
```

$$-(\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\,\alpha_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\,\alpha_2 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\,\alpha_3 - (\alpha_3 \ominus \beta_2)\,\alpha_1 \wedge \alpha_2 \wedge \beta_1 + (\alpha_3 \ominus \beta_1)\,\alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_2 \ominus \beta_2)\,\alpha_1 \wedge \alpha_3 \wedge \beta_1 - (\alpha_2 \ominus \beta_1)\,\alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_1 \ominus \beta_2)\,\alpha_2 \wedge \alpha_3 \wedge \beta_1 + (\alpha_1 \ominus \beta_1)\,\alpha_2 \wedge \alpha_3 \wedge \beta_2 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2$$

Clifford products in terms of scalar products

Finally, Clifford products may be expressed in terms of scalar and exterior products only.

```
ToScalarProducts[α ⋄ β]
                 3   2
```

$$(\alpha_2 \ominus \beta_2)\,(\alpha_3 \ominus \beta_1)\,\alpha_1 - (\alpha_2 \ominus \beta_1)\,(\alpha_3 \ominus \beta_2)\,\alpha_1 - (\alpha_1 \ominus \beta_2)\,(\alpha_3 \ominus \beta_1)\,\alpha_2 + (\alpha_1 \ominus \beta_1)\,(\alpha_3 \ominus \beta_2)\,\alpha_2 + (\alpha_1 \ominus \beta_2)\,(\alpha_2 \ominus \beta_1)\,\alpha_3 - (\alpha_1 \ominus \beta_1)\,(\alpha_2 \ominus \beta_2)\,\alpha_3 - (\alpha_3 \ominus \beta_2)\,\alpha_1 \wedge \alpha_2 \wedge \beta_1 + (\alpha_3 \ominus \beta_1)\,\alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_2 \ominus \beta_2)\,\alpha_1 \wedge \alpha_3 \wedge \beta_1 - (\alpha_2 \ominus \beta_1)\,\alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_1 \ominus \beta_2)\,\alpha_2 \wedge \alpha_3 \wedge \beta_1 + (\alpha_1 \ominus \beta_1)\,\alpha_2 \wedge \alpha_3 \wedge \beta_2 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2$$

12.3 The Reverse of an Exterior Product

Defining the reverse

We will find in our discussion of Clifford products that many operations and formulae are simplified by expressing some of the exterior products in a form which reverses the order of the 1-element factors in the products.

We denote the *reverse* of a simple m-element $\underset{m}{\alpha}$ by $\underset{m}{\alpha}^{\dagger}$, and define it to be:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m)^{\dagger} == \alpha_m \wedge \alpha_{m-1} \wedge \cdots \wedge \alpha_1 \qquad \mathbf{12.2}$$

We can easily work out the number of permutations to achieve this rearrangement as $\frac{1}{2}\,m\,(m-1)$. *Mathematica* automatically simplifies $(-1)^{\frac{1}{2}m\,(m-1)}$ to $\mathbb{i}^{m\,(m-1)}$.

$$\underset{m}{\alpha}^{\dagger} == (-1)^{\frac{1}{2} m (m-1)} \underset{m}{\alpha} == \mathbb{i}^{m (m-1)} \underset{m}{\alpha} \qquad 12.3$$

The operation of taking the reverse of an element is called *reversion*.

In *Mathematica*, the superscript dagger is represented by `SuperDagger`. Thus **`SuperDagger`**$[\underset{m}{\alpha}]$ returns $\underset{m}{\alpha}^{\dagger}$.

The pattern of signs, as *m* increases from zero, alternates in pairs:

1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, …

Computing the reverse

In *GrassmannAlgebra* you can take the reverse of the exterior products in a Grassmann expression or list of Grassmann expressions by using the function `GrassmannReverse`. `GrassmannReverse` expands products and factors scalars before reversing the factors. It will operate with graded variables of either numeric or symbolic grade. For example:

$$\texttt{GrassmannReverse}\left[\left\{1, x, x\wedge y, \underset{3}{\alpha}\wedge 3, \underset{k}{\beta}, \underset{m}{\alpha}\ominus\left(\underset{k}{\beta}-\underset{p}{\gamma}\right)\right\}\right]$$

$$\{1, x, y\wedge x, 3\,\alpha_3\wedge\alpha_2\wedge\alpha_1, \beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1, \alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1 - \alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\gamma_p\wedge\cdots\wedge\gamma_2\wedge\gamma_1\}$$

On the other hand, if we wish just to put exterior products into reverse *form* (that is, reversing the factors, and changing the sign of the product so that the result remains equal to the original), we can use the *GrassmannAlgebra* function `ReverseForm`.

$$\texttt{ReverseForm}\left[\left\{1, x, x\wedge y, \underset{3}{\alpha}, \underset{k}{\beta}, \underset{m}{\alpha}\ominus\left(\underset{k}{\beta}-\underset{p}{\gamma}\right)\right\}\right]$$

$$\{1, x, -(y\wedge x), -(\alpha_3\wedge\alpha_2\wedge\alpha_1), \mathbb{i}^{(-1+k)\,k}\,\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1, \mathbb{i}^{(-1+k)\,k+(-1+m)\,m}\,(\alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1) - \mathbb{i}^{(-1+m)\,m+(-1+p)\,p}\,(\alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\gamma_p\wedge\cdots\wedge\gamma_2\wedge\gamma_1)\}$$

12.4 Special Cases of Clifford Products

The Clifford product with scalars

The Clifford product of a scalar with any element simplifies to the usual (field) product.

```
ToScalarProducts[a ⋄ α_m]
```

$a\ \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$

$$a \diamond \underset{m}{\alpha} == a\ \underset{m}{\alpha} \qquad 12.4$$

Hence the Clifford product of any number of scalars is just their underlying field product.

```
ToScalarProducts[a ⋄ b ⋄ c]
```

$a\ b\ c$

The Clifford product of 1-elements

The Clifford product of two 1-elements is just the sum of their interior (here scalar) and exterior products. Hence it is of grade {0, 2}.

```
ToScalarProducts[x ⋄ y]
```

$x \ominus y + x \wedge y$

$$x \diamond y == x \wedge y + x \ominus y \qquad 12.5$$

The Clifford product of any number of 1-elements can be computed.

```
ToScalarProducts[x ⋄ y ⋄ z]
```

$z\ (x \ominus y) - y\ (x \ominus z) + x\ (y \ominus z) + x \wedge y \wedge z$

```
ToScalarProducts[w ⋄ x ⋄ y ⋄ z]
```

$(w \ominus z)\ (x \ominus y) - (w \ominus y)\ (x \ominus z) + (w \ominus x)\ (y \ominus z) + (y \ominus z)\ w \wedge x - (x \ominus z)\ w \wedge y +$
$(x \ominus y)\ w \wedge z + (w \ominus z)\ x \wedge y - (w \ominus y)\ x \wedge z + (w \ominus x)\ y \wedge z + w \wedge x \wedge y \wedge z$

The Clifford product of an *m*-element and a 1-element

The Clifford product of an arbitrary simple *m*-element and a 1-element results in just two terms: their exterior and interior products.

$$x \diamond \underset{m}{\alpha} == x \wedge \underset{m}{\alpha} + \underset{m}{\alpha} \ominus x \qquad 12.6$$

$$\underset{m}{\alpha} \diamond x == \underset{m}{\alpha} \wedge x - (-1)^m \underset{m}{\alpha} \ominus x \qquad 12.7$$

By rewriting equation 12.7 as:

$$(-1)^m \underset{m}{\alpha} \diamond x == x \wedge \underset{m}{\alpha} - \underset{m}{\alpha} \ominus x$$

we can add and subtract equations 12.6 and 12.7 to express the exterior product and interior products of a 1-element and an *m*-element in terms of Clifford products

$$x \wedge \underset{m}{\alpha} == \frac{1}{2} \left(x \diamond \underset{m}{\alpha} + (-1)^m \underset{m}{\alpha} \diamond x \right) \qquad 12.8$$

$$\underset{m}{\alpha} \ominus x == \frac{1}{2} \left(x \diamond \underset{m}{\alpha} - (-1)^m \underset{m}{\alpha} \diamond x \right) \qquad 12.9$$

The Clifford product of an *m*-element and a 2-element

The Clifford product of an arbitrary *m*-element and a 2-element is given by just three terms, an exterior product, a generalized product of order 1, and an interior product.

$$\underset{m}{\alpha} \diamond \underset{2}{\beta} == \underset{m}{\alpha} \wedge \underset{2}{\beta} - (-1)^m \underset{m}{\alpha} \underset{1}{\triangle} \underset{2}{\beta} - \underset{m}{\alpha} \ominus \underset{2}{\beta} \qquad 12.10$$

The Clifford product of two 2-elements

By way of example we explore the various forms into which we can cast a Clifford product of two 2-elements. The highest level is into a sum of generalized products. From this we can expand the terms into interior, inner or scalar products where appropriate.

We expect the grade of the Clifford product to be a composite one.

```
RawGrade[(x ∧ y) ⋄ (u ∧ v)]
```

{0, 2, 4}

```
ToGeneralizedProducts[(x ∧ y) ⋄ (u ∧ v)]
```

$$x \wedge y \underset{0}{\triangle} u \wedge v - x \wedge y \underset{1}{\triangle} u \wedge v - x \wedge y \underset{2}{\triangle} u \wedge v$$

The first of these generalized products is equivalent to an exterior product of grade 4. The second is a generalized product of grade 2. The third reduces to an interior product of grade 0. We can see this more explicitly by converting to interior products.

```
ToInteriorProducts[(x∧y)⋄(u∧v)]
```

$-(u\wedge v\ominus x\wedge y)-(x\wedge y\ominus u)\wedge v+(x\wedge y\ominus v)\wedge u+u\wedge v\wedge x\wedge y$

We can convert the middle terms to inner (in this case scalar) products.

```
ToInnerProducts[(x∧y)⋄(u∧v)]
```

$-(u\wedge v\ominus x\wedge y)+(v\ominus y)\,u\wedge x-(v\ominus x)\,u\wedge y-(u\ominus y)\,v\wedge x+(u\ominus x)\,v\wedge y+u\wedge v\wedge x\wedge y$

Finally we can express the Clifford number in terms only of exterior and scalar products.

```
ToScalarProducts[(x∧y)⋄(u∧v)]
```

$(u\ominus y)\,(v\ominus x)-(u\ominus x)\,(v\ominus y)+(v\ominus y)\,u\wedge x-(v\ominus x)\,u\wedge y-(u\ominus y)\,v\wedge x+(u\ominus x)\,v\wedge y+u\wedge v\wedge x\wedge y$

The Clifford product of two identical elements

The Clifford product of two identical elements $\underset{p}{\gamma}$ is, by definition

$$\underset{p}{\gamma}\diamond\underset{p}{\gamma}=\sum_{\lambda=0}^{p}(-1)^{\lambda(p-\lambda)+\frac{1}{2}\lambda(\lambda-1)}\left(\underset{p}{\gamma}\underset{\lambda}{\Delta}\underset{p}{\gamma}\right)$$

Since the only non-zero generalized product of the form $\underset{p}{\gamma}\underset{\lambda}{\Delta}\underset{p}{\gamma}$ is that for which $\lambda = p$, that is $\underset{p}{\gamma}\ominus\underset{p}{\gamma}$, we have immediately that:

$$\underset{p}{\gamma}\diamond\underset{p}{\gamma}=(-1)^{\frac{1}{2}p(p-1)}\underset{p}{\gamma}\ominus\underset{p}{\gamma}$$

Or, alternatively:

$$\underset{p}{\gamma}\ominus\underset{p}{\gamma}=\underset{p}{\gamma}^{\dagger}\diamond\underset{p}{\gamma}=\underset{p}{\gamma}\diamond\underset{p}{\gamma}^{\dagger} \qquad 12.11$$

12.5 Alternate Forms for the Clifford Product

Alternate expansions of the Clifford product

The Clifford product has been defined in Section 12.2 as:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2} \lambda (\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

Alternate forms for the generalized product have been discussed in Chapter 10. The generalized product, and hence the Clifford product, may be expanded by decomposition of $\underset{m}{\alpha}$, by decomposition of $\underset{k}{\beta}$, or by decomposition of both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$.

The Clifford product expressed by decomposition of the first factor

The generalized product, expressed by decomposition of $\underset{m}{\alpha}$ is:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}\right) \qquad \underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots$$

Substituting this into the expression for the Clifford product gives:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} (-1)^{\lambda (m-\lambda)+\frac{1}{2} \lambda (\lambda-1)} \underset{m-\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}\right)$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots$$

Our objective is to rearrange the formula for the Clifford product so that the signs are absorbed into the formula, thus making the form of the formula independent of the values of *m* and λ. We can do this by writing $(-1)^{\frac{1}{2} \lambda (\lambda-1)} \underset{\lambda}{\alpha^{i}} == \underset{\lambda}{\alpha^{i}}^{\dagger}$ (where $\underset{\lambda}{\alpha^{i}}^{\dagger}$ is the reverse of $\underset{\lambda}{\alpha^{i}}$) and interchanging the order of the decomposition of $\underset{m}{\alpha}$ into a λ-element and a (m-λ)-element to absorb the $(-1)^{\lambda (m-\lambda)}$ factor: $(-1)^{\lambda (m-\lambda)} \underset{m}{\alpha} == \underset{m-\lambda}{\alpha^{1}} \wedge \underset{\lambda}{\alpha^{1}} == \underset{m-\lambda}{\alpha^{2}} \wedge \underset{\lambda}{\alpha^{2}} == \cdots$.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}^{\dagger}\right)$$

$$\underset{m}{\alpha} == \underset{m-\lambda}{\alpha^{1}} \wedge \underset{\lambda}{\alpha^{1}} == \underset{m-\lambda}{\alpha^{2}} \wedge \underset{\lambda}{\alpha^{2}} == \cdots$$

Because of the symmetry of the expression with respect to λ and (m-λ), we can write μ equal to (m-λ) and then λ becomes (m-μ). This enables us to write the formula a little simpler by arranging for the factors of grade μ to come before those of (m-μ). Finally, because of the inherent arbitrariness of the symbol μ, we can change it back to λ to get the formula in a more accustomed form.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \underset{\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{m-\lambda}{\alpha^{i}}{}^{\dagger}\right)$$
$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots \qquad \mathbf{12.12}$$

The right hand side of this expression is a sum of interior products. In *GrassmannAlgebra* we can develop the Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ as a sum of interior products in this particular form by using `ToInteriorProductsD` (note the final '`D`'). Since $\underset{m}{\alpha}$ is to be decomposed, it must have an explicit numerical grade.

Example 1

We can expand any explicit elements.

```
ToInteriorProductsD[(x ∧ y) ⋄ (u ∧ v)]
```

```
x ∧ y ⊖ v ∧ u + x ∧ (u ∧ v ⊖ y) − y ∧ (u ∧ v ⊖ x) + u ∧ v ∧ x ∧ y
```

Note that this is a different (although equivalent) result from that obtained using `ToInteriorProducts`.

```
ToInteriorProducts[(x ∧ y) ⋄ (u ∧ v)]
```

```
−(u ∧ v ⊖ x ∧ y) − (x ∧ y ⊖ u) ∧ v + (x ∧ y ⊖ v) ∧ u + u ∧ v ∧ x ∧ y
```

Example 2

We can also enter the elements as graded variables and have *GrassmannAlgebra* create the requisite products.

ToInteriorProductsD$\left[\underset{2}{\alpha} \diamond \underset{2}{\beta}\right]$

$$\beta_1 \wedge \beta_2 \ominus \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge (\beta_1 \wedge \beta_2 \ominus \alpha_2) - \alpha_2 \wedge (\beta_1 \wedge \beta_2 \ominus \alpha_1) + \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$$

Example 3

The formula shows that it does not depend on the grade of $\underset{k}{\beta}$ for its form. Thus we can still obtain an expansion for general $\underset{k}{\beta}$. For example we can take:

A = ToInteriorProductsD$\left[\underset{2}{\alpha} \diamond \underset{k}{\beta}\right]$

$$\underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge \left(\underset{k}{\beta} \ominus \alpha_2\right) - \alpha_2 \wedge \left(\underset{k}{\beta} \ominus \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \underset{k}{\beta}$$

Example 4

Note that if *k* is less than the grade of the first factor, some of the interior product terms may be zero, thus simplifying the expression.

B = ToInteriorProductsD$\left[\underset{2}{\alpha} \diamond \beta\right]$

$$-(\alpha_1 \wedge \alpha_2 \ominus \beta) + \beta \wedge \alpha_1 \wedge \alpha_2$$

If we put *k* equal to 1 in the expression derived for general *k* in Example 3 we get:

```
A1 = A /. k → 1
```

$$\beta \ominus \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge (\beta \ominus \alpha_2) - \alpha_2 \wedge (\beta \ominus \alpha_1) + \alpha_1 \wedge \alpha_2 \wedge \beta$$

Although this does not immediately look like the expression **B** above, we can see that it is the same by noting that the first term of **A1** is zero, and expanding their difference to scalar products.

```
ToScalarProducts[B - A1]
```

0

Alternative expression by decomposition of the first factor

An alternative expression for the Clifford product expressed by a decomposition of $\underset{m}{\alpha}$ is obtained by reversing the order of the factors in the generalized products, and then expanding the generalized products in their B form expansion.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)+(m-\lambda)(k-\lambda)} \left(\underset{k}{\beta} \underset{\lambda}{\Delta} \underset{m}{\alpha}\right)$$

$$== \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} (-1)^{k (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{k}{\beta} \wedge \underset{m-\lambda}{\alpha^i}\right) \ominus \underset{\lambda}{\alpha^i}$$

As before we write $(-1)^{\frac{1}{2}\lambda(\lambda-1)} \underset{\lambda}{\alpha^i} == \underset{\lambda}{\alpha^i}^{\dagger}$ and interchange the order of $\underset{k}{\beta}$ and $\underset{m-\lambda}{\alpha^i}$ to absorb the sign $(-1)^{k(m-\lambda)}$ to get:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{m-\lambda}{\alpha^i} \wedge \underset{k}{\beta}\right) \ominus \underset{\lambda}{\alpha^i}^{\dagger}$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots$$

In order to get this into a form for direct comparison of our previously derived result in formula 12.12, we write $(-1)^{\lambda(m-\lambda)}\ \underset{m}{\alpha} == \underset{m-\lambda}{\alpha^1} \wedge \underset{\lambda}{\alpha^1} == \underset{m-\lambda}{\alpha^2} \wedge \underset{\lambda}{\alpha^2} == \cdots$, then interchange λ and (m-λ) as before to get finally:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{\lambda}{\alpha^i} \wedge \underset{k}{\beta}\right) \ominus \underset{m-\lambda}{\alpha^i}{}^{\dagger} \qquad 12.13$$
$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots$$

As before, the right hand side of this expression is a sum of interior products. In *GrassmannAlgebra* we can develop the Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ in this form by using `ToInteriorProductsC`. This is done in the Section 12.6 below.

The Clifford product expressed by decomposition of the second factor

If we wish to expand a Clifford product in terms of the second factor $\underset{k}{\beta}$, we can use formulas A and B of the generalized product theorem (Section 10.5) and substitute directly to get either of:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j}{}^{\dagger}\right) \wedge \underset{k-\lambda}{\beta^j} \qquad 12.14$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{m}{\alpha} \wedge \underset{k-\lambda}{\beta^j}\right) \ominus \underset{\lambda}{\beta^j}{}^{\dagger} \qquad 12.15$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

The Clifford product expressed by decomposition of both factors

We can similarly decompose the Clifford product in terms of both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$. The sign $(-1)^{\frac{1}{2}\lambda(\lambda-1)}$ can be absorbed by taking the reverse of either $\underset{\lambda}{\alpha^{i}}$ or $\underset{\lambda}{\beta^{j}}$.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\text{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{\lambda}{\alpha^{i}}^{\dagger} \ominus \underset{\lambda}{\beta^{j}}\right) \underset{m-\lambda}{\alpha^{i}} \wedge \underset{k-\lambda}{\beta^{j}}$$

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\text{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{\lambda}{\alpha^{i}} \ominus \underset{\lambda}{\beta^{j}}^{\dagger}\right) \underset{m-\lambda}{\alpha^{i}} \wedge \underset{k-\lambda}{\beta^{j}} \qquad 12.16$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots \qquad \underset{k}{\beta} == \underset{\lambda}{\beta^{1}} \wedge \underset{k-\lambda}{\beta^{1}} == \underset{\lambda}{\beta^{2}} \wedge \underset{k-\lambda}{\beta^{2}} == \cdots$$

12.6 Writing Down a General Clifford Product

The form of a Clifford product expansion

We take as example the Clifford product $\underset{4}{\alpha} \diamond \underset{k}{\beta}$ and expand it in terms of the factors of its first factor in both the D form and the C form. We see that all terms except the interior and exterior products at the ends of the expression appear to be different. There are of course equal numbers of terms in either form. Note that if the parentheses were not there, *the expressions would be identical* except for (possibly) the signs of the terms. Although these central terms differ between the two forms, we can show by reducing them to scalar products that their *sums* are the same.

The D form

```
X = ToInteriorProductsD[α₄ ◇ βₖ]
```

$$\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1+\alpha_1\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\right)-\alpha_2\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_1\right)+\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\wedge\alpha_1\right)-\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2\wedge\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\right)-\alpha_1\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\right)+\alpha_1\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2\right)+\alpha_2\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_1\right)-\alpha_2\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_1\right)+\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\right)-\alpha_1\wedge\alpha_2\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\right)+\alpha_1\wedge\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_2\right)-\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}$$

The C form

Note that since the exterior product has higher precedence in *Mathematica* than the interior product, $\alpha_1\wedge\underset{k}{\beta}\ominus\alpha_2$ is equivalent to $\left(\alpha_1\wedge\underset{k}{\beta}\right)\ominus\alpha_2$, and no parentheses are necessary in the terms of the C form.

```
ToInteriorProductsC[α₄ ◇ βₖ]
```

$$\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1-\alpha_1\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2+\alpha_2\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_1-\alpha_3\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\wedge\alpha_1+\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2\wedge\alpha_1+\alpha_1\wedge\alpha_2\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3-\alpha_1\wedge\alpha_3\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2+\alpha_1\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2+\alpha_2\wedge\alpha_3\wedge\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_1-\alpha_2\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_1+\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1-\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\underset{k}{\beta}\ominus\alpha_4+\alpha_1\wedge\alpha_2\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_3-\alpha_1\wedge\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_2+\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}\ominus\alpha_1+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}$$

Either form is easy to write down directly by observing simple mnemonic rules.

A mnemonic way to write down a general Clifford product

We can begin developing the D form mnemonically by listing the $\underset{\lambda}{\alpha}^{i}$. This list has the same form as the basis of the Grassmann algebra of a 4-space.

◆ 1. Calculate the $\underset{\lambda}{\alpha}^{i}$

These are all the essentially different combinations of factors of all grades. The list has the same form as the basis of the Grassmann algebra whose n-element is $\underset{m}{\alpha}$.

```
DeclareBasis[4, α]; B = BasisΛ[]
```

$\{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1\wedge\alpha_2, \alpha_1\wedge\alpha_3, \alpha_1\wedge\alpha_4, \alpha_2\wedge\alpha_3, \alpha_2\wedge\alpha_4, \alpha_3\wedge\alpha_4, \alpha_1\wedge\alpha_2\wedge\alpha_3, \alpha_1\wedge\alpha_2\wedge\alpha_4, \alpha_1\wedge\alpha_3\wedge\alpha_4, \alpha_2\wedge\alpha_3\wedge\alpha_4, \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\}$

◆ 2. Calculate the cobasis with respect to $\underset{m}{\alpha}$

```
A = CobasisΛ[]
```

$\{\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4, \alpha_2\wedge\alpha_3\wedge\alpha_4, -(\alpha_1\wedge\alpha_3\wedge\alpha_4), \alpha_1\wedge\alpha_2\wedge\alpha_4, -(\alpha_1\wedge\alpha_2\wedge\alpha_3), \alpha_3\wedge\alpha_4, -(\alpha_2\wedge\alpha_4), \alpha_2\wedge\alpha_3, \alpha_1\wedge\alpha_4, -(\alpha_1\wedge\alpha_3), \alpha_1\wedge\alpha_2, \alpha_4, -\alpha_3, \alpha_2, -\alpha_1, 1\}$

◆ 3. Take the reverse of the cobasis

```
R = GrassmannReverse[A]
```

$\{\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1, \alpha_4\wedge\alpha_3\wedge\alpha_2, -(\alpha_4\wedge\alpha_3\wedge\alpha_1), \alpha_4\wedge\alpha_2\wedge\alpha_1, -(\alpha_3\wedge\alpha_2\wedge\alpha_1), \alpha_4\wedge\alpha_3, -(\alpha_4\wedge\alpha_2), \alpha_3\wedge\alpha_2, \alpha_4\wedge\alpha_1, -(\alpha_3\wedge\alpha_1), \alpha_2\wedge\alpha_1, \alpha_4, -\alpha_3, \alpha_2, -\alpha_1, 1\}$

◆ 4. Take the interior product of each of these elements with $\underset{k}{\beta}$

```
S = Thread[β⊖R]
          k
```

$\{\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1, \underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2, \underset{k}{\beta}\ominus-(\alpha_4\wedge\alpha_3\wedge\alpha_1), \underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\wedge\alpha_1, \underset{k}{\beta}\ominus-(\alpha_3\wedge\alpha_2\wedge\alpha_1), \underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3, \underset{k}{\beta}\ominus-(\alpha_4\wedge\alpha_2), \underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2, \underset{k}{\beta}\ominus\alpha_4\wedge\alpha_1, \underset{k}{\beta}\ominus-(\alpha_3\wedge\alpha_1), \underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1, \underset{k}{\beta}\ominus\alpha_4, \underset{k}{\beta}\ominus-\alpha_3, \underset{k}{\beta}\ominus\alpha_2, \underset{k}{\beta}\ominus-\alpha_1, \underset{k}{\beta}\ominus 1\}$

◆ 5. Take the exterior product of the elements of the two lists B and S

```
T = Thread[B ∧ S]
```

$\{1\wedge(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1), \alpha_1\wedge(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2), \alpha_2\wedge(\underset{k}{\beta}\ominus-(\alpha_4\wedge\alpha_3\wedge\alpha_1)), \alpha_3\wedge(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\wedge\alpha_1), \alpha_4\wedge(\underset{k}{\beta}\ominus-(\alpha_3\wedge\alpha_2\wedge\alpha_1)), \alpha_1\wedge\alpha_2\wedge(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3), \alpha_1\wedge\alpha_3\wedge(\underset{k}{\beta}\ominus-(\alpha_4\wedge\alpha_2)), \alpha_1\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2), \alpha_2\wedge\alpha_3\wedge(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_1), \alpha_2\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus-(\alpha_3\wedge\alpha_1)), \alpha_3\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1), \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge(\underset{k}{\beta}\ominus\alpha_4), \alpha_1\wedge\alpha_2\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus-\alpha_3), \alpha_1\wedge\alpha_3\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus\alpha_2), \alpha_2\wedge\alpha_3\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus-\alpha_1), \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge(\underset{k}{\beta}\ominus 1)\}$

◆ 6. Add the terms and simplify the result by factoring out the scalars

```
U = FactorScalars[Plus @@ T]
```

$$\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\wedge\alpha_1+\alpha_1\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_2\right)-\alpha_2\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\wedge\alpha_1\right)+\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\wedge\alpha_1\right)-\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2\wedge\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_3\right)-\alpha_1\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_2\right)+\alpha_1\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_2\right)+\alpha_2\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\wedge\alpha_1\right)-\alpha_2\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\wedge\alpha_1\right)+\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\left(\underset{k}{\beta}\ominus\alpha_4\right)-\alpha_1\wedge\alpha_2\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_3\right)+\alpha_1\wedge\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_2\right)-\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\left(\underset{k}{\beta}\ominus\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\wedge\underset{k}{\beta}$$

◆ 7. Compare to the original expression

```
X == U
```

```
True
```

12.7 The Clifford Product of Intersecting Elements

General formulae for intersecting elements

Suppose two simple elements $\underset{p}{\gamma}\wedge\underset{m}{\alpha}$ and $\underset{p}{\gamma}\wedge\underset{k}{\beta}$ which have a simple element $\underset{p}{\gamma}$ in common. Then by definition their Clifford product may be written as a sum of generalized products.

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p}(-1)^{\lambda(p+m-\lambda)+\frac{1}{2}\lambda(\lambda-1)}\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\Delta}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)$$

But it has been shown in Section 10.12 that for $\lambda \geq p$ that:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\Delta}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{p(\lambda-p)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{-p+\lambda}{\Delta}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

Substituting in the formula above gives:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p}(-1)^{p+\lambda(m-\lambda)+\frac{1}{2}\lambda(\lambda-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda-p}{\Delta}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

Since the terms on the right hand side are zero for $\lambda < p$, we can define $\mu = \lambda - p$ and rewrite the right hand side as:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\omega}\sum_{\mu=0}^{\mathrm{Min}[m,k]}(-1)^{\mu(p+m-\mu)+\frac{1}{2}\mu(\mu-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\mu}{\Delta}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

where $\omega = m\,p + \frac{1}{2}\,p\,(p-1)$. Hence we can finally cast the right hand side as a Clifford product by taking out the second $\underset{p}{\gamma}$ factor.

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p+\frac{1}{2}p\,(p-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.17$$

In a similar fashion we can show that the right hand side can be written in the alternative form:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\frac{1}{2}p\,(p-1)}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.18$$

By reversing the order of $\underset{m}{\alpha}$ and $\underset{p}{\gamma}$ in the first factor, formula 12.17 can be written as:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\frac{1}{2}p\,(p-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.19$$

And by noting that the reverse, $\underset{p}{\gamma}^{\dagger}$ of $\underset{p}{\gamma}$ is $(-1)^{\frac{1}{2}p\,(p-1)}\underset{p}{\gamma}$, we can absorb the sign into the formulae by changing one of the $\underset{p}{\gamma}$ to $\underset{p}{\gamma}^{\dagger}$. For example:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.20$$

In sum: we can write:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.21$$

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.22$$

$$\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} == \left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.23$$

The relations derived up to this point are completely general and apply to Clifford products in an arbitrary space with an arbitrary metric. For example they do not require any of the elements to be orthogonal.

Special cases of intersecting elements

We can derive special cases of the formulae derived above by putting $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ equal to unity. First, put $\underset{k}{\beta}$ equal to unity. Remembering that the Clifford product with a scalar reduces to the ordinary product we obtain:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\underset{p}{\gamma} == (-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma} == (-1)^{m\,p}\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}$$

Next, put $\underset{m}{\alpha}$ equal to unity, and then, to enable comparison with the previous formulae, replace $\underset{k}{\beta}$ by $\underset{m}{\alpha}$.

$$\underset{p}{\gamma}^{\dagger}\diamond\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right) == \left(\underset{p}{\gamma}\diamond\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma} == \left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma}$$

Finally we note that, since the far right hand sides of these two equations are equal, all the expressions are equal.

$$\begin{gathered}\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger}\diamond\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\\ == \left(\underset{p}{\gamma}\diamond\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma} == \left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma} == (-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}\end{gathered} \qquad 12.24$$

By putting $\underset{m}{\alpha}$ to unity in these equations we can recover the relation 12.11 between the Clifford product of identical elements and their interior product.

12.8 The Clifford Product of Orthogonal Elements

The Clifford product of totally orthogonal elements

In Chapter 10 on generalized products we showed that if $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal (that is, $\alpha_i \ominus \beta_j == 0$ for each α_i belonging to $\underset{m}{\alpha}$ and β_j belonging to $\underset{k}{\beta}$, then $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == 0$, *except* when λ = 0.

Thus we see immediately from the definition of the Clifford product that the Clifford product of two totally orthogonal elements is equal to their exterior product.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.25$$

The Clifford product of partially orthogonal elements

Suppose now we introduce an arbitrary element $\underset{p}{\gamma}$ into $\underset{m}{\alpha} \diamond \underset{k}{\beta}$, and expand the expression in terms of generalized products.

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k+p]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \underset{m}{\alpha} \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)$$

But from formula 10.35 we have that:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad \lambda \le \mathrm{Min}[m, p]$$

Hence:

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.26$$

Similarly, expressing $\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta}$ in terms of generalized products and substituting from equation 10.37 gives:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[p,k]} (-1)^{\lambda (m+p-\lambda)+\frac{1}{2}\lambda(\lambda-1)+m\lambda} \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

But $(-1)^{\lambda(m+p-\lambda)+\frac{1}{2}\lambda(\lambda-1)+m\lambda} == (-1)^{\lambda(p-\lambda)+\frac{1}{2}\lambda(\lambda-1)}$, hence we can write:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta} == \underset{m}{\alpha}\wedge\left(\underset{p}{\gamma}\diamond\underset{k}{\beta}\right) \qquad \alpha_i\ominus\beta_j == 0 \qquad 12.27$$

Testing the formulae

To test any of these formulae we may always tabulate specific cases. Here we convert the difference between the sides of equation 12.27 to scalar products and then put to zero any products whose factors are orthogonal. To do this we use the *GrassmannAlgebra* function `OrthogonalSimplificationRules`. We verify the formula for the first 50 cases.

```
? OrthogonalSimplificationRules
```

```
OrthogonalSimplificationRules[{{X1,Y1},{X2,Y2},⋯}] develops a list of
  rules which put to zero all the scalar products of a 1-element from
  Xi and a 1-element from Yi. Xi and Yi may be either variables,
  basis elements, exterior products of these, or graded variables.
```

$$\text{Flatten}\left[\text{Table}\left[\text{ToScalarProducts}\left[\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}-\underset{m}{\alpha}\wedge\left(\underset{p}{\gamma}\diamond\underset{k}{\beta}\right)\right]\ /.\ \text{OrthogonalSimplificationRules}\left[\left\{\left\{\underset{m}{\alpha},\underset{k}{\beta}\right\}\right\}\right],\ \{m,0,3\},\{k,0,m\},\{p,0,m+2\}\right]\right]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

12.9 The Clifford Product of Intersecting Orthogonal Elements

Orthogonal union

We have shown in Section 12.7 above that the Clifford product of arbitrary intersecting elements may be expressed by:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{mp}\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{mp}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}$$

But we have also shown in Section 12.8 that if $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal, then:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)$$

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}$$

Hence the right hand sides of the first equations may be written, in the orthogonal case, as:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad \alpha_{i} \ominus \beta_{j} == 0 \qquad 12.28$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad \alpha_{i} \ominus \beta_{j} == 0 \qquad 12.29$$

Orthogonal intersection

Consider the case of three simple elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$ where $\underset{p}{\gamma}$ is *totally orthogonal* to both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ (and hence to $\underset{m}{\alpha} \wedge \underset{k}{\beta}$). A simple element $\underset{p}{\gamma}$ is totally orthogonal to an element $\underset{m}{\alpha}$ if and only if $\underset{m}{\alpha} \ominus \gamma_{i} == 0$ for all γ_{i} belonging to $\underset{p}{\gamma}$. Here, we do *not* assume that $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are orthogonal.

As before we consider the Clifford product of intersecting elements, but because of the orthogonality relationships, we can replace the generalized products on the right hand side by the right hand side of equation 10.39.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p} (-1)^{\lambda\,(p+m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)} \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)$$

$$== \sum_{\lambda=p}^{\mathrm{Min}[m,k]+p} (-1)^{\lambda\,(p+m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \underset{\lambda-p}{\triangle} \underset{k}{\beta}\right)$$

Let $\mu = \lambda - p$, then this equals:

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \sum_{\mu=0}^{\mathrm{Min}[m,k]} (-1)^{(\mu+p)\,(m-\mu)+\frac{1}{2}(\mu+p)\,(\mu+p-1)} \left(\underset{m}{\alpha} \underset{\mu}{\triangle} \underset{k}{\beta}\right)$$

$$== (-1)^{m\,p+\frac{1}{2}\,p\,(p-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \sum_{\mu=0}^{\mathrm{Min}[m,k]} (-1)^{\mu\,(m-\mu)+\frac{1}{2}\,\mu\,(\mu-1)} \left(\underset{m}{\alpha} \underset{\mu}{\triangle} \underset{k}{\beta}\right)$$

Hence we can cast the right hand side as a Clifford product.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p+\frac{1}{2}\,p\,(p-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)$$

Or finally, by absorbing the signs into the left-hand side as:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) ==$$
$$\left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \qquad \underset{m}{\alpha} \ominus \gamma_i == \underset{k}{\beta} \ominus \gamma_i == 0 \qquad 12.30$$

Note carefully, that for this formula to be true, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are not necessarily orthogonal. However, if they are orthogonal we can express the Clifford product on the right hand side as an exterior product.

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) ==$$
$$\left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0 \qquad 12.31$$

12.10 Summary of Special Cases of Clifford Products

Arbitrary elements

The main results of the preceding sections concerning Clifford Products of intersecting and orthogonal elements are summarized below.

◆ $\underset{p}{\gamma}$ is an arbitrary repeated factor

$$\underset{p}{\gamma}^{\dagger} \diamond \underset{p}{\gamma} == \underset{p}{\gamma} \diamond \underset{p}{\gamma}^{\dagger} == \underset{p}{\gamma} \ominus \underset{p}{\gamma} \qquad 12.32$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma} == \underset{p}{\gamma} \ominus \underset{p}{\gamma}^{\dagger} \qquad 12.33$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} == \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \underset{p}{\gamma} == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}^{\dagger}\right) \underset{p}{\gamma} \qquad 12.34$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right)^{2} \qquad 12.35$$

- **$\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ is arbitrary**

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger} \diamond \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right)$$
$$== \left(\underset{p}{\gamma} \diamond \underset{m}{\alpha}\right) \ominus \underset{p}{\gamma} == \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \ominus \underset{p}{\gamma} == (-1)^{m\,p} \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \ominus \underset{p}{\gamma} \qquad 12.36$$

- **$\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are arbitrary**

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad 12.37$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad 12.38$$

Arbitrary and orthogonal elements

- **$\underset{p}{\gamma}$ is arbitrary, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal**

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.39$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.40$$

◆ $\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.41$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.42$$

◆ $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are arbitrary, $\underset{p}{\gamma}$ is a repeated factor totally orthogonal to both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \qquad \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0 \qquad 12.43$$

Orthogonal elements

◆ $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.44$$

◆ $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{p}{\gamma}$ are totally orthogonal

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0 \tag{12.45}$$

Calculating with Clifford products

The previous section summarizes some alternative formulations for a Clifford product when its factors contain a common element, and when they have an orthogonality relationship between them. In this section we discuss how these relations can make it easy to calculate with these type of Clifford products.

◆ The Clifford product of totally orthogonal elements reduces to the exterior product

If we know that *all* the factors of a Clifford product are totally orthogonal, then we can interchange the Clifford product and the exterior product at will. Hence, *for totally orthogonal elements, the Clifford and exterior products are associative*, and we do not need to include parentheses.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} \diamond \underset{p}{\gamma} == \underset{m}{\alpha} \diamond \underset{k}{\beta} \wedge \underset{p}{\gamma} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \diamond \underset{p}{\gamma} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0 \tag{12.46}$$

Note carefully however, that this associativity does not extend to the *factors* of the *m*-, *k*-, or *p*-elements unless the factors of the *m*-, *k*-, or *p*-element concerned are mutually orthogonal. In which case we could for example then write:

$$\alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_m == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \qquad \alpha_i \ominus \alpha_j == 0 \quad i \neq j \tag{12.47}$$

■ Example

For example if $x \wedge y$ and z are totally orthogonal, that is $x \ominus z == 0$ and $y \ominus z == 0$, then we can write

$$(x \wedge y) \diamond z == x \wedge y \wedge z == x \wedge (y \diamond z) == -y \wedge (x \diamond z)$$

But since $\mathbf{x}$ *is not necessarily orthogonal to* $\mathbf{y}$, these are *not* the same as

$$x \diamond y \diamond z == (x \diamond y) \wedge z == x \diamond (y \wedge z)$$

We can see the difference by reducing the Clifford products to scalar products.

```
ToScalarProducts[{(x ∧ y) ⋄ z, x ∧ y ∧ z, x ∧ (y ⋄ z), -y ∧ (x ⋄ z)}] /.
 OrthogonalSimplificationRules[{{x ∧ y, z}}]
```

```
{x ∧ y ∧ z, x ∧ y ∧ z, x ∧ y ∧ z, x ∧ y ∧ z}
```

```
ToScalarProducts[{x ⋄ y ⋄ z, (x ⋄ y) ∧ z, x ⋄ (y ∧ z)}] /.
 OrthogonalSimplificationRules[{{x ∧ y, z}}]
```

```
{z (x ⊖ y) + x ∧ y ∧ z, z (x ⊖ y) + x ∧ y ∧ z, z (x ⊖ y) + x ∧ y ∧ z}
```

12.11 Associativity of the Clifford Product

Associativity of orthogonal elements

In what follows we use formula 12.45 to show that the Clifford product as defined at the beginning of this chapter is associative for totally orthogonal elements. This result will then enable us to show that by re-expressing the elements in terms in an orthogonal basis, the Clifford product is, *in general*, associative. Note that the approach we have adopted in this book has not required us to adopt *ab initio* the associativity of the Clifford product as an axiom, but rather show that associativity is a natural consequence of its definition.

Formula 12.45 tells us that the Clifford product of (possibly) intersecting and totally orthogonal elements is given by:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right)$$
$$\alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0$$

Note that it is not necessary that the factors of $\underset{m}{\alpha}$ be orthogonal to each other. This is true also for the factors of $\underset{k}{\beta}$ or of $\underset{p}{\gamma}$.

We begin by writing a Clifford product of three elements, and associating the first pair. The elements contain factors which are specific to the element ($\underset{m}{\alpha}$, $\underset{k}{\beta}$, $\underset{r}{\omega}$), pairwise common ($\underset{p}{\gamma}$, $\underset{q}{\delta}$, $\underset{s}{\epsilon}$), and common to all three ($\underset{z}{\theta}$). This product will therefore represent the most general case since other cases may be obtained by letting one or more factors reduce to unity.

$$\left(\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$==\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$==\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)(-1)^{sk}\left(\left(\underset{s}{\epsilon}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\wedge\left(\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$==(-1)^{sk}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\left(\underset{s}{\epsilon}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

On the other hand we can associate the second pair to obtain the same result.

$$\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)\right)$$

$$==\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond(-1)^{zk+z\,(s+r)}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\wedge\left(\underset{s}{\epsilon}\wedge\underset{r}{\omega}\right)$$

$$==(-1)^{zk+z\,(s+r)}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\right)$$

$$==(-1)^{zk+z\,(s+r)+sz+ks}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{z}{\theta}\wedge\underset{s}{\epsilon}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{s}{\epsilon}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\right)$$

$$==(-1)^{zk+zr+ks}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\left(\underset{s}{\epsilon}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{p}{\gamma}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{z}{\theta}\right)\wedge\left(\underset{k}{\beta}\wedge\underset{r}{\omega}\right)$$

$$==(-1)^{sk}\left(\left(\underset{z}{\theta}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\left(\underset{s}{\epsilon}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

In sum: We have shown that the Clifford product of possibly intersecting but otherwise totally orthogonal elements is associative. Any factors which make up a given individual element are not specifically involved, and hence it is of no consequence to the associativity of the element with another whether or not these factors are mutually orthogonal.

A mnemonic formula for products of orthogonal elements

Because $\underset{z}{\theta}$ and $\underset{p}{\gamma}$ are orthogonal to each other, and $\underset{s}{\epsilon}$ and $\underset{q}{\delta}$ are orthogonal to each other we can rewrite the inner products in the alternative forms:

$$\left(\underset{z}{\theta}^{\dagger}\wedge\underset{p}{\gamma}^{\dagger}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)==\left(\underset{z}{\theta}^{\dagger}\ominus\underset{z}{\theta}\right)\left(\underset{p}{\gamma}^{\dagger}\ominus\underset{p}{\gamma}\right)$$

$$\left(\underset{s}{\epsilon}^{\dagger}\wedge\underset{q}{\delta}^{\dagger}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)==\left(\underset{s}{\epsilon}^{\dagger}\ominus\underset{s}{\epsilon}\right)\left(\underset{q}{\delta}^{\dagger}\ominus\underset{q}{\delta}\right)$$

The results of the previous section may then summarized as:

$$\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$
$$=(-1)^{sk}\left(\underset{p}{\gamma}^{\dagger}\ominus\underset{p}{\gamma}\right)\left(\underset{z}{\theta}^{\dagger}\ominus\underset{z}{\theta}\right)$$
$$\left(\underset{s}{\epsilon}^{\dagger}\ominus\underset{s}{\epsilon}\right)\left(\underset{q}{\delta}^{\dagger}\ominus\underset{q}{\delta}\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right) \qquad 12.48$$

A mnemonic for making this transformation is then
1. Rearrange the factors in a Clifford product to get common factors adjacent to the Clifford product symbol, taking care to include any change of sign due to the quasi-commutativity of the exterior product.
2. Replace the common factors by their inner product, *but with one copy being reversed*.
3. If there are no common factors in a Clifford product, the Clifford product can be replaced by the exterior product.

Remember that for these relations to hold all the elements must be totally orthogonal to each other.

Note that if, in addition, the 1-element factors of any of these elements, $\underset{p}{\gamma}$ say, are orthogonal to each other, then:

$$\underset{p}{\gamma}\ominus\underset{p}{\gamma}=(\gamma_1\ominus\gamma_1)\,(\gamma_2\ominus\gamma_2)\cdots(\gamma_p\ominus\gamma_p)$$

Associativity of non-orthogonal elements

Consider the Clifford product $\left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)\diamond\underset{p}{\gamma}$ where there are no restrictions on the factors $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$. It has been shown in Section 6.3 that an arbitrary simple *m*-element may be expressed in terms of *m* orthogonal 1-element factors (the Gram-Schmidt orthogonalization process). Suppose that ν such orthogonal 1-elements $e_1, e_2, \cdots, e_\nu$ have been found in terms of which $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{p}{\gamma}$ can be expressed. Writing the *m*-elements, *k*-elements and *p*-elements formed from the e_i as $\underset{m}{e_j}$, $\underset{k}{e_r}$, and $\underset{p}{e_s}$ respectively, we can write:

$$\underset{m}{\alpha}=\Sigma\, a^j\,\underset{m}{e_j} \qquad \underset{k}{\beta}=\Sigma\, b^r\,\underset{k}{e_r} \qquad \underset{p}{\gamma}=\Sigma\, c^s\,\underset{p}{e_s}$$

Thus we can write the Clifford product as:

$$\left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)\diamond\underset{p}{\gamma}=\left(\Sigma\, a^j\,\underset{m}{e_j}\diamond\Sigma\, b^r\,\underset{k}{e_r}\right)\diamond\Sigma\, c^s\,\underset{p}{e_s}=\Sigma\Sigma\Sigma\, a^j\, b^r\, c^s\left(\underset{m}{e_j}\diamond\underset{k}{e_r}\right)\diamond\underset{p}{e_s}$$

But we have already shown in the previous section that the Clifford product of orthogonal elements is associative. That is:

$$\left(\underset{m}{e_j} \diamond \underset{k}{e_r}\right) \diamond \underset{p}{e_s} == \underset{m}{e_j} \diamond \left(\underset{k}{e_r} \diamond \underset{p}{e_s}\right) == \underset{m}{e_j} \diamond \underset{k}{e_r} \diamond \underset{p}{e_s}$$

Hence we can write:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \diamond \underset{p}{\gamma} == \Sigma\,\Sigma\,\Sigma\, a^j\, b^r\, c^s\, \underset{m}{e_j} \diamond \left(\underset{k}{e_r} \diamond \underset{p}{e_s}\right) == \underset{m}{\alpha} \diamond \left(\underset{k}{\beta} \diamond \underset{p}{\gamma}\right)$$

We thus see that *the Clifford product of general elements is associative*.

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \diamond \underset{p}{\gamma} == \underset{m}{\alpha} \diamond \left(\underset{k}{\beta} \diamond \underset{p}{\gamma}\right) == \underset{m}{\alpha} \diamond \underset{k}{\beta} \diamond \underset{p}{\gamma} \qquad 12.49$$

The associativity of the Clifford product is usually taken as an axiom. However, in this book we have chosen to show that associativity is a consequence of our definition of the Clifford product in terms of exterior and interior products. In this way we can ensure that the Grassmann and Clifford algebras are consistent.

Testing the general associativity of the Clifford product

We can easily create a function in *Mathematica* to test the associativity of Clifford products of elements of different grades in spaces of different dimensions. Below we define a function called `CliffordAssociativityTest` which takes the dimension of the space and the grades of three elements as arguments. The steps are as follows:

- Declare a space of the given dimension. It does not matter what the metric is since we do not use it.
- Create general elements of the given grades in a space of the given dimension.
- To make the code more readable, define a function which converts a product to scalar product form.
- Compute the products associated in the two different ways, and subtract them.
- The associativity test is successful if a result of 0 is returned.

```
CliffordAssociativityTest[n_][m_, k_, p_] :=
 Module[{X, Y, Z, S}, 𝕍n; X = CreateElement[ξ_m];
  Y = CreateElement[ψ_k]; Z = CreateElement[ζ_p];
  S[x_] := ToScalarProducts[x]; S[S[X ⋄ Y] ⋄ Z] - S[X ⋄ S[Y ⋄ Z]]]
```

We can either test individual cases, for example 2-elements in a 4-space:

```
CliffordAssociativityTest[4][2, 2, 2]
```

```
0
```

Or, we can tabulate a number of results together. For example, elements of all grades in all spaces of dimension 0, 1, 2, and 3.

```
Table[CliffordAssociativityTest[n][m, k, p],
 {n, 0, 3}, {m, 0, n}, {k, 0, n}, {p, 0, m}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

12.12 Decomposing a Clifford Product

Even and odd generalized products

A Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ can be decomposed into two sums, those whose terms are generalized products of even order $\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e$, and those whose terms are generalized products of odd order $\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o$. That is:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e + \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.50$$

where, from the definition of the Clifford product:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2} \lambda (\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

we can take just the even generalized products to get:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e == \sum_{\lambda = 0,2,4,\cdots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.51$$

(Here, the limit of the sum is understood to mean the greatest even integer less than or equal to **Min[m,k]**.)

The odd generalized products are:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o == (-1)^m \sum_{\lambda = 1,3,5,\cdots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.52$$

(In this case, the limit of the sum is understood to mean the greatest odd integer less than or equal to **Min[m,k]**.)

Note carefully that the evenness and oddness to which we refer is to the *order of the generalized product* not to the grade of the Clifford product.

The Clifford product in reverse order

The expression for the even and odd components of the Clifford product taken in the reverse order is simply obtained from the formulae above by using the quasi-commutativity of the generalized product.

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_e == \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{k}{\beta} \underset{\lambda}{\triangle} \underset{m}{\alpha}\right) == \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}+(m-\lambda)(k-\lambda)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

But, since λ is even, this simplifies to:

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_e == (-1)^{m\,k} \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.53$$

Hence, the even terms of both products are equal, except when both factors of the product are of odd grade.

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_e == (-1)^{m\,k} \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e \qquad 12.54$$

In a like manner we can show that for λ odd:

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_o == (-1)^{k} \sum_{\lambda=1,3,5,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}+(m-\lambda)(k-\lambda)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_o == -(-1)^{m\,k} (-1)^{m} \sum_{\lambda=1,3,5,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.55$$

$$\left(\underset{k}{\beta} \diamond \underset{m}{\alpha}\right)_o == -(-1)^{m\,k} \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.56$$

The decomposition of a Clifford product

Finally, therefore, we can write:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e + \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.57$$

$$(-1)^{m k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e - \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.58$$

which we can add and subtract to give:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{k}{\beta} + (-1)^{m k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha}\right) \qquad 12.59$$

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{k}{\beta} - (-1)^{m k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha}\right) \qquad 12.60$$

Example: An *m*-element and a 1-element

Putting *k* equal to 1 gives:

$$\underset{m}{\alpha} \wedge \beta == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \beta + (-1)^{m}\, \beta \diamond \underset{m}{\alpha}\right) \qquad 12.61$$

$$\frac{1}{2}\left(\underset{m}{\alpha} \diamond \beta - (-1)^{m}\, \beta \diamond \underset{m}{\alpha}\right) == -(-1)^{m}\, \underset{m}{\alpha} \ominus \beta \qquad 12.62$$

Example: An *m*-element and a 2-element

Putting *k* equal to 2 gives:

$$\frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{2}{\beta} + \underset{2}{\beta} \diamond \underset{m}{\alpha}\right) == \underset{m}{\alpha} \wedge \underset{2}{\beta} - \underset{m}{\alpha} \ominus \underset{2}{\beta} \qquad 12.63$$

$$\frac{1}{2}\left(\underset{m}{\alpha}\diamond\underset{2}{\beta}-\underset{2}{\beta}\diamond\underset{m}{\alpha}\right)==-(-1)^{m}\,\underset{m}{\alpha}\underset{1}{\triangle}\underset{2}{\beta} \qquad 12.64$$

12.13 Clifford Algebra

Generating Clifford algebras

Up to this point we have concentrated on the definition and properties of the Clifford product. We are now able to turn our attention to the algebras that such a product is capable of generating in different spaces.

Broadly speaking, an algebra can be constructed from a set of elements of a linear space which has a product operation, *provided* all products of the elements are again elements of the set.

In what follows we shall discuss Clifford algebras. The generating elements will be selected subsets of the basis elements of the full Grassmann algebra on the space. The product operation will be the Clifford product which we have defined in the first part of this chapter in terms of the generalized Grassmann product. Thus, Clifford algebras may viewed as living very much within the Grassmann algebra, relying on it for both its elements and its operations.

In this development therefore, a particular Clifford algebra is ultimately defined by the values of the scalar products of basis vectors of the underlying Grassmann algebra, and thus by the metric on the space.

In many cases we will be defining the specific Clifford algebras in terms of orthogonal (not necessarily orthonormal) basis elements.

Clifford algebras include the real numbers, complex numbers, quaternions, biquaternions, and the Pauli and Dirac algebras.

Real algebra

All the products that we have developed up to this stage, the exterior, interior, generalized Grassmann and Clifford products possess the valuable and consistent property that when applied to scalars yield results equivalent to the usual (underlying field) product. The field has been identified with a space of zero dimensions, and the scalars identified with its elements.

Thus, if **a** and **b** are scalars, then:

$$a \diamond b == a \wedge b == a \ominus b == a \underset{0}{\triangle} b == a\, b \qquad 12.65$$

Thus the real algebra is isomorphic with the Clifford algebra of a space of zero dimensions.

Clifford algebras of a 1-space

We begin our discussion of Clifford algebras with the simplest case: the Clifford algebras of 1-space. Suppose the basis for the 1-space is e_1, then the basis for the associated Grassmann algebra is $\{1, e_1\}$.

```
𝕍1; Basis∧[]
```

```
{1, e1}
```

There are only four possible Clifford products of these basis elements. We can construct a table of these products by using the *GrassmannAlgebra* function **CliffordProductTable**.

```
CliffordProductTable[]
```

```
{{1 ⋄ 1, 1 ⋄ e1}, {e1 ⋄ 1, e1 ⋄ e1}}
```

Usually however, to make the products easier to read and use, we will display them in the form of a palette using the *GrassmannAlgebra* function **PaletteForm**. (We can click on the palette to enter any of its expressions into the notebook).

```
C1 = CliffordProductTable[]; PaletteForm[C1]
```

$1 \diamond 1$	$1 \diamond e_1$
$e_1 \diamond 1$	$e_1 \diamond e_1$

In the general case any Clifford product may be expressed in terms of exterior and interior products. We can see this by applying **ToInteriorProducts** to the table (although only interior (here scalar) products result from this simple case),

```
C2 = ToInteriorProducts[C1]; PaletteForm[C2]
```

1	e_1
e_1	$e_1 \ominus e_1$

Different Clifford algebras may be generated depending on the metric chosen for the space. In this example we can see that the types of Clifford algebra which we can generate in a 1-space are dependent only on the choice of a single scalar value for the scalar product $e_1 \ominus e_1$. The Clifford product of two general elements of the algebra is

```
(a + b e1) ⋄ (c + d e1) // ToScalarProducts
```

$a\,c + b\,d\,(e_1 \ominus e_1) + b\,c\,e_1 + a\,d\,e_1$

It is clear to see that if we choose $e_1 \ominus e_1 == -1$, we have an algebra isomorphic to the complex algebra. The basis 1-element e_1 then plays the role of the imaginary unit *i*. We can generate this particular algebra immediately by declaring the metric, and then generating the product table.

```
DeclareBasis[{i}]; DeclareMetric[{{-1}}];
CliffordProductTable[] // ToScalarProducts // ToMetricForm //
 PaletteForm
```

1	i
i	-1

However, our main purpose in discussing this very simple example in so much detail is to emphasize that even in this case, there are an *infinite* number of Clifford algebras on a 1-space depending on the choice of the scalar value for $e_1 \ominus e_1$. The complex algebra, although it has surely proven itself to be the most useful, is just one among many.

Finally, we note that all Clifford algebras possess the real algebra as their simplest even subalgebra.

12.14 Clifford Algebras of a 2-Space

The Clifford product table in 2-space

In this section we explore the Clifford algebras of 2-space. As might be expected, the Clifford algebras of 2-space are significantly richer than those of 1-space. First we declare a (not necessarily orthogonal) basis for the 2-space, and generate the associated Clifford product table.

```
V2; C1 = CliffordProductTable[]; PaletteForm[C1]
```

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond (e_1 \wedge e_2)$
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond (e_1 \wedge e_2)$
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond (e_1 \wedge e_2)$
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e_2) \diamond (e_1 \wedge e_2)$

To see the way in which these Clifford products reduce to generalized Grassmann products we can apply the *GrassmannAlgebra* function `ToGeneralizedProducts`.

```
C2 = ToGeneralizedProducts[C1]; PaletteForm[C2]
```

$1 \underset{0}{\triangle} 1$	$1 \underset{0}{\triangle} e_1$	$1 \underset{0}{\triangle} e_2$	
$e_1 \underset{0}{\triangle} 1$	$e_1 \underset{0}{\triangle} e_1 + e_1 \underset{1}{\triangle} e_1$	$e_1 \underset{0}{\triangle} e_2 + e_1 \underset{1}{\triangle} e_2$	
$e_2 \underset{0}{\triangle} 1$	$e_2 \underset{0}{\triangle} e_1 + e_2 \underset{1}{\triangle} e_1$	$e_2 \underset{0}{\triangle} e_2 + e_2 \underset{1}{\triangle} e_2$	
$(e_1 \wedge e_2) \underset{0}{\triangle} 1$	$(e_1 \wedge e_2) \underset{0}{\triangle} e_1 - (e_1 \wedge e_2) \underset{1}{\triangle} e_1$	$(e_1 \wedge e_2) \underset{0}{\triangle} e_2 - (e_1 \wedge e_2) \underset{1}{\triangle} e_2$	$(e_1$

The next level of expression is obtained by applying `ToInteriorProducts` to reduce the generalized products to exterior and interior products.

```
C3 = ToInteriorProducts[C2]; PaletteForm[C3]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	$e_1 \ominus e_1$	$e_1 \ominus e_2 + e_1 \wedge e_2$	$e_1 \wedge e_2 \ominus e_1$
e_2	$e_1 \ominus e_2 - e_1 \wedge e_2$	$e_2 \ominus e_2$	$e_1 \wedge e_2 \ominus e_2$
$e_1 \wedge e_2$	$-(e_1 \wedge e_2 \ominus e_1)$	$-(e_1 \wedge e_2 \ominus e_2)$	$-(e_1 \wedge e_2 \ominus e_1 \wedge e_2) - (e_1 \wedge e_2 \ominus e_1) \wedge e_2$

Finally, we can expand the interior products to scalar products.

```
C4 = ToScalarProducts[C3]; PaletteForm[C4]
```

1	e_1	e_2	$e_1 \wedge e$
e_1	$e_1 \ominus e_1$	$e_1 \ominus e_2 + e_1 \wedge e_2$	$-(e_1 \ominus e_2)\, e_1 +$
e_2	$e_1 \ominus e_2 - e_1 \wedge e_2$	$e_2 \ominus e_2$	$-(e_2 \ominus e_2)\, e_1 +$
$e_1 \wedge e_2$	$(e_1 \ominus e_2)\, e_1 - (e_1 \ominus e_1)\, e_2$	$(e_2 \ominus e_2)\, e_1 - (e_1 \ominus e_2)\, e_2$	$(e_1 \ominus e_2)^2 - (e_1 \ominus$

It should be noted that the *GrassmannAlgebra* operation `ToScalarProducts` also applies the function `ToStandardOrdering`. Hence scalar products are reordered to present the basis element with lower index first. For example the scalar product $\mathbf{e_2 \ominus e_1}$ does not appear in the table above.

Product tables in a 2-space with an orthogonal basis

This is the Clifford product table for a general basis. If however,we choose an orthogonal basis in which $\mathbf{e_1 \ominus e_2}$ is zero, the table simplifies to:

```
C5 = C4 /. e1 ⊖ e2 → 0; PaletteForm[C5]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	$e_1 \ominus e_1$	$e_1 \wedge e_2$	$(e_1 \ominus e_1)\, e_2$
e_2	$-(e_1 \wedge e_2)$	$e_2 \ominus e_2$	$-(e_2 \ominus e_2)\, e_1$
$e_1 \wedge e_2$	$-(e_1 \ominus e_1)\, e_2$	$(e_2 \ominus e_2)\, e_1$	$-(e_1 \ominus e_1)\,(e_2 \ominus e_2)$

This table defines *all* the Clifford algebras on the 2-space. Different Clifford algebras may be generated by choosing different metrics for the space, that is, by choosing the two scalar values

$\mathbf{e_1} \ominus \mathbf{e_1}$ and $\mathbf{e_2} \ominus \mathbf{e_2}$.
Note however that allocation of scalar values **a** to $\mathbf{e_1} \ominus \mathbf{e_1}$ and **b** to $\mathbf{e_2} \ominus \mathbf{e_2}$ would lead to essentially the same structure as allocating **b** to $\mathbf{e_1} \ominus \mathbf{e_1}$ and **a** to $\mathbf{e_2} \ominus \mathbf{e_2}$.

In the rest of what follows however, we will restrict ourselves to metrics in which $\mathbf{e_1} \ominus \mathbf{e_1} == \pm 1$ and $\mathbf{e_2} \ominus \mathbf{e_2} == \pm 1$, whence there are three cases of interest.

$$\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to +1\}$$
$$\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$$
$$\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1\}$$

As observed previously the case $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to +1\}$ is isomorphic to the case $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$, so we do not need to consider it.

Case 1: $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to +1\}$

This is the standard case of a 2-space with an orthonormal basis. Making the replacements in the table gives:

```
C6 = C5 /. {e1 ⊖ e1 → +1, e2 ⊖ e2 → +1}; PaletteForm[C6]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	1	$-e_1$
$e_1 \wedge e_2$	$-e_2$	e_1	-1

Inspecting this table for interesting structures or substructures, we note first that the even subalgebra (that is, the algebra based on products of the even basis elements) is isomorphic to the complex algebra. For our own explorations we can use the palette to construct a product table for the subalgebra, or we can create a table using the *GrassmannAlgebra* function `TableTemplate`, and edit it by deleting the middle rows and columns.

```
TableTemplate[C6]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	1	$-e_1$
$e_1 \wedge e_2$	$-e_2$	e_1	-1

1	$e_1 \wedge e_2$
$e_1 \wedge e_2$	-1

If we want to create a palette for the subalgebra, we have to edit the normal list (matrix) form and then apply `PaletteForm`. For even subalgebras we can also apply the *GrassmannAlgebra* function `EvenCliffordProductTable` which creates a Clifford product table from just the basis elements of even grade. We then set the metric we want, convert the Clifford product to scalar products, evaluate the scalar products according to the chosen metric, and display the resulting table as a palette.

```
C7 = EvenCliffordProductTable[]; DeclareMetric[{1, 0}, {0, 1}];
PaletteForm[ToMetricForm[ToScalarProducts[C7]]]
```

1	$e_1 \wedge e_2$
$e_1 \wedge e_2$	-1

Here we see that the basis element $e_1 \wedge e_2$ has the property that its (Clifford) square is **-1**. We can see how this arises by carrying out the elementary operations on the product. Note that $e_1 \ominus e_1 == e_2 \ominus e_2 == 1$ since we have assumed the 2-space under consideration is Euclidean.

$$(e_1 \wedge e_2) \diamond (e_1 \wedge e_2) == -(e_2 \wedge e_1) \diamond (e_1 \wedge e_2) == -(e_1 \ominus e_1)\ (e_2 \ominus e_2) == -1$$

Thus the pair $\{1, e_1 \wedge e_2\}$ generates an algebra under the Clifford product operation, isomorphic to the complex algebra. It is also the basis of the even Grassmann algebra of $\underset{2}{\Lambda}$.

Case 2: $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$

Here is an example of a Clifford algebra which does not have any popular applications of which the author is aware.

```
C7 = C5 /. {e1 ⊖ e1 → +1, e2 ⊖ e2 → -1}; PaletteForm[C7]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	-1	e_1
$e_1 \wedge e_2$	$-e_2$	$-e_1$	1

Case 3: $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1\}$

Our third case in which the metric is **{{-1,0},{0,-1}}**, is isomorphic to the quaternions.

```
C8 = C5 /. {e1 ⊖ e1 → -1, e2 ⊖ e2 → -1}; PaletteForm[C8]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	-1	$e_1 \wedge e_2$	$-e_2$
e_2	$-(e_1 \wedge e_2)$	-1	e_1
$e_1 \wedge e_2$	e_2	$-e_1$	-1

We can see this isomorphism more clearly by substituting the usual quaternion symbols (here we *Mathematica*'s double-struck symbols and choose the correspondence $\{e_1 \to \mathbb{i},\ e_2 \to \mathbb{j},\ e_1 \wedge e_2 \to \mathbb{k}\}$.

```
C9 = C8 /. {e1 → 𝕚, e2 → 𝕛, e1 ∧ e2 → 𝕜}; PaletteForm[C9]
```

1	𝕚	𝕛	𝕜
𝕚	-1	𝕜	-𝕛
𝕛	-𝕜	-1	𝕚
𝕜	𝕛	-𝕚	-1

Having verified that the structure is indeed quaternionic, let us return to the original specification in terms of the basis of the Grassmann algebra. A quaternion can be written in terms of these basis elements as:

$$Q == a + b\, e_1 + c\, e_2 + d\, e_1 \wedge e_2 == (a + b\, e_1) + (c + d\, e_1) \wedge e_2$$

Now, because e_1 and e_2 are orthogonal, $e_1 \wedge e_2$ is equal to $e_1 \diamond e_2$. But for any further calculations we will need to use the Clifford product form. Hence we write

$$Q == a + b\, e_1 + c\, e_2 + d\, e_1 \diamond e_2 == (a + b\, e_1) + (c + d\, e_1) \diamond e_2 \qquad 12.66$$

Hence under one interpretation, each of e_1 and e_2 *and* their Clifford product $e_1 \diamond e_2$ behaves as a different imaginary unit. Under the second interpretation, a quaternion is a complex number with imaginary unit e_2, whose components are complex numbers based on e_1 as the imaginary unit.

12.15 Clifford Algebras of a 3-Space

The Clifford product table in 3-space

In this section we explore the Clifford algebras of 3-space. First we declare a (not necessarily orthogonal) basis for the 3-space, and generate the associated Clifford product table. Because of the size of the table, only the first few columns are shown in the print version.

```
𝕍3; C1 = CliffordProductTable[]; PaletteForm[C1]
```

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond e_3$	
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond e_3$	
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond e_3$	
$e_3 \diamond 1$	$e_3 \diamond e_1$	$e_3 \diamond e_2$	$e_3 \diamond e_3$	
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e_2) \diamond e_3$	$(e_1$
$(e_1 \wedge e_3) \diamond 1$	$(e_1 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_3) \diamond e_3$	$(e_1$
$(e_2 \wedge e_3) \diamond 1$	$(e_2 \wedge e_3) \diamond e_1$	$(e_2 \wedge e_3) \diamond e_2$	$(e_2 \wedge e_3) \diamond e_3$	$(e_2$
$(e_1 \wedge e_2 \wedge e_3) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_3$	$(e_1 \wedge$

✡ $\mathcal{C}\ell_3$: The Pauli algebra

Our first exploration is to Clifford algebras in Euclidean space, hence we accept the default metric which was automatically declared when we declared the basis.

```
MatrixForm[Metric]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the basis elements are orthogonal, we can use the *GrassmannAlgebra* function `CliffordToOrthogonalScalarProducts` for computing the Clifford products. This function is faster for larger calculations than the more general `ToScalarProducts` used in the previous examples.

```
C2 = ToMetricForm[CliffordToOrthogonalScalarProducts[C1]];
PaletteForm[C2]
```

1	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge$
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_2	e
e_2	$-(e_1 \wedge e_2)$	1	$e_2 \wedge e_3$	$-e_1$	$-(e_1 \wedge$ e
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	1	$e_1 \wedge e_2 \wedge e_3$	−e
$e_1 \wedge e_2$	$-e_2$	e_1	$e_1 \wedge e_2 \wedge e_3$	-1	$-(e_2$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	e_1	$e_2 \wedge e_3$	−
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	$-e_3$	e_2	$-(e_1 \wedge e_3)$	$e_1 \wedge$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	$-e_3$	e

We can show that this Clifford algebra of Euclidean 3-space is isomorphic to the Pauli algebra. Pauli's representation of the algebra was by means of 2×2 matrices over the complex field:

$$\sigma_1 == \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 == \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 == \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The isomorphism is direct and straightforward:

$$\{I \leftrightarrow 1,\ \sigma_1 \leftrightarrow e_1,\ \sigma_2 \leftrightarrow e_2,\ \sigma_3 \leftrightarrow e_3,\ \sigma_1\,\sigma_2 \leftrightarrow e_1 \wedge e_2,$$
$$\sigma_1\,\sigma_3 \leftrightarrow e_1 \wedge e_3,\ \sigma_2\,\sigma_3 \leftrightarrow e_2 \wedge e_3,\ \sigma_1\,\sigma_2\,\sigma_3 \leftrightarrow e_1 \wedge e_2 \wedge e_3\}$$

To show this we can:

1) Replace the symbols for the basis elements of the Grassmann algebra in the table above by symbols for the Pauli matrices. Replace the exterior product operation by the matrix product operation.

2) Replace the symbols by the actual matrices, allowing *Mathematica* to perform the matrix products.

3) Replace the resulting Pauli matrices with the corresponding basis elements of the Grassmann

algebra.
4) Verify that the resulting table is the same as the original table.

We perform these steps in sequence. Because of the size of the output, only the electronic version will show the complete tables.

Step 1: Replace symbols for entities and operations

```
C3 = (C2 // ReplaceNegativeUnit) /. {1 → I, e1 → σ1, e2 → σ2, e3 → σ3} /.
  Wedge → Dot; PaletteForm[C3]
```

$\mathbb{I}$	σ_1	σ_2	σ_3	$\sigma_1 . \sigma_2$	$\sigma_1 . \sigma_3$	$\sigma_2 .$
σ_1	$\mathbb{I}$	$\sigma_1 . \sigma_2$	$\sigma_1 . \sigma_3$	σ_2	σ_3	$\sigma_1 . \sigma$
σ_2	$-\sigma_1 . \sigma_2$	$\mathbb{I}$	$\sigma_2 . \sigma_3$	$-\sigma_1$	$-\sigma_1 . \sigma_2 . \sigma_3$	σ
σ_3	$-\sigma_1 . \sigma_3$	$-\sigma_2 . \sigma_3$	$\mathbb{I}$	$\sigma_1 . \sigma_2 . \sigma_3$	$-\sigma_1$	$-$
$\sigma_1 . \sigma_2$	$-\sigma_2$	σ_1	$\sigma_1 . \sigma_2 . \sigma_3$	$-\mathbb{I}$	$-\sigma_2 . \sigma_3$	$\sigma_1 .$
$\sigma_1 . \sigma_3$	$-\sigma_3$	$-\sigma_1 . \sigma_2 . \sigma_3$	σ_1	$\sigma_2 . \sigma_3$	$-\mathbb{I}$	$-\sigma_1$
$\sigma_2 . \sigma_3$	$\sigma_1 . \sigma_2 . \sigma_3$	$-\sigma_3$	σ_2	$-\sigma_1 . \sigma_3$	$\sigma_1 . \sigma_2$	$-$
$\sigma_1 . \sigma_2 . \sigma_3$	$\sigma_2 . \sigma_3$	$-\sigma_1 . \sigma_3$	$\sigma_1 . \sigma_2$	$-\sigma_3$	σ_2	$-$

Step 2: Substitute matrices and calculate

$$C_4 = C_3 \,/.\, \left\{\mathbb{I} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\};$$

```
MatrixForm[C4]
```

$$\begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\end{pmatrix}$$

Step 3:

Here we let the first row (column) of the product table correspond back to the basis elements of the Grassmann representation, and make the substitution: throughout the whole table.

```
C5 = C4 /. Thread[First[C4] → BasisΛ[]] /.
  Thread[-First[C4] → -BasisΛ[]]
```

```
{{1, e1, e2, e3, e1∧e2, e1∧e3, e2∧e3, e1∧e2∧e3},
 {e1, 1, e1∧e2, e1∧e3, e2, e3, e1∧e2∧e3, e2∧e3},
 {e2, -(e1∧e2), 1, e2∧e3, -e1, -(e1∧e2∧e3), e3, -(e1∧e3)},
 {e3, -(e1∧e3), -(e2∧e3), 1, e1∧e2∧e3, -e1, -e2, e1∧e2},
 {e1∧e2, -e2, e1, e1∧e2∧e3, -1, -(e2∧e3), e1∧e3, -e3},
 {e1∧e3, -e3, -(e1∧e2∧e3), e1, e2∧e3, -1, -(e1∧e2), e2},
 {e2∧e3, e1∧e2∧e3, -e3, e2, -(e1∧e3), e1∧e2, -1, -e1},
 {e1∧e2∧e3, e2∧e3, -(e1∧e3), e1∧e2, -e3, e2, -e1, -1}}
```

Step 4: Verification

Verify that this is the table with which we began.

```
C5 == C2
```

```
True
```

$C\ell_3{}^+$: The Quaternions

Multiplication of two even elements always generates an even element, hence the even elements form a subalgebra. In this case the basis for the subalgebra is composed of the unit **1** and the bivectors $e_i \wedge e_j$.

```
C4 = EvenCliffordProductTable[] //
   CliffordToOrthogonalScalarProducts //
  ToMetricForm; PaletteForm[C4]
```

1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_2 \wedge e_3$
$e_1 \wedge e_2$	-1	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$
$e_1 \wedge e_3$	$e_2 \wedge e_3$	-1	$-(e_1 \wedge e_2)$
$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	-1

From this multiplication table we can see that the even subalgebra of the Clifford algebra of 3-space is isomorphic to the quaternions. To see the isomorphism more clearly, replace the bivectors by 𝕚, 𝕛, and 𝕜.

```
C5 = C4 /. {e2∧e3 → 𝕚, e1∧e3 → 𝕛, e1∧e2 → 𝕜}; PaletteForm[C5]
```

1	𝕜	𝕛	𝕚
𝕜	-1	-𝕚	𝕛
𝕛	𝕚	-1	-𝕜
𝕚	-𝕛	𝕜	-1

The Complex subalgebra

The subalgebra generated by the pair of basis elements **{1, $e_1 \wedge e_2 \wedge e_3$}** is isomorphic to the complex algebra. Although, under the Clifford product, each of the 2-elements behaves *like* an imaginary unit, it is only the 3-element $e_1 \wedge e_2 \wedge e_3$ that also commutes with each of the other basis elements.

Biquaternions

We now explore the metric in which $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1,\ e_3 \ominus e_3 \to -1\}$.

To generate the Clifford product table for this metric we enter:

```
DeclareMetric[DiagonalMatrix[{-1, -1, -1}]];
C6 = CliffordProductTable[] // ToScalarProducts // ToMetricForm;
PaletteForm[C6]
```

1	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge e_3$
e_1	-1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$-e_2$	$-e_3$
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_1	$-(e_1 \wedge e_2 \wedge e_3)$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	$e_1 \wedge e_2 \wedge e_3$	e_1
$e_1 \wedge e_2$	e_2	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	-1	$e_2 \wedge e_3$
$e_1 \wedge e_3$	e_3	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$-(e_2 \wedge e_3)$	-1
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$
$e_1 \wedge e_2 \wedge e_3$	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$-e_3$	e_2

Note that every one of the basis elements of the Grassmann algebra (except **1** and $e_1 \wedge e_2 \wedge e_3$) acts as an imaginary unit under the Clifford product. This enables us to build up a general element of the algebra as a sum of nested complex numbers, or of nested quaternions. To show this, we begin by writing a general element in terms of the basis elements of the Grassmann algebra:

$$QQ == a + b\, e_1 + c\, e_2 + d\, e_3 + e\, e_1 \wedge e_2 + f\, e_1 \wedge e_3 + g\, e_2 \wedge e_3 + h\, e_1 \wedge e_2 \wedge e_3$$

Or, as previously argued, since the basis 1-elements are orthogonal, we can replace the exterior product by the Clifford product and rearrange the terms in the sum to give

$$QQ == (a + b\, e_1 + c\, e_2 + e\, e_1 \diamond e_2) + (d + f\, e_1 + g\, e_2 + h\, e_1 \diamond e_2) \diamond e_3$$

This sum may be viewed as the complex sum of two quaternions

$$Q_1 == a + b\, e_1 + c\, e_2 + e\, e_1 \diamond e_2 == (a + b\, e_1) + (c + e\, e_1) \diamond e_2$$
$$Q_2 == d + f\, e_1 + g\, e_2 + h\, e_1 \diamond e_2 == (d + f\, e_1) + (g + h\, e_1) \diamond e_2$$

$$QQ == Q_1 + Q_2 \diamond e_3$$

Historical Note

This complex sum of two quaternions was called a biquaternion by Hamilton [Hamilton, Elements of Quaternions, p133] but Clifford in a footnote to his *Preliminary Sketch of Biquaternions* [Clifford, Mathematical Papers, Chelsea] says 'Hamilton's biquaternion is a quaternion with complex coefficients; but it is convenient (as Prof. Pierce remarks) to suppose from the beginning that all scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one."

Hamilton uses the word *biscalar* for a *complex number* and *bivector* [p 225 Elements of Quaternions] for a *complex vector*, that is, for a vector $\mathbf{x} + i\,\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors; and the word *biquaternion* for a complex quaternion $\mathbf{q_0} + i\,\mathbf{q_1}$, where $\mathbf{q_0}$ and $\mathbf{q_1}$ are quaternions. He emphasizes here that "… i is the (scalar) *imaginary of algebra*, and *not* a symbol for a *geometrically real right versor* …"

Hamilton introduces his biquaternion as the quotient of a bivector (his usage) by a (real) vector.

12.16 Clifford Algebras of a 4-Space

The Clifford product table in 4-space

In this section we explore the Clifford algebras of 4-space. First we declare a (not necessarily orthogonal) basis for the 4-space, and generate the associated Clifford product table. Because of the size of the table, only the first few columns are shown in the print version.

$\mathbb{V}_4$; C_1 = CliffordProductTable[]; PaletteForm[C_1]

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond$
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond$
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond$
$e_3 \diamond 1$	$e_3 \diamond e_1$	$e_3 \diamond e_2$	$e_3 \diamond$
$e_4 \diamond 1$	$e_4 \diamond e_1$	$e_4 \diamond e_2$	$e_4 \diamond$
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e$
$(e_1 \wedge e_3) \diamond 1$	$(e_1 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_3) \diamond e_2$	$(e_1 \wedge e$
$(e_1 \wedge e_4) \diamond 1$	$(e_1 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_4) \diamond e_2$	$(e_1 \wedge e$
$(e_2 \wedge e_3) \diamond 1$	$(e_2 \wedge e_3) \diamond e_1$	$(e_2 \wedge e_3) \diamond e_2$	$(e_2 \wedge e$
$(e_2 \wedge e_4) \diamond 1$	$(e_2 \wedge e_4) \diamond e_1$	$(e_2 \wedge e_4) \diamond e_2$	$(e_2 \wedge e$
$(e_3 \wedge e_4) \diamond 1$	$(e_3 \wedge e_4) \diamond e_1$	$(e_3 \wedge e_4) \diamond e_2$	$(e_3 \wedge e$
$(e_1 \wedge e_2 \wedge e_3) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_2 \wedge$
$(e_1 \wedge e_2 \wedge e_4) \diamond 1$	$(e_1 \wedge e_2 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_2 \wedge$
$(e_1 \wedge e_3 \wedge e_4) \diamond 1$	$(e_1 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_3 \wedge$
$(e_2 \wedge e_3 \wedge e_4) \diamond 1$	$(e_2 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_2 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_2 \wedge e_3 \wedge$
$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_2 \wedge e$

$\mathcal{C}\ell_4$: The Clifford algebra of Euclidean 4-space

Our first exploration is to the Clifford algebra of Euclidean space, hence we accept the default metric which was automatically declared when we declared the basis.

Metric

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

Since the basis elements are orthogonal, we can use the faster *GrassmannAlgebra* function `CliffordToOrthogonalScalarProducts` for computing the Clifford products.

```
C₂ = CliffordToOrthogonalScalarProducts[C₁] // ToMetricForm;
PaletteForm[C₂]
```

1	e_1	e_2	e_3
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$
e_2	$-(e_1 \wedge e_2)$	1	$e_2 \wedge e_3$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	1
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$
$e_1 \wedge e_2$	$-e_2$	e_1	$e_1 \wedge e_2 \wedge e_3$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	e_1
$e_1 \wedge e_4$	$-e_4$	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	$-e_3$	e_2
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	$-e_4$	$-(e_2 \wedge e_3 \wedge e_4)$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	$-e_4$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$e_2 \wedge e_4$	$-(e_1 \wedge e_4)$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$
$e_1 \wedge e_3 \wedge e_4$	$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_4)$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$e_3 \wedge e_4$	$-(e_2 \wedge e_4)$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_2 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_4)$

This table is well off the page in the printed version, but we can condense the notation for the basis elements of Λ by replacing $\mathbf{e_i \wedge \cdots \wedge e_j}$ by $\mathbf{e_{i,\ldots,j}}$. To do this we can use the *GrassmannAlgebra* function `ToBasisIndexedForm`. For example

```
1 - (e₁ ∧ e₂ ∧ e₃) // ToBasisIndexedForm
```

$1 - e_{1,2,3}$

To display the table in condensed notation we make up a rule for each of the basis elements.

```
C3 = C2 /. Reverse[Thread[BasisΛ[] → ToBasisIndexedForm[BasisΛ[]]]];
PaletteForm[C3]
```

1	e_1	e_2	e_3	e_4	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$
e_1	1	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$	e_2	e_3	e_4
e_2	$-e_{1,2}$	1	$e_{2,3}$	$e_{2,4}$	$-e_1$	$-e_{1,2,3}$	$-e_{1,2,}$
e_3	$-e_{1,3}$	$-e_{2,3}$	1	$e_{3,4}$	$e_{1,2,3}$	$-e_1$	$-e_{1,3,}$
e_4	$-e_{1,4}$	$-e_{2,4}$	$-e_{3,4}$	1	$e_{1,2,4}$	$e_{1,3,4}$	$-e_1$
$e_{1,2}$	$-e_2$	e_1	$e_{1,2,3}$	$e_{1,2,4}$	-1	$-e_{2,3}$	$-e_{2,4}$
$e_{1,3}$	$-e_3$	$-e_{1,2,3}$	e_1	$e_{1,3,4}$	$e_{2,3}$	-1	$-e_{3,4}$
$e_{1,4}$	$-e_4$	$-e_{1,2,4}$	$-e_{1,3,4}$	e_1	$e_{2,4}$	$e_{3,4}$	-1
$e_{2,3}$	$e_{1,2,3}$	$-e_3$	e_2	$e_{2,3,4}$	$-e_{1,3}$	$e_{1,2}$	$e_{1,2,3,}$
$e_{2,4}$	$e_{1,2,4}$	$-e_4$	$-e_{2,3,4}$	e_2	$-e_{1,4}$	$-e_{1,2,3,4}$	$e_{1,2}$
$e_{3,4}$	$e_{1,3,4}$	$e_{2,3,4}$	$-e_4$	e_3	$e_{1,2,3,4}$	$-e_{1,4}$	$e_{1,3}$
$e_{1,2,3}$	$e_{2,3}$	$-e_{1,3}$	$e_{1,2}$	$e_{1,2,3,4}$	$-e_3$	e_2	$e_{2,3,4}$
$e_{1,2,4}$	$e_{2,4}$	$-e_{1,4}$	$-e_{1,2,3,4}$	$e_{1,2}$	$-e_4$	$-e_{2,3,4}$	e_2
$e_{1,3,4}$	$e_{3,4}$	$e_{1,2,3,4}$	$-e_{1,4}$	$e_{1,3}$	$e_{2,3,4}$	$-e_4$	e_3
$e_{2,3,4}$	$-e_{1,2,3,4}$	$e_{3,4}$	$-e_{2,4}$	$e_{2,3}$	$-e_{1,3,4}$	$e_{1,2,4}$	$-e_{1,2,}$
$e_{1,2,3,4}$	$-e_{2,3,4}$	$e_{1,3,4}$	$-e_{1,2,4}$	$e_{1,2,3}$	$-e_{3,4}$	$e_{2,4}$	$-e_{2,3}$

$C\ell_4{}^+$: The even Clifford algebra of Euclidean 4-space

The even subalgebra is composed of the unit **1**, the bivectors $\mathbf{e_i \wedge e_j}$, and the single basis 4-element $\mathbf{e_1 \wedge e_2 \wedge e_3 \wedge e_4}$.

```
C3 = EvenCliffordProductTable[] //
   CliffordToOrthogonalScalarProducts //
  ToMetricForm; PaletteForm[C3]
```

1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_1 \wedge e_4$	$e_2 \wedge e_3$
$e_1 \wedge e_2$	-1	$-(e_2 \wedge e_3)$	$-(e_2 \wedge e_4)$	$e_1 \wedge e_3$
$e_1 \wedge e_3$	$e_2 \wedge e_3$	-1	$-(e_3 \wedge e_4)$	$-(e_1 \wedge e$
$e_1 \wedge e_4$	$e_2 \wedge e_4$	$e_3 \wedge e_4$	-1	$e_1 \wedge e_2 \wedge e_3$
$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	-1
$e_2 \wedge e_4$	$-(e_1 \wedge e_4)$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2$	$e_3 \wedge e_4$
$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_4)$	$e_1 \wedge e_3$	$-(e_2 \wedge e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e_2 \wedge e_3)$	$-(e_1 \wedge e$

$\mathcal{C}\ell_{1,3}$: The Dirac algebra

To generate the Dirac algebra we need the Minkowski metric in which there is one time-like basis element $e_1 \ominus e_1 == +1$, and three space-like basis elements $e_i \ominus e_i == -1$.

$$\{e_1 \ominus e_1 == 1,\ e_2 \ominus e_2 == -1,\ e_3 \ominus e_3 == -1,\ e_4 \ominus e_4 == -1\}$$

To generate the Clifford product table for this metric we enter:

$\mathbb{V}_4$**; DeclareMetric[DiagonalMatrix[{1, -1, -1, -1}]]**

{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}

C_6 **= CliffordToOrthogonalScalarProducts[**C_1**] // ToMetricForm;**
PaletteForm[C_6**]**

1	e_1	e_2	e_3	
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_1
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_2
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	e_3
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$	
$e_1 \wedge e_2$	$-e_2$	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$e_1 \wedge$
$e_1 \wedge e_4$	$-e_4$	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$-$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_2 \wedge$
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	e_4	$-(e_2 \wedge e_3 \wedge e_4)$	$-$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	e_4	$-$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$e_2 \wedge e_4$	$e_1 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e$
$e_1 \wedge e_3 \wedge e_4$	$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_1 \wedge e_4$	$-(e$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_2 \wedge e_3 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_4$	$-(e_1 \wedge$

To confirm that this structure is isomorphic to the Dirac algebra we can go through the same procedure that we followed in the case of the Pauli algebra.

Step 1: Replace symbols for entities and operations

$$C_7 = (C_6 \text{ // ReplaceNegativeUnit}) \text{ /. } \{1 \to I,\ e_1 \to \gamma_0,\ e_2 \to \gamma_1,\ e_3 \to \gamma_2,\ e_4 \to \gamma_3\} \text{ /. Wedge} \to \text{Dot; PaletteForm}[C_7]$$

I	γ_0	γ_1	γ_2	γ_3
γ_0	I	$\gamma_0 \cdot \gamma_1$	$\gamma_0 \cdot \gamma_2$	$\gamma_0 \cdot \gamma_3$
γ_1	$-\gamma_0 \cdot \gamma_1$	$-I$	$\gamma_1 \cdot \gamma_2$	$\gamma_1 \cdot \gamma_3$
γ_2	$-\gamma_0 \cdot \gamma_2$	$-\gamma_1 \cdot \gamma_2$	$-I$	$\gamma_2 \cdot \gamma_3$
γ_3	$-\gamma_0 \cdot \gamma_3$	$-\gamma_1 \cdot \gamma_3$	$-\gamma_2 \cdot \gamma_3$	$-I$
$\gamma_0 \cdot \gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_2$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_3$
$\gamma_0 \cdot \gamma_2$	$-\gamma_2$	$-\gamma_0 \cdot \gamma_1 \cdot \gamma_2$	$-\gamma_0$	$\gamma_0 \cdot \gamma_2 \cdot \gamma_3$
$\gamma_0 \cdot \gamma_3$	$-\gamma_3$	$-\gamma_0 \cdot \gamma_1 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_0$
$\gamma_1 \cdot \gamma_2$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_2$	γ_2	$-\gamma_1$	$\gamma_1 \cdot \gamma_2 \cdot \gamma_3$
$\gamma_1 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_3$	γ_3	$-\gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_1$
$\gamma_2 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_2 \cdot \gamma_3$	$\gamma_1 \cdot \gamma_2 \cdot \gamma_3$	γ_3	$-\gamma_2$
$\gamma_0 \cdot \gamma_1 \cdot \gamma_2$	$\gamma_1 \cdot \gamma_2$	$\gamma_0 \cdot \gamma_2$	$-\gamma_0 \cdot \gamma_1$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3$
$\gamma_0 \cdot \gamma_1 \cdot \gamma_3$	$\gamma_1 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_1$
$\gamma_0 \cdot \gamma_2 \cdot \gamma_3$	$\gamma_2 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_2$
$\gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_2 \cdot \gamma_3$	$\gamma_1 \cdot \gamma_3$	$-\gamma_1 \cdot \gamma_2$
$\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_1 \cdot \gamma_2 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_2 \cdot \gamma_3$	$\gamma_0 \cdot \gamma_1 \cdot \gamma_3$	$-\gamma_0 \cdot \gamma_1 \cdot \gamma_2$

Step 2: Substitute matrices and calculate

$$C_8 = C_7 \text{ /. } \left\{I \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \gamma_0 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma_1 \to \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\right.$$
$$\left.\gamma_2 \to \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},\ \gamma_3 \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\}; \text{ MatrixForm}[C_8]$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\ \begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\ \begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\ \begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\ \begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\ \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\ \begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\ \begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\ \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\ \begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}\ \begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}\ \begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}\ \begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}\ \begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}\ \begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}\ \begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\ \begin{pmatrix}0&0&i&0\\0&0&0&i\\i&0&0&0\\0&i&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-i&0\\0&0&0&-i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}\ \begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}\ \begin{pmatrix}0&0&i&0\\0&0&0&i\\i&0&0&0\\0&i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-i&0\\0&0&0&-i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}\ \begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\ \begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}$$

$$\begin{pmatrix}0&0&i&0\\0&0&0&i\\i&0&0&0\\0&i&0&0\end{pmatrix}\ \begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix}\ \begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}\ \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\ \begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}$$

Step 3:

```
C9 = C8 /. Thread[First[C8] → Basis∧[]] /.
   Thread[-First[C8] → -Basis∧[]];
```

Step 4: Verification

```
C9 == C6
```

```
True
```

$\mathcal{C}\ell_{0,4}$: The Clifford algebra of complex quaternions

To generate the Clifford algebra of complex quaternions we need a metric in which all the scalar products of orthogonal basis elements are of the form $\mathbf{e_i} \ominus \mathbf{e_j}$ (*i* not equal to *j*) are equal to **-1**. That is

$$\{e_1 \ominus e_1 == -1,\ e_2 \ominus e_2 == -1,\ e_3 \ominus e_3 == -1,\ e_4 \ominus e_4 == -1\}$$

To generate the Clifford product table for this metric we enter:

```
V4; DeclareMetric[DiagonalMatrix[{-1, -1, -1, -1}]]
```

```
{{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

```
C6 = CliffordToOrthogonalScalarProducts[C1] // ToMetricForm;
PaletteForm[C6]
```

1	e_1	e_2	e_3	
e_1	-1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_1
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_2
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	e_3
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$	
$e_1 \wedge e_2$	e_2	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge$
$e_1 \wedge e_3$	e_3	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$e_1 \wedge$
$e_1 \wedge e_4$	e_4	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$-$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_2 \wedge$
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	e_4	$-(e_2 \wedge e_3 \wedge e_4)$	$-$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	e_4	$-$
$e_1 \wedge e_2 \wedge e_3$	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$-(e_2 \wedge e_4)$	$e_1 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e$
$e_1 \wedge e_3 \wedge e_4$	$-(e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_1 \wedge e_4$	$-(e$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_4$	$-(e_1 \wedge$

Note that both the vectors and bivectors act as an imaginary units under the Clifford product. This enables us to build up a general element of the algebra as a sum of nested complex numbers, quaternions, or complex quaternions. To show this, we begin by writing a general element in terms of the basis elements of the Grassmann algebra:

```
X = CreateGrassmannNumber[a]
```

$$a^0 + e_1\, a^1 + e_2\, a^2 + e_3\, a^3 + e_4\, a^4 + a^5\, e_1 \wedge e_2 + a^6\, e_1 \wedge e_3 + a^7\, e_1 \wedge e_4 + a^8\, e_2 \wedge e_3 + a^9\, e_2 \wedge e_4 + a^{10}\, e_3 \wedge e_4 + a^{11}\, e_1 \wedge e_2 \wedge e_3 + a^{12}\, e_1 \wedge e_2 \wedge e_4 + a^{13}\, e_1 \wedge e_3 \wedge e_4 + a^{14}\, e_2 \wedge e_3 \wedge e_4 + a^{15}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

We can then factor this expression to display it as a nested sum of numbers of the form $C_k == a^i + a^j\, e_1$. (Of course any basis element will do upon which to base the factorization.)

$$X == (a^0 + a^1\, e_1) + (a^2 + a^5\, e_1) \wedge e_2 + ((a^3 + a^6\, e_1) + (a^8 + a^{11}\, e_1) \wedge e_2) \wedge e_3 + ((a^4 + a^7\, e_1) + (a^9 + a^{12}\, e_1) \wedge e_2 + ((a^8 + a^{13}\, e_1) + (a^{14} + a^{15}\, e_1) \wedge e_2) \wedge e_3) \wedge e_4$$

Which can be rewritten in terms of the C_i as

$$X == C_1 + C_2 \wedge e_2 + (C_3 + C_4 \wedge e_2) \wedge e_3 + (C_5 + C_6 \wedge e_2 + (C_7 + C_8 \wedge e_2) \wedge e_3) \wedge e_4$$

Again, we can rewrite each of these elements as $Q_k == C_i + C_j \wedge e_2$ to get

$$X == Q_1 + Q_2 \wedge e_3 + (Q_3 + Q_4 \wedge e_3) \wedge e_4$$

Finally, we write $QQ_k == Q_i + Q_j \wedge e_3$ to get

$$X == QQ_1 + QQ_2 \wedge e_4$$

Since the basis 1-elements are orthogonal, we can replace the exterior product by the Clifford product to give

$$X == (a^0 + a^1\, e_1) + (a^2 + a^5\, e_1) \diamond e_2 + ((a^3 + a^6\, e_1) + (a^8 + a^{11}\, e_1) \diamond e_2) \diamond e_3 + ((a^4 + a^7\, e_1) + (a^9 + a^{12}\, e_1) \diamond e_2 + ((a^8 + a^{13}\, e_1) + (a^{14} + a^{15}\, e_1) \diamond e_2) \diamond e_3) \diamond e_4$$

$$== C_1 + C_2 \diamond e_2 + (C_3 + C_4 \diamond e_2) \diamond e_3 + (C_5 + C_6 \diamond e_2 + (C_7 + C_8 \diamond e_2) \diamond e_3) \diamond e_4$$

$$== Q_1 + Q_2 \diamond e_3 + (Q_3 + Q_4 \diamond e_3) \diamond e_4$$

$$== QQ_1 + QQ_2 \diamond e_4$$

12.17 Rotations

To be completed