Fluid Dynamics

Dr. J.R. Lister

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Contents

Introduction

These notes are based on the course "Fluid Dynamics" given by Dr. J.R. Lister in Cambridge in the Michælmas Term 1996. These typeset notes are totally unconnected with Dr. Lister.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
Analysis	Further Analysis
Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems
Combinatorics	Bifurcations in Nonlinear Convection

They may be downloaded from

`http://www.istari.ucam.org/maths/`.

Chapter 1

Kinematics

1.1 Continuum Fields

Everyday experience suggests that at a macroscopic scale, liquids and gases look like smooth continua with density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ fields.

Since fluids are made of molecules this is of course only an approximate description. On large lengthscales we can define these fields by averaging over a volume V smaller than the scale of interest but large enough to contain many molecules. The effect of this averaging is to exchange an enormous number of ODEs that describe the motion of each molecule for a few PDEs that describe the averaged fields.

This *continuum approximation* is not always appropriate. For instance the velocity structure about a spacecraft during re-entry has a lengthscale comparable with the molecular mean free path. Similarly, blood flow in capillaries must take the red blood cells into account.

1.2 Flow Visualization

There are many experimental techniques for obtaining a description of the velocity field $\mathbf{u}(\mathbf{x}, t)$. Three simple visualisation techniques give rise to the ideas of *streamlines*, *pathlines* and *streaklines*. We will illustrate these ideas by application to the simple two-dimensional example $\mathbf{u}(\mathbf{x}, t) = (t, y)$.

Streamlines are curves that are everywhere parallel to the instantaneous flow. They are visualised experimentally by the short-time exposure of many brightly-lit particles — the streamlines are obtained by joining the resulting short segments in a manner analogous to obtaining magnetic fields from iron filings.

Mathematically, a streamline is a curve $\mathbf{x}(s; \mathbf{x}_0, t)$ at a given fixed time t with s varying along the curve and passing through a given point $\mathbf{x}_0$ that satisfies

$$\frac{\partial \mathbf{x}}{\partial s} = \mathbf{u}(\mathbf{x}, t) \qquad \mathbf{x}(0; x_0, t) = \mathbf{x}_0.$$

For our example we have

$$x(s; \mathbf{x}_0, t) = x_0 + ts \qquad y(s; \mathbf{x}_0, t) = y_0 e^s.$$

This gives the curve

$$\frac{x - x_0}{t} = \log \frac{y}{y_0}.$$

Pathlines are particle paths: paths traversed by particles moving with the flow. They are visualised experimentally by the long-time exposure of a few brightly-lit particles.

Mathematically, a pathline is a curve $\mathbf{x}(t; \mathbf{x}_0, t_0)$ corresponding to a particle released from $\mathbf{x} = \mathbf{x}_0$ at $t = t_0$. The differential equation is

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{u}(\mathbf{x}, t) \qquad \mathbf{x}(t_0; x_0, t_0) = \mathbf{x}_0.$$

Our example gives

$$x(t; \mathbf{x}_0, t_0) = x_0 + \frac{t^2 - t_0^2}{2} \qquad y(t; \mathbf{x}_0, t_0) = y_0 e^{t - t_0}.$$

For a particle released at $t_0 = 0$ this gives a curve

$$y = y_0 e^{\sqrt{2(x - x_0)}}.$$

Streaklines give the position at some fixed time of dye released over a range of previous times from a fixed source (e.g. an oil spill).

Mathematically, a streakline is a curve $\mathbf{x}(t_0; \mathbf{x}, t)$ with t_0 varying along the curve and a fixed observation time t. To obtain it, we still solve

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{u}(\mathbf{x}, t) \qquad \mathbf{x}(t_0; x_0, t_0) = \mathbf{x}_0,$$

but then fix t instead of t_0. Suppose we observe our flow at $t = 0$ — we can use our previous solution to get a streakline

$$y = y_0 e^{-t_0} = y_0 e^{-\sqrt{2(x_0 - x)}}.$$

Note that for this unsteady flow we get different results from each method of visualisation. The different methods give the same result if the flow is steady.

1.3 Material Derivative

This is a rate of change "moving with the fluid". For any quantity F, the rate of change in that quantity seen by an observer moving with the fluid is the material or Lagrangian derivative $\frac{\mathrm{D}F}{\mathrm{D}t}$.

$$\begin{aligned}\delta F &= \frac{F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - F(\mathbf{x}, t)}{\delta t} \\ &= (\delta\mathbf{x} \cdot \nabla)\, F + \delta t \frac{\partial F}{\partial t} + \text{ smaller terms.}\end{aligned}$$

Hence

$$\frac{\mathrm{D}F}{\mathrm{D}t} = (\mathbf{u} \cdot \nabla)\, F + \frac{\partial F}{\partial t}.$$

1.4 Conservation of Mass

Consider an arbitrary (at least smooth – this is applied maths) volume V, fixed in space with bounding surface A and outward normal $\mathbf{n}$. The mass inside V is

$$M = \int_V \rho \, \mathrm{d}V,$$

and the mass changes due to the flow over the boundary, so

$$\frac{\partial M}{\partial t} = -\int_A \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}A.$$

Application of the divergence theorem gives

$$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_V \nabla \cdot (\rho \mathbf{u}) \, \mathrm{d}V = 0.$$

Since V is arbitrarily small,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

or rewritten using the material derivative

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} = 0.$$

1.5 Kinematic Boundary Condition

This is an expression of mass conservation at a boundary. If the velocity of the boundary is $\mathbf{u}^A$, the condition for no mass flux is

$$\rho \left(\mathbf{u}(\mathbf{x}, t) - \mathbf{u}^A(\mathbf{x}, t) \right) \cdot \mathbf{n} \delta A \delta t = 0,$$

which gives that $\mathbf{u} \cdot \mathbf{n} = \mathbf{u}^A \cdot \mathbf{n}$. For a fixed surface, $\mathbf{u}^A = 0$, so the surface is a streamline.

1.6 Incompressible Fluids

For this course, we restrict ourselves to fluids with $\rho = const$. Mass conservation reduces to $\nabla \cdot \mathbf{u} = 0$. Such a velocity field $\mathbf{u}$ is said to be solenoidal.

1.7 Streamfunctions

This gives a representation of the flow satisfying $\nabla \cdot \mathbf{u} = 0$ automatically. For example, in 2D Cartesians, any velocity field $\mathbf{u} = (u, v, 0)$ is solenoidal if there exists $\psi(x, y, t)$ such that $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$.

In 2D polars, we want ψ such that $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$ and $u_\theta = -\frac{\partial \psi}{\partial r}$.

In axisymmetric cylindrical polars, we want Ψ such that $u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}$ and $u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}$. Ψ is called a Stokes streamfunction.

In axisymmetric spherical polars, we want Ψ such that $u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta}$ and $u_\theta = \frac{-1}{r \sin \theta} \frac{\partial \Psi}{\partial r}$.

Chapter 2

Dynamics

2.1 Surface and volume forces

Two types of force are considered to act on a fluid: those proportional to volume (e.g. gravity) and those proportional to area (e.g. pressure). This is a simplification appropriate to the continuum level description — e.g. surface forces in a gas are the average result of many molecules transferring momentum by collision with other molecules over the very short distance of the mean free path.

Volume forces

We denote the force on a small volume element δV by $\mathbf{F}^V(\mathbf{x}, t)\delta V$. The volume force is often conservative, with a potential energy per unit volume χ, so that $\mathbf{F}^V = -\nabla\chi$ (or potential energy per unit mass Φ, so that $\mathbf{F}^V = -\rho\nabla\chi$.

The most common case is

$$\mathbf{F}^V(\mathbf{x}, t)\delta V = \rho\mathbf{g}\delta V.$$

Surface forces

We denote the force on a small surface element $\mathbf{n}\delta A$ by $\mathbf{F}^A(\mathbf{x}, t, \mathbf{n})\delta A$, which depends on the orientation $\mathbf{n}$ of the surface element. A full description of surface forces includes the effects of friction of layers of water sliding over each other or over rigid boundaries (viscosity).

Viscous effects are important when the *Reynolds number*

$$\frac{UL}{\nu} \leq 1,$$

where U is a typical velocity, L a typical length and ν is the dynamic viscosity, which is a property of the fluid.

In many cases fluids act as nearly frictionless and in this course we neglect frictional forces completely. For a treatment of viscous fluids see the Fluid Dynamics 2 course in Part IIB.

For inviscid (frictionless) fluids the surface force is simply perpendicular to the surface with a magnitude independent of orientation:

$$\mathbf{F}^A(\mathbf{x},t)\delta A = -p(\mathbf{x},t)\mathbf{n}\delta A,$$

where p is the pressure. The minus sign is so that pressure is positive.

2.2 Momentum Equation

Consider an arbitrary (at least smooth – this is applied maths) volume V, fixed in space with bounding surface A and outward normal $\mathbf{n}$. The momentum inside V is

$$\int_V \rho\mathbf{u}\,\mathrm{d}V,$$

and the momentum changes due to the flow over the boundary, surface forces and volume forces, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = -\int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}A + \int_A -p\mathbf{n}\,\mathrm{d}A + \int_V \mathbf{F}^V\,\mathrm{d}V, \tag{2.1}$$

which is the momentum integral equation. Written in component form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i\,\mathrm{d}V = -\int_A \rho u_i u_j n_j\,\mathrm{d}A + \int_A -pn_i\,\mathrm{d}A + \int_V F_i^V\,\mathrm{d}V$$

$\rho u_i u_j$ is called the momentum flux tensor.

V is fixed, so LHS is $\int_V \frac{\partial \rho u_i}{\partial t}\,\mathrm{d}V$, and using the divergence theorem on the RHS, then letting V be arbitrarily small, we achieve the Euler momentum equation

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = -\nabla p + \mathbf{F}^V. \tag{2.2}$$

The associated dynamic boundary condition is that given forces are applied at the boundary (i.e. p is given).

2.2.1 Applications of integral form

Uncoiling of hosepipes

Assume a steady uniform flow U through a pipe of constant cross-section A. Neglect gravity. Now (2.1) becomes

$$\int_{\text{walls}} + \int_{\text{ends}} \left(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\right)\,\mathrm{d}A = 0.$$

The integral over the walls is

$$\int_{\text{walls}} p\mathbf{n}\,\mathrm{d}A = \text{force on pipe},$$

since $\mathbf{u}\cdot\mathbf{n} = 0$ on the walls. The integral over the ends is

$$(\rho U^2 + p)A(\mathbf{2} - \mathbf{1})$$

and so the force on the pipe is $\mathbf{F} = (\rho U^2 + p)A(\mathbf{1} - \mathbf{2})$.

Pressure change at abrupt junction

Apply a momentum balance to the sketched shape. Neglect gravity, and also neglect the time derivative, which is zero on average.

The momentum integral equation (2.1) becomes

$$\int \left(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\right)\,\mathrm{d}A = 0.$$

The horizontal component gives

$$\rho u_1^2 A_1 + p_1 A_1 = \rho u_2^2 A_2 + p_2 A_2,$$

and mass conservation gives $u_1 A_1 = u_2 A_2$. Then we see that

$$p_2 - p_1 = \rho u_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right) > 0.$$

2.3 Bernoulli's Theorem

For a steady flow ($\frac{\partial\mathbf{u}}{\partial t} = 0$) with potential forces ($\mathbf{F}_V = -\nabla\chi$),

$$\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla(p + \chi),$$

which can be written

$$\rho\left(\frac{1}{2}\nabla u^2 - \mathbf{u}\wedge(\nabla\wedge\mathbf{u})\right) = -\nabla(p+\chi). \tag{2.3}$$

We define the *vorticity* $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$ and then let $H = \frac{1}{2}\rho u^2 + p + \chi$. Then $\nabla H = \rho\mathbf{u}\wedge\boldsymbol{\omega}$. Now $\mathbf{u}\cdot\nabla H = 0$, so H is constant on streamlines. This is Bernoulli's theorem.

Note also that $\boldsymbol{\omega}\cdot\nabla H = 0$ and that H is constant on vortex lines.

The constancy of H means that p is low at high speeds.

2.3.1 Application

Consider a water jet hitting an inclined plane.

Neglect gravity, so that on the surface streamline $p = p_a$ the speed is constant. Let this speed be U.

Now apply the momentum integral equation (2.1) to get

$$\int_A \rho\mathbf{u}\mathbf{u}\cdot\mathbf{n} + (p-p_a)\cdot\mathbf{n}\,\mathrm{d}A = 0.$$

Now $\mathbf{u}\cdot\mathbf{n} = 0$ except at the ends. Mass conservation gives

$$\rho a U = \rho a_1 U + \rho a_2 U.$$

Now balance the momentum parallel to the wall to get

$$\rho a U^2\cos\beta = \rho a_2 U^2 - \rho a_1 U^2.$$

Thus

$$a_1 = \frac{1+\cos\beta}{2}a \qquad \text{and} \qquad a_2 = \frac{1-\cos\beta}{2}a.$$

Balancing the momentum perpendicular to the wall we get $F = \rho a U^2\sin\beta$.

2.4 Vorticity and Circulation

2.4.1 Vorticity Equation

Start with the Euler momentum equation (2.2) with potential forces

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) = -\nabla(p+\chi)$$

and take its curl to obtain

$$\frac{\partial \omega}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} - (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} + \mathbf{u} \nabla \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \nabla \cdot \mathbf{u}.$$

Now $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = 0$, so we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} - (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega}$$

or

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u}. \tag{2.4}$$

2.4.2 Interpretation of vorticity

Consider a material line element (ie a line element moving with the fluid). Then in a time δt, $\delta \mathbf{l} \to \delta \mathbf{l} + (\delta \mathbf{l} \cdot \nabla)\, \mathbf{u} \delta t$, which gives that $\frac{\partial \delta \mathbf{l}}{\partial t} = (\delta \mathbf{l} \cdot \nabla)\, \mathbf{u}$. Hence the tensor $\frac{\partial u_i}{\partial x_j}$ determines the local rate of deformation of line elements.

$$\begin{aligned}\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial x_j}{\partial u_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial x_j}{\partial u_i}\right) \\ &= e_{ij} + \frac{1}{2}\epsilon_{jik}\omega_k.\end{aligned}$$

The local motion due to e_{ij} is called the strain. The motion due to the second term $\frac{1}{2}\epsilon_{jik}\omega_k \delta l_j = \frac{1}{2}\left(\boldsymbol{\omega} \wedge \delta \mathbf{l}\right)$ is rotation with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

2.4.3 Ballerina effect and vortex line stretching

The vorticity equation (2.4) can be interpreted as saying that vorticity changes just like the rotation and stretching of material line elements. This is just the conservation of angular momentum.

Consider a rotating fluid cylinder, initially with angular velocity ω_1, radius a_1 and length l_1. Conservation of mass gives $a_1^2 l_1 = a_2^2 l_2$ and conservation of angular momentum gives $a_1^4 l_1 \omega_1 = a_2^4 l_2 \omega_2$. These combine to give

$$\frac{\omega_1}{\omega_2} = \frac{l_1}{l_2},$$

which says that vorticity increases as the fluid is stretched. This explains the bathtub vortex.

2.4.4 Kelvin's Circulation Theorem

Assume ρ constant and $\mathbf{F}^V = -\nabla \chi$. Define the circulation $C(t)$ around a closed material curve $\Gamma(t)$ by

$$C(t) = \oint_{\Gamma(t)} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{dl}.$$

Then

$$\frac{\partial C(t)}{\partial t} = \oint_{\Gamma(t)} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \mathbf{dl} + \mathbf{u} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, \mathbf{dl}$$

since Γ moves with the fluid. But from the momentum equation $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla\frac{p+\chi}{\rho}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{dl} = (\mathbf{dl}\cdot\nabla)\,\mathbf{u}$. Hence

$$\begin{aligned} \frac{\partial C(t)}{\partial t} &= \oint_{\Gamma(t)} \left(\nabla\left(\frac{1}{2}u^2 - \frac{p+\chi}{\rho}\right)\right)\cdot\mathbf{dl} \\ &= 0 \qquad \text{since } \Gamma \text{ is closed} \end{aligned}$$

So, for an inviscid fluid of constant density with potential forces, the circulation around a closed material curve is constant.

2.4.5 Irrotational flow remains so

A flow with $\boldsymbol{\omega} = 0$ is said to be irrotational. If $\boldsymbol{\omega} = 0$ everywhere at $t = 0$, then the vorticity equation (2.4) becomes $\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = 0$, implying that $\boldsymbol{\omega} = 0$ for all times $t \geq 0$.

This isn't quite true, vorticity can leave a stagnation point, especially at sharp trailing edges.

Chapter 3

Irrotational Flows

You will want to find a table of vector differential operators in various co-ordinate systems. There is one in the back of Acheson.

3.1 Velocity potential

The vorticity equation tells us that an initially irrotational flow remains so for all time. Since $\nabla \wedge \mathbf{u} = 0$ for all time there exists a velocity potential $\phi(\mathbf{x}, t)$ such that $\mathbf{u} = \nabla\phi$. Note both the $+$ sign and that any function $f(t)$ can be added to ϕ. Given $\mathbf{u}$, we can find ϕ from

$$\phi = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{dl}.$$

The result is independent of path since $\nabla \wedge \mathbf{u} = 0$. Note that ϕ can be multivalued in 2D if there are holes with

$$\oint_{\text{hole}} \mathbf{u} \cdot \mathbf{dl} \neq 0.$$

Mass conservation for an incompressible fluid reduces to $\cdot\phi = 0$, and so if $\mathbf{u} = \nabla\phi$ we have to solve $\nabla^2\phi = 0$.[1]

The kinematic boundary condition $\mathbf{u}^A \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}$ is

$$\mathbf{u}^A \cdot \mathbf{n} = \mathbf{n} \cdot \nabla\phi \equiv \frac{\partial\phi}{\partial n}.$$

Thus solving the Euler momentum equation (2.2) reduces to solving the more familiar Laplace's equation with Neumann boundary conditions. The flow is only non-zero because of the applied boundary conditions on $\frac{\partial\phi}{\partial n}$ (on bodies, surfaces and at infinity).

3.2 Some simple solutions

For simple geometries it is often possible to write down solutions of $\nabla^2\phi = 0$ as a sum of separable solutions in suitable co-ordinate systems. See the Methods course for details.

[1]Hence irrotational flows are sometimes called potential or harmonic flows.

Cartesians

- $\phi = \mathbf{U}\cdot\mathbf{x}$ gives $\mathbf{u} = \mathbf{U}$. This is uniform flow with velocity $\mathbf{U}$ (e.g. flow in a straight pipe).
- $\phi = (e^{\pm kz}$ or $\cosh kz$ or $\sinh kz) \times (e^{\pm \imath kx}$ or $\cos kx$ or $\sin kx)$ gives a flow which is periodic in x (e.g. waves).

Spherical polars

The general axisymmetric solution of $\nabla^2\phi = 0$ in spherical polars is

$$\phi = \sum_{n\geq 0} \left(A_n r^n + B_n r^{-n-1}\right) P_n(\cos\theta),$$

where the P_n are the Legendre polynomials. We will only use the first few modes.

- $\phi = -\frac{m}{4\pi r}$ gives $\mathbf{u} = \frac{m}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}$. This is a radial flow. The total outflow over a sphere of radius R is $4\pi R^2 u_r = m$, which is independent of R by mass conservation. This is a *point source* of strength m. (If $m < 0$ it is a *point sink*.)
- $\phi = Ur\cos\theta \equiv Uz$ gives uniform flow again.
- $\phi \propto r^{-2}\cos\theta$ gives *dipole flow*.

Plane polars

The general solution of $\nabla^2\phi = 0$ in plane polars is

$$\phi = K + A_0 \log r + B_0\theta + \sum_{n\geq 1} \left(A_n r^n + B_n r^{-n}\right) e^{\imath n\theta}.$$

We again use only the first few modes.

- $\phi = \frac{m}{2\pi}\log r$ gives $\mathbf{u} = \frac{m}{2\pi}\frac{\hat{\mathbf{r}}}{r}$. This is a radial flow, a line source of strength m.
- $\phi = \frac{\kappa\theta}{2\pi}$ gives $\mathbf{u} = \frac{\kappa}{2\pi}\frac{\hat{\boldsymbol{\theta}}}{r}$. The circulation about a circle of radius R is κ, which is independent of R by $\nabla\wedge\mathbf{u} = 0$. This is a *line vortex* of circulation κ.
- $\phi \propto r^{-1}\cos\theta$ is a 2D dipole.
- $\phi \propto r^2\cos\theta \equiv x^2 - y^2$ is a 2D straining flow.

3.3 Applications

3.3.1 Uniform flow past a sphere

Consider a hard sphere of radius a in a fluid having a uniform velocity U at infinity.

We have to solve the equations

$$\begin{aligned} &\nabla^2\phi = 0 && \text{in } r > a,\\ &\frac{\partial\phi}{\partial r} = 0 && \text{at } r = a,\\ &\text{and } \phi \sim Ur\cos\theta && \text{as } r \to \infty. \end{aligned}$$

We can satisfy the Laplace equation and boundary condition at infinity with

$$\phi = U\cos\theta\left(r + \frac{B}{r^2}\right).$$

The boundary condition at $r = a$ yields $B = \frac{a^3}{2}$. Thus

$$\mathbf{u} = \left(U\cos\theta\left(1 - \frac{a^3}{r^3}\right), -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), 0\right)$$

in spherical polars (r, θ, ϕ).

3.3.2 Uniform flow past a cylinder

Consider a hard cylinder of radius a in a fluid with a uniform velocity U at infinity and with a circulation κ.

We have to solve

$$\begin{aligned}
\nabla^2\phi &= 0 && \text{in } r > a,\\
\frac{\partial\phi}{\partial r} &= 0 && \text{at } r = a,\\
\text{and } \phi &\sim Ur\cos\theta && \text{as } r \to \infty.
\end{aligned}$$

We further need

$$\oint_{r=a} \mathbf{u} \cdot \mathbf{dl} = \kappa = [\phi]_{r=a}$$

to obtain a unique solution. These conditions give

$$\phi = U \cos\theta \left(r + \frac{a^2}{r} \right) + \frac{\kappa\theta}{2\pi}.$$

In plane polars (r, θ), we therefore have

$$\mathbf{u} = \left(U \cos\theta \left(1 - \frac{a^2}{r^2} \right), -U \sin\theta \left(1 + \frac{a^2}{r^2} \right) + \frac{\kappa}{2\pi r} \right).$$

3.4 The pressure in irrotational potential flow with potential forces

The momentum and vorticity equations (2.2, 2.4) become

$$\rho \left(\frac{\partial \nabla \phi}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right) = -\nabla \left(p + \chi \right)$$

and

$$(\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nabla \left(\frac{1}{2} u^2 \right)$$

respectively, and these combine to give

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho u^2 + p + \chi \right) = 0,$$

which integrates to give

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho u^2 + p + \chi = f(t) \text{ independent of } \mathbf{x}. \tag{3.1}$$

Application

We can apply this theory to the free oscillations of a manometer. We need to calculate

$$\phi(\mathbf{x}, t) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u} \cdot \mathbf{dl}.$$

Let s be the arc length from the bottom, with the equilibrium points at $s = -l_1, l_2$. From mass conservation the flow is uniform, so that $u = \dot{h}$ everywhere. Therefore $\phi = \dot{h}s$, and so

$$\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}} = \ddot{h}s.$$

At the interfaces the pressure is constant. Using the equation for potential flow (3.1) we get

$$\rho(-l_1 + h)\ddot{h} + \tfrac{1}{2}\rho\dot{h}^2 + p_a - \rho g h \sin\alpha = \rho(l_2 + h) + \tfrac{1}{2}\rho\dot{h}^2 + p_a + \rho g h \sin\beta.$$

This simplifies to give

$$\ddot{h} = \frac{g(\sin\alpha + \sin\beta)}{l_1 + l_2}h,$$

and so we have SHM (even for large oscillations).

3.5 Bubbles

3.5.1 General theory for spherically symmetric motion

The pressure is $p(r,t)$, and the far-field pressure is $p(\infty,t)$. We have radial motion, $u \propto \frac{1}{r^2}$. If the radius of the bubble is $a(t)$, then $\dot{a} = u_r$ at $r = a$, which gives that $\mathbf{u} = \frac{\hat{\mathbf{r}}}{r^2}\dot{a}a^2 = \nabla\phi$. So $\phi = -\frac{\dot{a}a^2}{r}$ and $\left.\frac{\partial\phi}{\partial t}\right|_r = -\frac{\ddot{a}a^2 + 2\dot{a}^2 a}{r}$. Putting all this together gives

$$-\rho\frac{\ddot{a}a^2 + 2\dot{a}^2 a}{r} + \frac{\rho}{2}\frac{\dot{a}^2 a^4}{r^4} + p(r,t) = p(\infty,t).$$

At $r = a$

$$-\rho\ddot{a}a - \frac{3}{2}\rho\dot{a}^2 = p(\infty,t) - p(a,t),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho a^3\dot{a}^2\right) = a^2\dot{a}\left(p(a,t) - p(\infty,t)\right).$$

This can be interpreted as “rate of change of kinetic energy equals rate of working by pressure forces”.

Another rewrite gives

$$p(r,t) - p(\infty,t) = \left(p(a,t) - p(\infty,t)\right)\frac{a}{r} + \frac{1}{2}\dot{a}^2\left(\frac{a}{r} - \frac{a^4}{r^4}\right)$$

3.5.2 Small oscillations of a gas bubble

Assume $a(t) = a_0 + \delta a(t)$ and that $\delta_a(t)$ is small, p_∞ is constant in time and that the gas in the bubble has pressure such that $\delta p_{\text{gas}} = -\gamma p_\infty \frac{3\delta_a}{a_0}$ (**which can be obtained from PV^γ constant for ideal adiabatic gas**). Neglect surface tension. Linearising

$$-\rho\ddot{a}a - \frac{3}{2}\rho\dot{a}^2 = p(\infty,t) - p(a,t)$$

about $a = a_0$ and $\dot{a} = 0$ we obtain

$$-\rho a_0\ddot{\delta}_a = \frac{3\gamma p_\infty \delta_a}{a_0},$$

which is SHM with $\omega = \left(\frac{3\gamma p_\infty}{\rho a_o^2}\right)^{\frac{1}{2}}$.

3.5.3 Total collapse of a void

Bernoulli implies that the pressure decreases as the speed increases. If this makes the pressure negative, then the liquid will break to form a void filled only with vapour. This has important consequences for valves and propellors (cavitation).

Consider a spherical void (ie $p(a,t) = 0$) at rest with $a = a_0$ and $\dot{a} = 0$ at $t = 0$ with a constant background pressure p_∞. Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho a^3 \dot{a}^2\right) = a^2 \dot{a}\left(p(a,t) - p(\infty,t)\right),$$

so

$$\frac{1}{2}\rho a^3 \dot{a}^2 = \frac{1}{3} p_\infty \left(a_0^3 - a^3\right).$$

Integrate this again (numerically!), to get

$$t_{\text{collapse}} = 0.92 \left(\frac{\rho a_0^2}{p_\infty}\right)^{\frac{1}{2}}.$$

3.6 Translating sphere & inertial reaction to acceleration

3.6.1 Steady motion

Use the inertial frame moving steadily with the sphere. It was shown in §3.3.1 that uniform flow past a fixed sphere has

$$\phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right) \text{ and}$$

$$\mathbf{u} = \left(U\cos\theta\left(1 - \frac{a^3}{r^3}\right), -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), 0,\right).$$

Hence at $r = a$, $|\mathbf{u}| = \frac{3}{2}U\sin\theta$. Potential flow means we can apply

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi \text{ independent of } \mathbf{x},$$

with $\frac{\partial\phi}{\partial t} = 0$ (since steady motion) to compare $r = a$ and $r = \infty$. This gives

$$p(a,\theta) = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right).$$

This pressure distribution is symmetric fore and aft and around the equator, so no force is exerted on a steadily moving sphere (**or indeed any 3D body**). This surprising result is called D'Alembert's paradox.

3.6.2 ** Effects of friction **

D'Alembert's paradox can be understood only by analogy with Newtonian dynamics: in the absence of friction forces are needed only for acceleration and not for uniform motion.

This potential flow result is a good approximation for bubbles in steady motion, because they have slippery surfaces. It is a bad approximation for rigid spheres in steady motion. What is observed in experiments is *separation* and a *wake*.

The small amount of friction produces a thin layer of fluid next to the rigid surface that is slowed down from the potential flow. This thin *boundary layer* is sensitive to the slowing down of the surrounding potential flow from the equator to the rear stagnation point, and detaches from the sphere into the body of the fluid to produce a shear layer of concentrated vorticity (this is called *separation*). Separation also occurs behind other bluff bodies in steady translation but can be suppressed to a certain extent behind streamlined/tapered bodies. Boundary layers are covered in more detail in Part IIB and Part III courses.

We can estimate the drag force as proportional to the projected area (A) times the pressure difference: applying Bernoulli gives

$$\text{drag force} = \tfrac{1}{2} C_D \rho U^2 A,$$

where C_D is a dimensionless coefficient that must be measured experimentally (0.4 for a sphere, 1.1 for a disc, 1.0 for a cylinder).

3.6.3 Accelerating spheres

Potential flow is useful for slippery bubbles, rapid accelerations of small (rigid) particles and small oscillations.

For the accelerating sphere, it is best to have the fluid at ∞ at rest. Consider a sphere of radius a and centre $\mathbf{x}_0(t)$ and velocity $\mathbf{u}(t) = \dot{\mathbf{x}}_0(t)$. The velocity potential problem is then

$$\begin{aligned} \nabla^2 \phi &= 0 \text{ in } r \geq a, \\ \phi &\to 0 \text{ as } r \to \infty, \\ \left.\frac{\partial \phi}{\partial r}\right|_t &= \mathbf{u}(t) \cdot \mathbf{n} \text{ on } r = a. \end{aligned}$$

with solution

$$\begin{aligned} \phi &= -\frac{U \cos\theta a^3}{2r^2} \\ &= -\frac{\mathbf{u} \cdot (\mathbf{x} - \mathbf{x}_0(t))\, a^3}{2\left|\mathbf{x} - \mathbf{x}_0(t)\right|^3} \end{aligned}$$

To calculate the force, we need

$$\left.\frac{\partial \phi}{\partial t}\right|_r = \frac{-\dot{\mathbf{u}} \cdot \mathbf{r} a^3}{2r^3} + \mathbf{u} \cdot (\text{terms linear in } \dot{\mathbf{x}}_0)\,.$$

Now $\nabla\phi =$ terms linear in $\mathbf{u}$. These linear terms must be the same as for steady motion. Hence the pressure on the sphere is $p = p_\infty + \rho\frac{\dot{\mathbf{u}}\cdot\mathbf{r}}{2} + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$. The force on the sphere is given by

$$\int_{r=a} -p\mathbf{n}\,\mathrm{d}A = -\frac{\rho}{2}\int_{r=a}(\dot{\mathbf{u}}\cdot\mathbf{r})\,\frac{\mathbf{r}}{a}\,\mathrm{d}A$$

Now by the isotropy of the sphere, $\int_{r=a} r_i r_j\,\mathrm{d}A = \frac{4\pi a^4}{3}\delta_{ij}$, so

$$\mathbf{F} = -\frac{1}{2}\rho\dot{\mathbf{u}}\frac{4\pi a^4}{3}\frac{1}{a} = -m^*\dot{\mathbf{u}}$$

where $m^* = \frac{1}{2}\rho_{\text{fluid}}\frac{4\pi a^3}{3}$, the added (or effective or virtual) mass.

3.6.4 Kinetic Energy

The kinetic energy of fluid motion in a volume V is

$$\begin{aligned} T &= \int_V \frac{1}{2}\rho u^2\,\mathrm{d}V \\ &= \frac{\rho}{2}\int_V (\nabla\phi)^2\,\mathrm{d}V \\ &= \frac{\rho}{2}\int_V \nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi\,\mathrm{d}V \\ &= \frac{\rho}{2}\int_S \phi\,(\nabla\phi)\cdot\mathbf{dS} \\ &= \int_S \frac{\rho}{2}\phi\,(\mathbf{n}\cdot\nabla)\,\phi\,\mathrm{d}S \end{aligned}$$

For the translating sphere we get $T = \frac{1}{2}m^*U^2$.

3.7 Translating cylinders with circulation

The potential for flow past a uniform cylinder with circulation κ is

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa\theta}{2\pi},$$

with

$$\mathbf{u} = \left(U\cos\theta\left(1 - \frac{a^2}{r^2}\right), -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}\right).$$

On $r = a$, $|\mathbf{u}| = -2\sin\theta U + \frac{\kappa}{2\pi a}$, thus

$$p(a,\theta) = p_\infty + \frac{1}{2}\rho\left(U^2 - \left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)^2\right)$$

and so (by doing the integral), $\mathbf{F} = (0, -\rho U\kappa)$ (in Cartesians). This is a lift force perpendicular to the velocity. The same lift force applies to arbitrary aerofoils (at least to the first approximation).

3.7.1 ** Generation of circulation **

In order to calculate the lift on an aerofoil we thus need to know the circulation κ. Potential flow without circulation would have very large velocities around the sharp trailing edge.

This is strongly opposed by friction, and the flow avoids these high velocities by shedding an eddy as the aerofoil starts to move. This eddy is of such a size as to streamline the flow on the aerofoil.

The circulation around $C_1 + C_2$ is initially zero and so remains zero at all times (by Kelvin's theorem). Hence the circulation around the aerofoil (the *bound vortex*) is equal in magnitude to the circulation in the shed *starting vortex* (but opposite in sign). The strength is chosen to avoid a singularity at the sharp trailing edge, a condition which gives $\kappa \approx \pi l U \sin \alpha$, provided $\alpha < 14°$. For $\alpha > 14°$ the flow separates and the aerofoil stalls.[2]

Since $\nabla \cdot \boldsymbol{\omega} = 0$ vortex lines cannot end in the fluid, and in fact the bound vortex on the wing is connected to the starting vortex by two vortices shed from the wingtips. These are responsible for the observed vapour trails.

[2] See Fluids IIB or Acheson for more details.

3.8 More solutions to Laplace's equation

3.8.1 Image vortices and 2D vortex dynamics

Recall that a single line vortex has

$$\phi = \frac{\kappa\theta}{2\pi}, \text{ with } \mathbf{u} = \frac{\kappa}{2\pi r}(-y, x).$$

Since the potential flow equations are linear we can superpose solutions.

This configuration would give

$$\phi = \sum_i \frac{\kappa_i \theta_i}{2\pi}.$$

This gives us a new method of finding solutions in "nice" geometries. For instance, consider a vortex near a plane wall as shown.

The velocity field is the same as if there was an *image vortex* of opposite strength so that normal velocities cancel. This image produces a velocity $\frac{\kappa}{4\pi d}$ at the real vortex, and so the vortex trundles along parallel to the wall as shown.

Application to dispersal of wingtip vortices

We have vortices at $(\pm x_0(t), y_0(t))$ and so we need to add images at $(\pm x_0(t), -y_0(t))$. Now

$$\dot{x}_0 = \frac{\kappa}{4\pi}\frac{x_0^2}{y_0(x_0^2+y_0^2)} \quad \text{and} \quad \dot{y}_0 = -\frac{\kappa}{4\pi}\frac{y_0^2}{x_0(x_0^2+y_0^2)}.$$

This gives $\frac{\mathrm{d}y_0}{\mathrm{d}x_0} = -\frac{y_0^3}{x_0^3}$, or

$$\frac{1}{x_0^2} + \frac{1}{y_0^2} = C.$$

3.8.2 Flow in corners

We use $\phi = r^\mu \sin \mu\theta$, which satisifies

$$\frac{\partial \phi}{\partial \theta} = 0 \text{ on } \theta = \pm\frac{\pi}{2\mu}.$$

Now $|\mathbf{u}| \propto r^{\mu-1}$, and so there are infinite velocities as $r \to 0$ if $\mu < 1$. The effect of friction here is to introduce a circulation.

Chapter 4

Free Surface Flows

4.1 Governing equations

The flow is assumed to start from rest and is thus irrotational (and remains so). Let the free surface be at $\zeta(x, y, t)$, $|\zeta|$ sufficiently small for nice things to happen. So

$$\nabla^2\phi = 0 \quad \zeta \geq z \geq -h,$$

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left|\nabla\phi\right|^2 + p + \rho g z = f(t),$$

$$p = p_{\text{air}} \text{ at } z = \zeta,$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x}\left.\frac{\partial\phi}{\partial x}\right|_{z=\zeta} + \frac{\partial\zeta}{\partial y}\left.\frac{\partial\phi}{\partial y}\right|_{z=\zeta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=\zeta},$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = 0.$$

We will restrict to the 1-D case but even so, to have any hope of solving this we have to linearise it.

1. Throw out the nonlinear terms.
2. Evaluate at $z = \zeta$ using information at $z = 0$.
3. Throw out new nonlinear terms.

$$\nabla^2\phi = 0 \quad 0 \geq z \geq -h,$$

$$\rho\left.\frac{\partial\phi}{\partial t}\right|_{z=0} + \rho g\zeta = f(t),$$

$$p = p_{\text{air}} \text{ at } z = \zeta,$$

$$\frac{\partial\zeta}{\partial t} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0},$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = 0.$$

We seek separable solutions and obtain $\phi(x,z,t) = A\cosh k\,(z+h)\,e^{i(kx-\omega t)}$. Using the dynamic boundary condition $\rho\,\frac{\partial\phi}{\partial t}\Big|_{z=0} + \rho g\zeta = f(t)$, we obtain the dispersion relation

$$\omega^2 = gk\tanh kh.$$

In deep water, $h \gg \frac{1}{k}$, so $\tanh kh \approx 1$, giving $\omega^2 = kh$ and $c = \sqrt{\frac{g}{k}}$[1]. In shallow water, $\tanh kh \approx kh$ giving $\omega = k\sqrt{gh}$ and $c = \sqrt{gh}$.

4.1.1 Particle paths under a wave

By linearising, we get that

$$x(t) = x_0 - \frac{ia\cosh k\,(z_0+h)}{\sinh kh}e^{i(kx_0-\omega t)}$$
$$z(t) = z_0 + \frac{a\sinh k\,(z_0+h)}{\sinh kh}e^{i(kx_0-\omega t)},$$

which is elliptic motion. x_0 and z_0 are the mean positions of the particles in the (x,z) plane.

In the deep water limit we get circular motion and in the shallow water limit we get mainly horizontal motion.

4.2 Standing Waves

Consider waves in a deep rectangular box, $0 \le x \le a$, $0 \le y \le b$ and $z \le 0$. Look for linearised waves with displacement $\zeta(x,y,t)$. Try separable solutions to obtain

$$\phi(x,y,z,t) = A\cos\frac{m\pi x}{s}\cos\frac{n\pi y}{b}e^{kz}e^{-i\omega t}.$$

The Laplace equation determines k from m and n by

$$k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$

and the dynamic boundary conditions determine $\omega^2 = gk$. This is to be expected, since a standing wave is the sum of progressive waves.

4.2.1 Rayleigh-Taylor instability

Turn the box in the previous section upside down (or equivalently replace g with $-g$). This makes ω imaginary, leading to a $e^{\sqrt{gk}t}$ term, which quickly magnifies any deviations from $\zeta = 0$. The largest growth rate is for large k, but this is cancelled by surface tension in the real world.

4.3 River Flows

These are nonlinear problems, but can be solved since the shallowness of the river in comparision to its length means that the flow is nearly unidirectional.

[1] The wave speed...

4.3.1 Steady flow over a bump

What happens to the free surface at the bump?

We assume that

- the river has vertical sides and constant width;
- it varies only slowly in the x direction, so that the flow is (to a good approximation) horizontal and uniform across any vertical section (since $\frac{\partial u}{\partial z} = \omega_y = 0$);
- the flow is steady;
- far from the bump ($x \to \pm\infty$), $\zeta \to 0$, $h \to h_\infty$ and $U \to U_\infty$.

Then mass conservation gives

$$U(\zeta + h) = U_\infty h_\infty, \tag{4.1}$$

whilst applying Bernoulli's equation (2.3) to the surface streamline gives

$$\frac{1}{2}\rho U^2 + p_{\text{atm}} + \rho g \zeta = \frac{1}{2}\rho U_\infty^2 + p_{\text{atm}}. \tag{4.2}$$

Now eliminate ζ between (4.1) and (4.2) to obtain

$$\frac{1}{2}U^2 + \frac{gU_\infty h_\infty}{U} = \frac{1}{2}U_\infty^2 + gh. \tag{4.3}$$

We can now extract information from (4.3) graphically.

There are thus two roots and three possibilities.

- If the bump is too big then the assumption of steady/slowly varying flow must fail. This gives a hydraulic jump — see §4.3.3.

- If the bump is just right then the flow can pass smoothly from one root the other (e.g. flow over a weir in §4.3.2).
- If the bump is not too large then the flow stays on the same root. This has two subcases.
  - On the left hand root: $U_\infty < \sqrt{gh_\infty}$ — flow slower than shallow water waves — we have

$$\text{bump up} \Rightarrow h \downarrow \Rightarrow \text{RHS} \downarrow \Rightarrow u \uparrow \Rightarrow \zeta \downarrow .$$

    i.e. slow deep flow converts PE to KE to maintain the mass flux.
  - On the right hand root: $U_\infty > \sqrt{gh_\infty}$ — flow faster than shallow water waves — we have

$$\text{bump up} \Rightarrow h \downarrow \Rightarrow \text{RHS} \downarrow \Rightarrow u \downarrow \Rightarrow \zeta \uparrow .$$

    i.e. fast shallow flow converts KE to PE to maintain the mass flux.

Normal rivers are in the slow deep state.

4.3.2 Flow out of a lake over a broad weir

This is the same as §4.3.1 but with a smooth change of branch. We need the bump "just right".

How fast is the outflow as a function of the minimum of $h(x)$?

The lake is large and deep, so we take the limit $h_\infty \to \infty$, $U_\infty \to 0$ with $U_\infty h_\infty = Q$ fixed. Mass conservation gives

$$U(\zeta + h) = Q$$

and Bernoulli on the surface streamline gives

$$\frac{1}{2}U^2 + g\zeta = 0.$$

Eliminating ζ as before gives

$$\frac{1}{2}U^2 + \frac{gQ}{U} = gh(x), \tag{4.4}$$

which is a cubic for unknown $U(x)$ given $h(x)$.

The flow starts high on the LH branch in the lake, but comes out of the weir high on the RH branch. The minimum of $h(x)$ (the crest of the weir) must be at the join of the branches, and so

$$Q = \left(\frac{8}{27} g h_{\text{min}}^3\right)^{\frac{1}{2}},$$

giving the flow rate as a function of the height of the weir.

At the crest of the weir, $U = \left(\frac{2}{3} g h_{\text{min}}^3\right)^{\frac{1}{2}}$ and mass conservation gives

$$\zeta_{\text{crest}} = -\frac{1}{3} h_{\text{min}}$$

and the fluid depth is $\frac{2}{3} h_{\text{min}}$. Also, $U_{\text{crest}}^2 = g(\text{fluid depth})_{\text{crest}}$, so that the flow is travelling at the speed of shallow water waves at the crest. There can therefore be no communication from after the weir back to the lake.

4.3.3 Hydraulic jumps

Given U_1 and h_1 from river data and h_2 from tidal theory can we predict the speed V of the jump (and the flow U_2)?

The energy loss in the turbulent jump makes Bernoulli inapplicable (friction is important in the small-scale unsteady motions). However the flat-bottomed case provides an application of the momentum integral equation.

Change to a frame moving with the jump. Away from the jump there is no vertical acceleration and so the pressure is hydrostatic there.

Taking the horizontal component of

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = -\int_A \left(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\right)\,\mathrm{d}A + \int_V \mathbf{F}^V\,\mathrm{d}V,$$

and noting that the flow is steady on average we obtain (from the area integral)

$$\rho(V+U_1)^2h_1 + (p_ah_1 + \frac{1}{2}\rho g h_1^2) + p_a(h_2-h_1)$$
$$= \rho(V-U_2)^2h_2 + (p_ah_2 + \frac{1}{2}\rho g h_2^2),$$

which boils down to

$$(V+U_1)^2h_1 - (V-U_2)^2h_2 = \frac{1}{2}g(h_2^2-h_1^2).$$

Mass conservation gives $(V+U_1)h_1 = (V-U_2)h_2$, and eliminating $V-U_2$ between these equations gives (eventually)

$$(V+U_1)^2 = \frac{g(h_1+h_2)h_2}{2h_1}.$$

We can solve this for V and then solve

$$(V-U_2)^2 = \frac{g(h_1+h_2)h_1}{2h_2}$$

for U_2.

Note that if $h_2 > h_1$ then $V-U_2 < \sqrt{gh_2}$ and $V+U_1 > \sqrt{gh_1}$ — the jump travels faster than shallow water waves on the river side and overtakes all information of its future arrival.

References

- D.J. Acheson, *Elementary Fluid Dynamics*, OUP, 1990.

  This is an excellent book, easy to read and with everything in. It is also good for Fluids IIB. Highly recommended.

- G.K. Batchelor, *An Introduction to Fluid Dynamics*, CUP, 1967.

  The lecturer recommended this, and it is a reasonably good book for Fluids IIB. Personally I think you'd be wrong in your head to buy it for this course, but YMMV.

- M. van Dyke, *An Album of Fluid Motion*, The Parabolic Press, 1982.

  Lots of pictures of flows. An excellent book. Go out and buy it. Now.

Related courses

In Part IIA there are courses on *Transport Processes* and *Theoretical Geophysics*. The IIB fluids courses are *Fluid Dynamics 2* and *Waves in Fluid and Solid Media*, both of which use the material in this course to some extent.