Geometry

Dr. N.I. Shepherd-Barron

Easter 1996

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Contents

Introduction

These notes are based on the course "Geometry" given by Dr. N.I. Shepherd-Barron in Cambridge in the Easter Term 1996. These typeset notes are totally unconnected with Dr. Shepherd-Barron.

These notes are incomplete. If you have a problem with this, then you can sort them out yourself. Dr. Shepherd-Barron has an updated version on his web page.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
Analysis	Further Analysis
Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems
Combinatorics	Bifurcations in Nonlinear Convection

They may be downloaded from

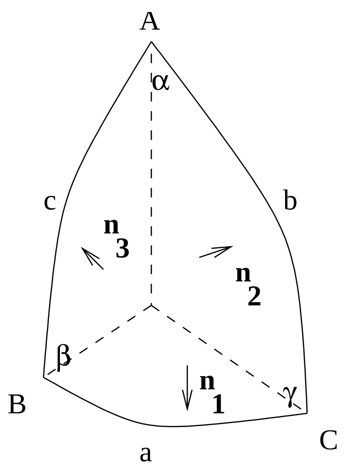
`http://www.istari.ucam.org/maths/`.

Chapter 1

Spherical Trigonometry

1.1 Introduction

Fix a sphere S in $\mathbb{R}^3$ with centre 0 and radius 1. A line on S is a great circle (e.g. the equator). Given any two non-antipodal points P and Q on S, there exists just one great circle through P and Q. A spherical triangle looks like



where AB, BC and AC are segments of great circles. The length of the line AB is the angle subtended at 0. Any great circle is $S \cap H$, where H is a plane through the origin. α is defined as the angle between the two relevant planes.

$\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are the unit normals and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are the position vectors of A, B and C. Note that

$$\mathbf{n}_1 = \frac{\mathbf{C} \times \mathbf{B}}{\sin a} \qquad \mathbf{n}_2 = \frac{\mathbf{A} \times \mathbf{C}}{\sin b} \qquad \text{and} \quad \mathbf{n}_3 = \frac{\mathbf{B} \times \mathbf{A}}{\sin c}.$$

Theorem 1.1. $\sin a \sin b \cos \gamma = \cos c - \cos a \cos b$.

Proof. Use $(\mathbf{C} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{C} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{A})$. Now $|\mathbf{C}| = 1$, so $(\mathbf{C} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{A})$.

Now

$$
\begin{aligned}
-\cos\gamma = \mathbf{n}_1 \cdot \mathbf{n}_2 &= \frac{(\mathbf{C}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C})}{\sin a \sin b} \\
&= \frac{(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}.\mathbf{C}) - (\mathbf{B}\cdot\mathbf{A})}{\sin a \sin b} \\
&= \frac{\cos b \cos a - \cos c}{\sin a \sin b}.
\end{aligned}
$$

□

Theorem 1.2. $\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha\cos\beta$.

Proof. Use the same identity on $\mathbf{n}_2 \times \mathbf{n}_3 = \mathbf{A}\sin\alpha$, $\mathbf{n}_3 \times \mathbf{n}_1 = \mathbf{B}\sin\beta$ and $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{C}\sin\gamma$. Now

$$
\begin{aligned}
\sin\alpha\sin\beta\cos c &= (\mathbf{n}_2 \times \mathbf{n}_3)\cdot(\mathbf{n}_3 \times \mathbf{n}_1) \\
&= (\mathbf{n}_1 \cdot \mathbf{n}_3)(\mathbf{n}_2 \cdot \mathbf{n}_3) - (\mathbf{n}_1 \cdot \mathbf{n}_2) \\
&= \cos(\pi - \beta)\cos(\pi - \alpha) - \cos(\pi - \gamma) \\
&= \cos\gamma + \cos\alpha\cos\beta.
\end{aligned}
$$

□

Theorem 1.3.

$$
\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}.
$$

Proof. Use $(\mathbf{A}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{B}) = (\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}))\mathbf{C}$.

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{B}) &= -(\mathbf{n}_1 \times \mathbf{n}_2)\sin a \sin b \quad \text{and} \\
\mathbf{n}_1 \times \mathbf{n}_2 &= \mathbf{C}\sin\gamma \quad \text{so} \\
-\sin a \sin b \sin\alpha\, \mathbf{C} &= (\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}))\mathbf{C}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) &= \sin a \sin b \sin\gamma \\
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sin b \sin c \sin\alpha \\
\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) &= \sin c \sin a \sin\beta.
\end{aligned}
$$

Divide by $\sin a \sin b \sin c$ to get result. □

These results can be compared to the Euclidean case, when a, b and c are very small. Theorem 1.1 gives the cosine rule, theorem 1.2 is uninteresting and theorem 1.3 gives the sine rule.

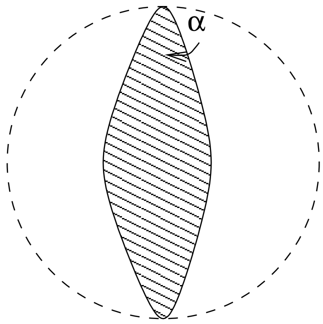
The triangle inequality ($c \leq a + b$) can also be deduced if α, β and γ are less than $\frac{\pi}{2}$ and a, b and c are less than π.

$$
\begin{aligned}
\cos c - \cos a \cos b &= \sin a \sin b \cos\gamma \quad \text{so} \\
\cos c &\geq \cos a \cos b \\
&\geq \cos(a + b) \quad \text{thus} \\
c &\leq a + b.
\end{aligned}
$$

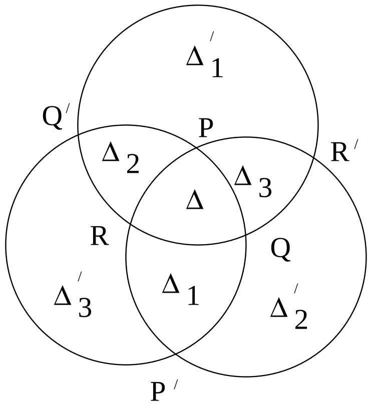
1.2 Areas of Spherical Triangles

Theorem 1.4. *Suppose Δ is a spherical triangle with angles α, β and γ. Then the area of Δ is $\alpha + \beta + \gamma - \pi$.*

Proof. Suppose A and B are antipodal points on the unit sphere S and suppose we have two great circles through A and B. These 2 circles cut S into 4 pieces called lunes.



The area of the lune is $4\pi\frac{\alpha}{2\pi} = 2\alpha$.



P', Q' and R' are the antipodes of P, Q and R respectively and Δ_1', Δ_2' and Δ_3' are the antipodal triangles of Δ_1, Δ_2 and Δ_3 respectively. Δ' is the antipodal triangle of Δ, which is the exterior of the figure shown. Note that $\Delta + \Delta_1$, $\Delta + \Delta_2$ and $\Delta + \Delta_3$ are lunes with areas of 2α, 2β and 2γ respectively[1].

Now $S \subset \mathbb{R}^3$, and the transformation sending x to its antipodes x' is the matrix $-I$, which is area-preserving. Thus $\Delta = \Delta'$ and so on. Hence $\Delta + \Delta_1 + \Delta_2 + \Delta_3 = \Delta' + \Delta_1' + \Delta_2' + \Delta_3'$. But these 8 triangles make up the whole sphere, and thus $\Delta + \Delta_1 + \Delta_2 + \Delta_3 = 2\pi$. From the lunes, $3\Delta + \Delta_1 + \Delta_2 + \Delta_3 = 2(\alpha + \beta + \gamma)$, and thus $\Delta = \alpha + \beta + \gamma - \pi$. □

The area thus depends only on the angles. But the sides determine the angles, and thus the area.

Theorem 1.5 (Polygons on the sphere). *Suppose Π is an n-gon on S with interior angles $\sigma_1, \ldots, \sigma_n$. Then the area of Π is $\sum_i \sigma_i - (n-2)\pi$.*

[1]In an abuse of notation Δ_X will be either the triangle or its area.

Proof. Cut Π into $n-2$ triangles (prove this is possible by induction). Suppose the angles of Δ_i are α_i, β_i and γ_i. Then the area of Π is

$$\begin{aligned}\sum_{i=1}^{n-2} \Delta_i &= \sum_{i=1}^{n-2} (\alpha_i + \beta_i + \gamma_i) - (n-2)\pi \\ &= \sum_{i=1}^{n} \sigma_n - (n-2)\pi.\end{aligned}$$

$\square$

Corollary 1.6 (Gauss-Bonnet Formula). *Suppose that S is cut into polygons labelled $\Pi_1, \dots, \Pi_F$. Say there are E edges and V vertices in total. Then $V - E + F = 2$.*

Proof. Suppose that Π_i has n_i edges and its interior angles sum to τ_i. Note that $\sum \tau_i = 2\pi V$ and $\sum_{i=1}^{F} n_i = 2E$. Then

$$\begin{aligned}4\pi = \sum_{i=1}^{F} \Pi_i &= \sum_{i=1}^{F} (\tau_i - (n_i - 2)\pi) \\ &= 2\pi(V - E + F).\end{aligned}$$

$\square$

1.3 Sterographic projection of S into $\mathbb{C}$

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. $\mathbb{C}$ has a co-ordinate ζ. Near the point at infinity, use the co-ordinate $\omega = 1/\zeta$. Thus to make calculations at or near infinity, use ω instead of ζ.

Consider $P \in S$ and $\phi : S \mapsto \mathbb{C}_\infty$ be the map defined by making N, P and $\phi(P)$ colinear. To get an explicit formula for ϕ, take $P = (x, y, z)$. We know that $\phi(P) = t(x, y, z) + (1 - t)(0, 0, 1)$ for some $t \in [0, 1]$. Thus $zt + 1 - t = 0$ and $t = 1/(1 - z)$ and

$$\phi(P) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

N.B. $\zeta = \frac{x+\imath y}{1-z}$ *and the north pole corresponds to the point at infinity.*

Recall that $\mathbb{C}_\infty$ has the group of Möbius transforms acting on it and S has $SO(3)$. If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a 2×2 complex matrix with non-zero determinant, then it acts on $\mathbb{C}_\infty$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \zeta = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}.$$

Theorem 1.7. *Via ϕ, every rotation of S gives rise to a Möbius transform on $\mathbb{C}_\infty$. (Not every Möbius transform comes from a rotation.)*

Proof. Step 1. Deal with rotations about the z axis through an arbitrary angle θ ($R_{z,\theta}$). This is the same as rotating the complex plane through θ about 0, accomplished by

$$\begin{pmatrix} e^{\imath\theta/2} & 0 \\ 0 & e^{-\imath\theta/2} \end{pmatrix}.$$

Step 2. Now look at a rotation $R_{y,-\frac{\pi}{2}}$. This is a 3×3 orthogonal matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and $\zeta \mapsto \zeta' = \frac{z+\imath y}{1-x}$. The Möbius transform $\zeta \mapsto \frac{\zeta-1}{\zeta+1}$ does the trick. (Proof by churn.)

Step 3. Now $R_{\nu,-\pi/2}$ for any horizontal ν. Set ψ to be the angle between ν and the y axis. Then

$$R_{\nu,-\pi/2} = R_{z,\phi} R_{y,-\pi/2} \left(R_{z,\psi}\right)^{-1},$$

and thus $R_{\nu,-\pi/2}$ gives a Möbius map.

Step 4. Now a general rotation $R_{\nu,\theta}$. Rotate ν about the x axis to ν', which is horizontal. Then $\nu' = R_{x,\psi}(\nu)$ for some ψ. Hence

$$R_{\nu,\theta} = R_{x,\psi} R_{\nu',\theta} \left(R_{x,\psi}\right)^{-1},$$

so the general rotation gives rise to a Möbius map. □

The question remains as to which Möbius transforms arise from rotations. Rotations have 3 real degrees of freedom, whereas Möbius transforms have 6 (0, 1 and ∞ can each go anywhere on $\mathbb{C}_\infty$). In fact, Möbius transforms arising from rotations are the ones given by $A \in SU(2)$. A proof is via quaternions.

Chapter 2

Reflexions and Tessellations

Suppose that in Euclidean space $\mathbb{R}^n$ we have hyperplanes $H_1, \dots, H_N$ (not necessarily containing the origin), with unit normals n_i. Each H_i divides $\mathbb{R}^n$ into two pieces; say $\mathbb{R}^n \setminus H_i = A_i^+ \cup A_i^-$, with A_i^+ being "the vectors on the same side as n_i". Put $\mathcal{C} = \cap_i A_i^+$.

Define the angle $\theta_{ij} \in [0, \pi)$ between H_i and H_j by $n_i.n_j = -\cos\theta_{ij}$. Let s_i denote reflexion in H_i and put $S = \{s_1, \dots, s_N\}$. We shall be interested in the group $W = W_S = \langle s_1, \dots, s_N \rangle$ generated by S and how the regions $w(\mathcal{C})$, where $w \in W$, fit together.

Lemma 2.1. *If H is a side of $\mathcal{C}$ and $w \in W$, then "reflexion in w(H)" is an element of W.*

Proof. If σ is reflexion in H, then $w\sigma w^{-1}$ is reflexion in $w(H)$. $\square$

Notation. *If $\sigma = s_i$, then we sometimes write $A_\sigma^\pm$, H_σ instead of $A_i^\pm$ and H_σ.*

N.B. $\sigma(A_\sigma^\pm) = A_\sigma^\mp$.

Definition 2.2. *For $w \in W$, define the* S-length *of w, $\ell_s(w)$ as the least $p \geq 0$ such that $w = s_{i_1}, \dots, s_{i_p}$, $s_{i_j} \in S$.*

N.B. *If $T \subset S$ and $u \in W_T$, then $\ell_S(u) \leq \ell_T(u)$.*

Assume now that every dihedral angle θ_{ij} is either a fraction of π, $\theta_{ij} = \pi/m_{ij}$ for some $m_{ij} \in \mathbb{N}$ or $\theta_{ij} = 0$. In this latter case, we write $m_{ij} = \infty$.

Lemma 2.3. *Suppose s, $s' \in S$, $s \neq s'$, $T = \{s, s'\}$ and $v \in W_T$. Put $A_s^+ \cap A_{s'}^+ = \mathcal{P}$. Then $v(P)$ is contained in either A_s^+ or A_s^- and in the latter case $\ell_T(sv) = \ell_T(v) - 1$.*

Proof. Suppose that H and H' are the hyperplanes corresponding to s and s' respectively. There are 2 cases to consider.

Case 1: H, H' are parallel. Label the images $v(\mathcal{P})$ by the element v. Clearly $v(\mathcal{P})$ lies in just one of the regions A_s^+, A_s^-. Also, $v(\mathcal{P}) \subset A_s^-$ iff

$$v \in \{s, ss', ss's, ss'ss', \dots\}$$

and in this case $\ell_T(sv) = \ell_T(v) - 1$.

Case 2. The dihedral angle between H and H' is π/m, $m \in \mathbb{N}$. Then take a 2-plane L perpendicular to $H \cap H'$ and divide L into $2m$ equal sectors by lines through

$L \cap H \cap H'$, which we will regard as the origin in L. One of these sectors corresponds to $\mathcal{P}$.

$W_T = \{1, s', s's, s'ss', \ldots, u = (s's\ldots)\} \cup \{s, ss', ss's, \ldots, w = (ss'\ldots)\}$, where $\ell_T(u) = m - 1$ and $\ell_T(w) = m$. Note that W_T is D_{2m}, the dihedral group with $2m$ elements or the symmetry group of a regular m-gon.

$v(\mathcal{P})$ is clearly one of these sectors (draw a picture to convince yourself), and so lies in just one of A_s^+, A_s^-. Moreover, $v(\mathcal{P}) \subset A_s^-$ iff $v \in \{s, ss', \ldots, w\}$, and thus $\ell_T(sv) = \ell_T(v) - 1$. □

This next result is the main step in constructing tessellations of Euclidean space and spheres. By definition, a tessellation of a space is a partition of it into disjoint congruent regions. Sometimes it is demanded that these regions have finite volume.

Theorem 2.4. *If $w \in W$ and $w(\mathcal{C}) \cap \mathcal{C}$ is nonempty, then $w = 1$.*

Proof. Non-examinable. □

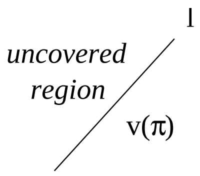
2.1 Regular Polyhedra

We are now back in $\mathbb{R}^3$. Assume $0 \in H_i \ \forall\, i$. Take the unit sphere S. Each H_i cuts S in a great circle, and $\mathcal{C} \cap S$ is a spherical polygon Π.

If $N = 3$ then Π is a triangle, with angles α, β, $\gamma = \pi/p$, π/q, π/r with p, q, $r \geq 2$. The area of Π is $\pi\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right)$, and so $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. Solve these equations to get

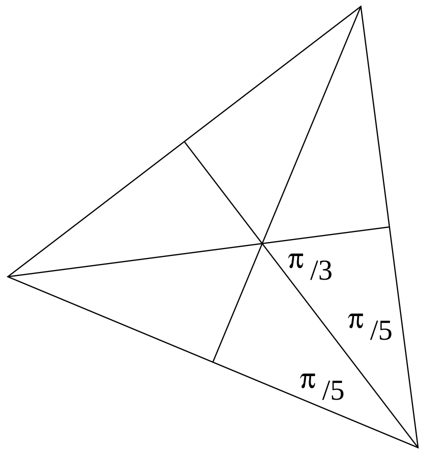
$$\begin{aligned}(p, q, r) =& (2, 2, n), n \geq 2,\\ & (2, 3, 3)\\ & (2, 3, 4)\\ & (2, 3, 5).\end{aligned}$$

Identify reflection of $\mathbb{R}^3$ in H with reflection of S in $S \cap H$. Let W be the group generated by reflections in the sides of Π. Claim that $w(\Pi)$ will cover the sphere. Suppose otherwise, then somewhere on S there is something like:



Now reflect in l. Thus we have covered the sphere with disjoint congruent spherical triangles.

Take $(p, q, r) = (2, 3, 5)$. Then the area of Π is $\pi/30$. Now $w(\Pi)$ tessellates S with $\frac{4\pi}{\pi/30} = 120$ triangles. Use these triangles to construct a regular icosahedron - that is a tessellation of S by 20 congruent equilateral triangles. Group together the 120 triangles 6 at at time as shown:



How many vertices does the icosahedron have? Now $V - E + F = 2$, $E = 30$, $F = 20$, so $V = 12$. So the sphere S is tessellated into 20 congruent equilateral triangles with angles $2\pi/5$. There are 12 vertices and 5 triangles around each vertex. W is a group of symmetries of the icosahedron because W preserves the tessellation. The 6-grouping is unique because the vertices of the big triangles are those points surrounded by 10 little triangles. So W preserves the tessellation into 20 big triangles.

Also, W acts transitively on faces, edges and vertices.

Proof for faces. The elements of W correspond to 120 small triangles, so $w \in W$ corresponds to $w(\Pi)$. Now $|W| = 120$, so $|\operatorname{Orb} F| = \frac{120}{\operatorname{Stab} F}$ and thus $|\operatorname{Orb} F| \geq 20$, so $|\operatorname{Orb} F| = 20$. There is just one orbit, so W acts transitively on the faces. The proof for edges and vertices is similar. $\square$

Also, given a vertex P, there are 5 faces around P. $\operatorname{Stab} P$ acts transitively on these 5 faces. $\operatorname{Stab} P \cong D_{2\times 5}$.

At the same time, we can construct a regular dodecahedron. Take 10 small triangles around P. They form a regular pentagon, and by repeating we get a tessellation of the sphere into 12 regular pentagons – a regular dodecahedron with symmetry properties analogous to those of the icosahedron.

(p, q, r)	shapes	number of little triangles
$(2, 3, 4)$	cube and octahedron	48
$(2, 3, 3)$	tetrahedron	24
$(2, 2, n)$		$4n$

Chapter 3

Hyperbolic Geometry

This is the third kind of 2D geometry where the group of isometries has 2 degrees of freedom.

3.1 Riemannian Metrics

Suppose $U \subseteq \mathbb{R}^2$ with co-ordinates x and y. Then a Riemannian metric on U is an expression $\mathrm{d}s^2 = A\,\mathrm{d}x^2 + 2B\,\mathrm{d}x\,\mathrm{d}y + C\,\mathrm{d}y^2$ such that the matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ is positive definite and } A > 0.$$

Note that A, B and C are not necessarily constant.

Now $\mathrm{d}s^2$ can be used to compute lengths of curves, angles between curves and areas as follows.

Suppose Γ is a path from P to Q in U, $\gamma : [0,1] \mapsto U$ with $\gamma(0) = P$ and $\gamma(1) = Q$. Then the length of Γ is

$$\begin{aligned} \int_\gamma \mathrm{d}s &= \int_{t=0}^{1} \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t \\ &= \int_{t=0}^{1} \sqrt{A\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 2B\,\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} + C\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t, \text{ where } \gamma(t) = (x(t), y(t)). \end{aligned}$$

It is easy to show that if γ' is a different parametrisation of Γ, the length is found to be the same.

Now, suppose that $v = (v_1, v_2)$ and $w = (w_1, w_2)$ starting at P. Then define the angle θ between them by $v.w = \|v\|\,\|w\| \cos\theta$, where $v.w$ =

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} A(P) & B(P) \\ B(P) & C(P) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Define $\|v\| = \sqrt{v.v}$.

If Γ_1 and Γ_2 are two curves meeting at P, then the angle between them is defined to be the angle between their tangent vectors.

As for areas: Suppose we have a small parallelogram in U. Measure the lengths and angles according to $\mathrm{d}s^2$. Then the area is $\sqrt{AC-B^2}\,\delta x\,\delta y$. So given some subset $\Omega \subset U$, the area of Ω is

$$\int_\Omega \sqrt{AC-B^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Definition 3.1. *Suppose* $\mathrm{d}s^2$ *and* $\mathrm{d}u^2$ *are 2 Riemannian metrics on* U*. They are said to be* conformal *if* $\mathrm{d}s^2 = \phi \mathrm{d}u^2$*, where* ϕ *is differentiable and greater than* 0 *on* U*.*

Lemma 3.2. *If* $\mathrm{d}s^2$ *and* $\mathrm{d}\sigma^2$ *are conformal then they define the same notion of angle.*

Proof. Let $\mathrm{d}s^2 = A\,\mathrm{d}x^2 + 2B\,\mathrm{d}x\,\mathrm{d}y + C\,\mathrm{d}y^2$ and $\mathrm{d}\sigma^2 = \alpha\,\mathrm{d}x^2 + 2\beta\,\mathrm{d}x\,\mathrm{d}y + \gamma\,\mathrm{d}y^2$.

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

Call the angle between v and w defined by $\mathrm{d}s^2$ θ_1 and the angle between v and w defined by $\mathrm{d}\sigma^2$ θ_2. Similarly, let $\|v\|_1$ be the norm defined by $\mathrm{d}s^2$ and $(v.w)_1$ be the dot product from $\mathrm{d}s^2$ (and so on for $\|v\|_2$ and $(v.w)_2$).

Now

$$\begin{aligned}
(v.w)_1 &= \|v\|_1 \|w\|_1 \cos\theta_1 \\
&= v^T \begin{pmatrix} A & B \\ B & C \end{pmatrix} w \\
&= \phi v^T \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} w \\
&= \phi (v.w)_2 \\
\Rightarrow \phi(P)(v.w)_2 &= \phi(P) \|v\|_2 \|w\|_2 \cos\theta_1 \\
&= \|v\|_2 \|w\|_2 \cos\theta_2 \\
\Rightarrow \cos\theta_2 &= \cos\theta_1
\end{aligned}$$

□

3.2 The Hyperbolic Plane

Definition 3.3. *Define the hyperbolic plane* H *as* $\{z \in \mathbb{C} \mid \Im z > 0\}$.

Definition 3.4. *Define* $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ *– the hyperbolic metric.*

The notion of angle is the same as in the Euclidean case, but lengths and areas are different.

An isometry of H is one which preserves the hyperbolic metric - that is if $g(x,y) = (\xi,\eta)$, then $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2} = \frac{\mathrm{d}\xi^2+\mathrm{d}\eta^2}{\eta^2}$.

Let

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha,\beta,\gamma,\delta \in \mathbb{R}, \alpha\delta - \beta\gamma = 1 \right\} = SL_2(\mathbb{R}).$$

Now G acts as a group of Möbius transforms on $\mathbb{C}_\infty$ and preserves the real line $\mathbb{R}$. Thus G acts as a group of Möbius transforms on H, and preserves $\mathbb{C} \setminus \mathbb{R} = H \cup H_-$. (Need to check that $g \in G$ cannot flip H and H_- – not hard.)

Proposition 3.5. *G preserves the hyperbolic metric.*

Proof. We will work with z and $\bar{z}$, thus

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{((z-\bar{z})/2\imath)^2} = \frac{-4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(z-\bar{z})^2}.$$

Now take $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and set $\zeta = g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$.

Now

$$\mathrm{d}\zeta = \frac{\alpha\,\mathrm{d}z\,(\gamma z+\delta) - \gamma\,\mathrm{d}z\,(\alpha z+\beta)}{(\gamma z+\delta)^2} = (\gamma z+\delta)^{-2}\mathrm{d}z,$$

and $\mathrm{d}\bar{\zeta} = (\gamma\bar{z}+\delta)^{-2}\mathrm{d}\bar{z}$. Then put everything together – it works! □

Definition 3.6. *A hyperbolic line in H (or a H-line) is either a semi-circle meeting $\mathbb{R}$ at right-angles or a vertical line. We shall see that these H-lines minimize distance in H.*

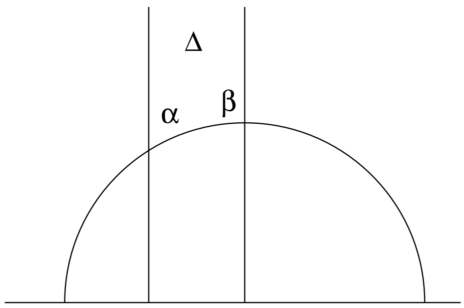
It follows from facts about circles that given two H-lines L and M there are 3 possibilities.

1. L meets M at 1 point in H.
2. L meets M at 1 point in $\mathbb{R} \cup \{\infty\}$. In this case, L and M are said to be parallel.
3. L and M do not meet – they are said to be ultraparallel.

If L and M are not ultraparallel then we can define an angle between them. In case 1, take it to be the Euclidean angle between them, otherwise the angle is 0.

Definition 3.7. *A hyperbolic triangle is a region defined by 3 H-lines, no two of which are ultraparallel.*

Example.



Δ has three angles, α, β and $\gamma - \gamma = 0$.

Proposition 3.8. *The area of Δ is $\pi - (\alpha+\beta+\gamma)$.*

To prove this, we need a few facts about maps preserving the Riemannian metric. Suppose γ is a curve from P to Q, and g takes γ to γ_1. Now g preserves $\mathrm{d}s^2$, so $\mathrm{d}s$ is preserved and so is

$$\int \mathrm{d}s = \text{length}.$$

Now given $\Omega \subset U$, the area of Ω is

$$\int_\Omega \sqrt{AC - B^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Let $g(x, y) = (\xi, \eta)$, so that

$$\mathrm{d}s^2 = \begin{pmatrix} \mathrm{d}\xi & \mathrm{d}\eta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix} = \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}.$$

Now

$$\begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = J \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}.$$

So

$$J^T \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} J = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

and thus $(\det J)^2(\alpha\gamma - \beta^2) = AC - B^2$ and $\sqrt{AC - B^2} = |\det J|\,\sqrt{\alpha\gamma - \beta^2}$. So the area of Ω is

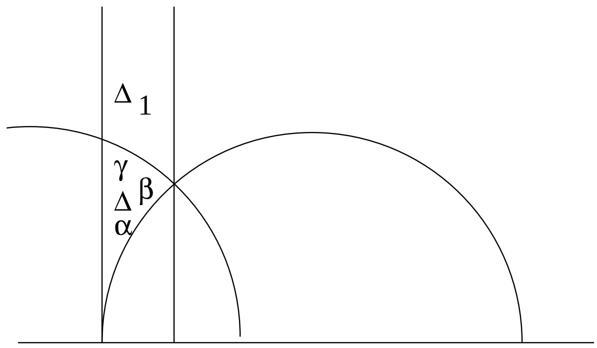
$$\int_\Omega \sqrt{AC - B^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{g(\Omega)} |\det J|\,\sqrt{\alpha\gamma - \beta^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{g(\Omega)} \sqrt{\alpha\gamma - \beta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

which is the area of $g(\Omega)$.

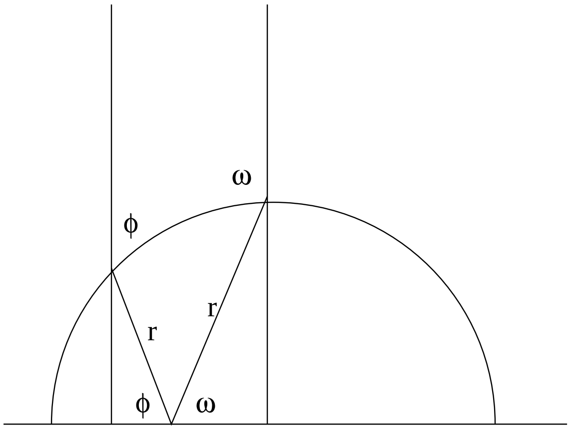
We are now in a position to prove the proposition.

Proof. $\exists\ g \in G$ taking a side of Δ to a vertical line. If one of the sides of Δ is a semi-circle from $P = (t, 0)$ to Q, then $g = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ shifts P to 0. Thus we may assume that $P = 0$. Now if $Q = (s, 0)$, $g = \begin{pmatrix} -1 & 0 \\ s^{-1} & -1 \end{pmatrix}$ shifts Q to ∞.

Now, we may assume we have something looking like:



$$\mathrm{Area}(\Delta + \Delta_1) = \int_{\Gamma_1+\Gamma_2+\Gamma_3} \frac{\mathrm{d}x}{y} = \int_{\Gamma_2} \frac{\mathrm{d}x}{y}.$$



Put $z = re^{\imath\theta}$, so $\mathrm{d}x = -r\sin\theta\mathrm{d}\theta$ and $y = r\sin\theta$. Thus the required integral is

$$\int_{\Gamma_2} -\mathrm{d}\theta = \int_{\pi-\phi}^{\omega} -\mathrm{d}\theta = \pi - (\phi + \omega).$$

Thus the area of Δ_1 is $\pi - (\pi - \beta + \delta)$ and the area of $\Delta + \Delta_1$ is $\pi - (\alpha + \gamma + \delta)$. Thus the area of Δ is $\pi - (\alpha + \beta + \gamma)$. □

3.3 Another look at the hyperbolic plane

Now introduce $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ with $\mathrm{d}\sigma^2 = \frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1-z\bar{z})^2}$. We firstly want to find a Möbius map ψ taking Δ to H and we then want to show that ψ is an isometry.

Now there exists a unique Möbius map with the properties that:

$$\begin{aligned} -1 &\mapsto 0, \\ 0 &\mapsto \imath \text{ and} \\ 1 &\mapsto \infty. \end{aligned}$$

Let $\psi(z)$ be this map, that is $\psi(z) = -\imath\frac{z+1}{z-1}$. Now $\psi(\imath)$ is real, so ψ must take the unit circle to $\mathbb{R} \cup \{\infty\}$. Thus ψ must take Δ to either H or H^-. But $\psi(0) = \imath$, so ψ takes Δ to H.

A hyperbolic line in Δ is a circle meeting $\partial\Delta$ at right-angles. Since ψ is Möbius, it takes hyperbolic lines in Δ to hyperbolic lines in H.

Proposition 3.9. *ψ is an isometry.*

Proof. Let $\psi(z) = \zeta$. Then $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2} = \frac{-4\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}}{(\zeta-\bar{\zeta})^2}$. Then

$$\mathrm{d}\zeta = \frac{-\imath\,\mathrm{d}z(z-1) + \imath(z+1)\mathrm{d}z}{(z-1)^2} = \frac{2\imath\,\mathrm{d}z}{(z-1)^2}$$
$$\mathrm{d}\bar{\zeta} = \frac{-2\imath\,\mathrm{d}\bar{z}}{(\bar{z}-1)^2}.$$

Now, substituting for $\mathrm{d}\zeta$ and $\mathrm{d}\bar{\zeta}$, we get

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 - z\bar{z})^2},$$

and thus ψ is an isometry. □

On H we have $G = SL_2(\mathbb{R})$ preserving $\mathrm{d}s^2$. Thus $\psi^{-1}g\psi$ is a Möbius transform from Δ to Δ preserving $\mathrm{d}\sigma^2$. So $\Gamma = \{\psi^{-1}g\psi \mid g \in G\}$ acts as a group of Möbius transforms on Δ preserving $\mathrm{d}\sigma^2$. These are the 2×2 complex matrices A such that

$$A^* J A = J, \text{ where } J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

N.B. *In* Γ, $\operatorname{Stab} 0 = \left\{ \begin{pmatrix} e^{\imath\theta/2} & 0 \\ 0 & e^{-\imath\theta/2} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$

Proposition 3.10. *In Δ and H, given $P \neq Q$, $\exists!$ hyperbolic line joining P to Q.*

Proof. We will prove this in Δ.

If $P = 0$ the hyperbolic lines are precisely the diameters, so given $Q \neq 0$ $\exists!$ diameter through Q.

If $P = \zeta \neq 0$, claim $\exists\ \gamma \in \Gamma$ such that $\gamma(\zeta) = 0$. To show this, go back to the upper half plane. We must show that given $z \in \mathbb{C}, \exists\ g \in G$ such that $g(z) = \imath$. Let $z = x + \imath y$, and then put

$$g = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}^{-1}.$$

This works, and reduces the problem to the previous case. □

Now we want to define a distance in Δ or H. If $P \neq Q$ we will define the distance from P to Q as the length[1] of the unique hyperbolic line joining P and Q, and 0 if $P = Q$. We will compute this in Δ when one point $P = 0$. Put Q on $\mathbb{R}^+$ (say $Q = X$). Call the H-line joining P and Q γ.

$$\ell(\gamma) = \int_P^Q \mathrm{d}s.$$

$$\text{Also, } \mathrm{d}s = \sqrt{\mathrm{d}\sigma^2} = \frac{2\sqrt{\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}}}{1 - \zeta\bar{\zeta}} = \frac{2\mathrm{d}x}{(1 - x^2)}. \quad \text{Thus}$$

$$\ell(\gamma) = 2\int_{t=0}^{X} \frac{\mathrm{d}t}{1 - t^2} = \log\frac{1 + X}{1 - X}.$$

[1] Measured according to $\mathrm{d}s^2$ or $\mathrm{d}\sigma^2$.

3.4 Geodesics

We want to find the paths which minimise distance in Δ. We shall use the calculus of variations to minimise

$$\int \mathrm{d}s = \int \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t.$$

Now $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2(\dot{x}^2+\dot{y}^2)^{1/2}}{1-(x^2+y^2)}$, thus we wish to minimise

$$\int \frac{2(\dot{x}^2+\dot{y}^2)^{1/2}}{1-(x^2+y^2)}\mathrm{d}t.$$

Use polars, so that $x = r\cos\theta$ and $y = r\sin\theta$, to get

$$F = \frac{2(\dot{r}^2 + r^2\dot{\theta}^2)^{1/2}}{1-r^2}.$$

Thus, by the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{r}} = F_r$$
$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{\theta}} = F_\theta.$$

By applying a Möbius map we may assume that $P = 0$.

Now $F_\theta = 0$, so $\frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{\theta}} = 0$. Now $F_{\dot{\theta}} = \frac{(\dot{r}^2+r^2\dot{\theta}^2}{1-r^2}$, and evaluating at $r = 0$ gives that $F_{\dot{\theta}} = 0$, so $\dot{\theta} = 0$.

Thus in Δ, the geodesics are the hyperbolic lines. It is an obvious corollary that the same result holds in H.

N.B. *Given $P, Q \in \Delta$, the distance from P to Q gives a metric (in metric space sense).*

Theorem 3.11. *Take $P \in \Delta$ and fix $r > 0$. Then the hyperbolic circle C with hyperbolic centre P and hyperbolic radius r (that is $\{z \in \Delta : d_{hyp}(P,z) = r\}$) is a Euclidean circle, but possibly with a different centre and radius.*

Proof. We may assume that $P = 0$ (otherwise apply a Möbius isometry – which preserves both Euclidean and hyperbolic circles).

If h is Möbius and in $\mathrm{Stab}\,0$ then $h(C) = C$. But then h is a rotation, so C is rotationally invariant and thus a Euclidean circle. □

Theorem 3.12. *In Δ, Euclidean circles are hyperbolic circles.*

Proof. Given a Euclidean circle C in Δ, we may rotate it so that its centre is on $\mathbb{R}$, so we may assume that its centre is on $\mathbb{R}$. Now consider the Möbius isometries

$$\left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Consider $g_t(C)$. For $t \gg 0$, the centre of $g_t(C) > 0$ and for $t \ll 0$ the centre of $g_t(C) < 0$. So $\exists\, t' \in \mathbb{R}$ such that the centre of $g_{t'}(C)$ is 0.

We may now assume that C is centred about 0. C is now rotationally invariant about 0 – but these are Möbius isometries, so that C is a hyperbolic circle. □

The obvious corollary is that in H, Euclidean circles are hyperbolic circles.

Theorem 3.13. *Any invertible holomorphic map $g : \Delta \mapsto \Delta$ (or $H \mapsto H$) is Möbius. Any isometry $\Delta \mapsto \Delta$ (or $H \mapsto H$) is either holomorphic or has a holomorphic conjugate.*

Idea of proof. g preserves $\mathrm{d}s^2$ and thus geodesics. So it takes H-lines to H-lines. Then show that g is determined by 3 points. Then show that 2 triples can be mapped to each other by a Möbius transformation. □

3.5 Hyperbolic Trigonometry

We now show another model of the hyperbolic plane. In $\mathbb{R}^3$ consider Ω defined by $x^2 + y^2 - z^2 = -1$ and $z > 0$. Define a map ψ from Ω to Δ,

$$(x, y, z) \mapsto \frac{x + \imath y}{z + 1}.$$

Then hyperbolic lines in the disc correspond to planes through 0 which cut Ω. We define a new dot product on $\mathbb{R}^3$,

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = ax + by - cz.$$

Then

$$\Omega = \left\{ u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid u.u = -1, z > 0 \right\}.$$

There is a notion of cross product such that

$$u.(v \times w) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix},$$

with the properties that $v \times w = -w \times v$ and $(x \times y).(z \times t) = (x.t)(y.z) - (x.z)(y.t)$.

Consider the group Γ of linear maps which preserve the dot product with determinant $+1$. Then $A \in \Gamma$ iff $A^T J A = J$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ as before.}$$

The positive determinant prevents A from flipping Ω to its opposite with $z < 0$.

Proposition 3.14. *Elements of Γ correspond by ψ to Möbius isometries of Δ. (That is, given $A \in \Gamma$, $\psi A \psi^{-1}$ is Möbius.)*

Proof. First we construct a Riemannian metric on Ω such that ψ is an isometry — $\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ does the trick. It is not Riemannian on $\mathbb{R}^3$ (it's Lorentzian), but its restriction to Ω is Riemannian. On Ω, $x\,\mathrm{d}x + y\,\mathrm{d}y - z\,\mathrm{d}z = 0$, substitute for $\mathrm{d}z$ in $\mathrm{d}\sigma^2$.

Let $\zeta = \psi(x, y, z) = \frac{x+\imath y}{z+1}$. Now

$$\mathrm{d}\zeta = \frac{(z+1)\,\mathrm{d}z - x\,\mathrm{d}z + \imath(z+1)\,\mathrm{d}y - y\,\mathrm{d}z}{(z+1)^2},$$

and on Δ $\mathrm{d}s^2 = \frac{4\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}}{(1-\zeta\zeta)^2}$. Substitute away... it works!

Now, suppose L is a plane in $\mathbb{R}^3$ with $0 \in L$ and $L \cap \Omega \neq \emptyset$. If $(0,0,1) \in L$, then $\psi(L \cap \Omega)$ is a straight line $\lambda\xi + \mu\eta = 0$, with $\zeta = \xi + \imath\eta$. For general L, $\exists$ many $A \in \Gamma$ such that $(0,0,1) \in A(L)$. Now $A(L) \mapsto$ a hyperbolic line, say M, so $L \mapsto \psi A^{-1}\psi^{-1}(M)$, which is a hyperbolic line. $\square$

We can use Ω to derive formulae in Δ.

Proposition 3.15. *If $\zeta, \zeta_1 \in \Delta$, with $\zeta = \psi(u)$ and $\zeta_1 = \psi(u_1)$ then* $\mathrm{dist}(\zeta, \zeta_1) = \cosh^{-1} -u.u_1$.

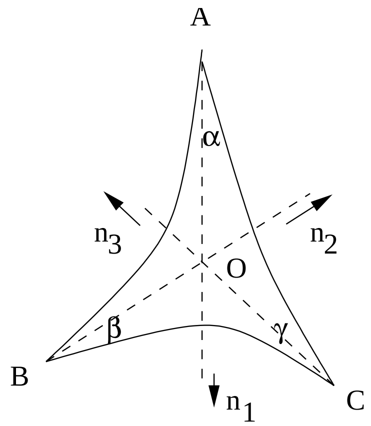
Proof. The LHS is invariant under Möbius isometries and the RHS is invariant under Γ. Therefore we may assume that $\zeta_1 = 1$ so that $u_1 = (0,0,1)$. Then if $\zeta = \frac{x+\imath y}{z+1}$, $-u.u_1 = z$. Now $\cosh \mathrm{dist}(0,\zeta) = z$, so we have proved that

$$\mathrm{dist}(0,\zeta) = \log\left|\frac{1+|\zeta|}{1-|\zeta|}\right|.$$

$\square$

3.6 Hyperbolic Trigonometry

In Ω we have:



A, B, $C \in \Omega$ and α, β, γ are the angles in the image in Δ.
We want to find these formulae, analogous to those found in the spherical case:

$$\sin\alpha\sin\beta\cosh c = \cos\gamma + \cos\alpha\cos\beta$$
$$\sinh a\sinh b\cos\gamma = -\cosh c + \cosh a\cosh b$$
$$\frac{\sin a}{\sinh\alpha} = \frac{\sin b}{\sinh\beta} = \frac{\sin c}{\sinh\gamma}$$

We can make $\mathbf{n}_i \cdot \mathbf{n}_i = 1$, since vectors pointing out of the cone are positive.

Lemma 3.16.

$$\mathbf{n}_1 \times \mathbf{n}_2 = -\sin\gamma C$$

Unfinished

If you've got this far... the book by Rees is probably the best for this course. When you've read and understood it, you can complete these notes and remove all of the errors in the previous N pages.

Have fun,
Paul