Methods

Paul Metcalfe

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These notes are maintained by Paul Metcalfe.
Comments and corrections to pdm23@cam.ac.uk.

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The following people have maintained these notes.

– date Paul Metcalfe

Contents

Introduction

These notes are based on the course "Methods" given by Dr. E.P. Shellard in Cambridge in the Michælmas Term 1996. These typeset notes are totally unconnected with Dr. Shellard. They are more vaguely based on the course than my notes usually are, and I have mainly used Dr. Shellard's notes to get a sense of ordering and content.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
Analysis	Further Analysis
Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems
Combinatorics	Bifurcations in Nonlinear Convection

They may be downloaded from

`http://www.istari.ucam.org/maths/`.

Chapter 1

Fourier series

1.1 Properties of sine and cosine

Consider the set of functions

$$g_n(x) = \cos\frac{n\pi x}{L} \qquad h_n(x) = \sin\frac{n\pi x}{L}$$

with $n \in \mathbb{N}$. These functions are periodic on $[0, 2L]$ and are also *mutually orthogonal*:

$$\int_0^{2L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} L\delta_{mn} & m, n \neq 0 \\ 0 & m = n = 0. \end{cases}$$

$$\int_0^{2L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = 0$$

$$\int_0^{2L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} L\delta_{mn} & m, n \neq 0 \\ 2L\delta_{0n} & m = 0. \end{cases}$$

These properties are easy to verify by direct integration.

In fact the functions g_n, h_n form a *complete orthogonal set*; they span the space of functions periodic on $[0, 2L]$.

1.2 Definition of Fourier series

We can expand any sufficiently well-behaved real periodic function $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{L}, \tag{1.1}$$

where a_n and b_n are constants such that the series is convergent for all x. They are called *Fourier coefficients* and can be found using the results on orthogonality of $\sin$ and $\cos$:

$$\int_0^{2L} f(x)\sin\frac{m\pi x}{L}\,\mathrm{d}x = \sum_{n=1}^{\infty} b_n\delta_{nm} = Lb_m$$
$$\int_0^{2L} f(x)\cos\frac{m\pi x}{L}\,\mathrm{d}x = La_m.$$

Note that the $\frac{1}{2}a_0$ is (1.1) is not a typo, but the $\frac{1}{2}$ is required for the above integral to work for all n. Note also that the particular interval used doesn't matter, provided it is of length $2L$.

Example: sawtooth wave

Define $f(x)$ by

$$f(x) = x \qquad -L < x \leq L$$

and let f be periodic elsewhere. We have

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,\mathrm{d}x = 0 \quad \text{(odd function)},$$

but

$$b_n = \frac{2}{L}\int_0^{L} x\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2L}{n\pi}(-1)^{n+1} \quad \text{(integrate by parts)}.$$

Therefore the Fourier series is

$$f(x) = \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \dots\right).$$

We can plot the approximation

$$f(x) \approx \frac{2L}{\pi}\sum_{i=1}^{N}(-1)^{i+1}\sin\frac{i\pi x}{L}.$$

This is shown in figure 1.1. We see that as N increases the following occurs.

- The approximation improves away from the discontinuity — it is convergent where f is continuous.
- The Fourier series tends to 0 at $x = L$ — the midpoint of the discontinuity.
- The Fourier series has a persistent overshoot at $x = L$ of approximately 9% (Gibbs' phenomenon).

1.2.1 The meaning of good behaviour

The Dirichlet conditions are sufficiency conditions for a well-behaved function $f(x)$ to have a convergent Fourier series.

Theorem 1.1. *If $f(x)$ is a bounded periodic function with period $2L$ with a finite number of maxima, minima and discontinuities in $[0, 2L]$ then its Fourier series converges to $f(x)$ at all points where f is continuous. At discontinuities the series converges to the midpoint of the discontinuity: $\frac{1}{2}\left(f(x^-) + f(x^+)\right)$.*

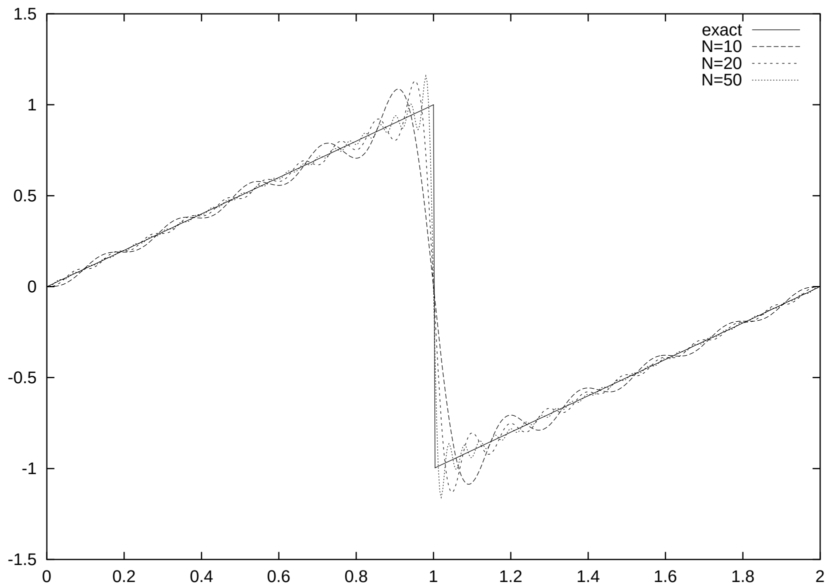

Figure 1.1: Fourier series approximation showing Gibbs' phenomenon ($L = 1$)

Proof. Omitted. □

Note that these are very weak conditions (compare Taylor's theorem). Pathological functions (e.g. x^{-1}, $\sin x^{-1}$) are excluded. The converse to this theorem is not true: $\sin x^{-1}$ has a convergent Fourier series.

1.3 Complex Fourier series

It is obvious that we can rewrite (1.1) as

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{\frac{\imath n \pi x}{L}},$$

where

$$c_n = \frac{1}{2L} \int_0^{2L} f(x) e^{-\frac{\imath n \pi x}{L}} \, \mathrm{d}x.$$

This is sometimes useful (and also makes the analogy with Fourier transforms slightly more obvious).

1.4 Sine and cosine series

Consider a function $f(x)$ defined only on the half interval $[0, L]$. We can extend its range in two obvious ways by making it either odd or even on $[-L, L]$.

If we make it odd then we put $a_n = 0$ and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, \mathrm{d}x$$

in (1.1).

If we make it even then $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x.$$

1.5 Parseval's theorem

This is a relation between the average of the square of a function and its Fourier coefficients.

$$\begin{aligned}\int_0^{2L} f(x)^2\,\mathrm{d}x &= \int_0^{2L}\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{\pi n x}{L}\right)^2\,\mathrm{d}x\\ &= \int_0^{2L}\left(\frac{1}{4}a_0^2 + \sum_{n=1}^{\infty} a_n^2\cos^2\frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n^2\sin^2\frac{\pi n x}{L}\right)\,\mathrm{d}x\\ &= L\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right).\end{aligned}$$

This is also called a *completeness relation*.

Example: sawtooth wave

Recall the sawtooth wave (page 2). Here we had $a_n = 0$ and $b_n = \frac{2L}{n\pi}(-1)^{n+1}$. Then applying Parseval's relation gives

$$\frac{2}{3}L^3 = \int_{-L}^{L} x^2\,\mathrm{d}x = L\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2},$$

and so

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}.$$

Chapter 2

The Wave Equation

2.1 Waves on an elastic string

Consider small displacements on a stretched string with the endpoints fixed and the initial conditions (displacement and velocity) given.

Resolve horizontally to get

$$T_1 \cos\theta_1 = T_2 \cos\theta_2.$$

Now for small θ, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, and so $T_1 = T_2$ with error $\mathcal{O}(\frac{\partial y}{\partial x})^2$.
Resolving vertically,

$$F_T = T_1 \sin\theta_1 + T_2 \sin\theta_2 = T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\mathrm{d}x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x.$$

Therefore (from Newton II)

$$\mu \mathrm{d}x\,\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\,\mathrm{d}x,$$

and so

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}.$$

This is the wave equation, with $c = \sqrt{\frac{T}{\mu}}$. In general, the 1D wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}. \tag{2.1}$$

2.2 Separation of variables

We want to solve (2.1) given the *boundary values*

$$y(0,t) = 0 \qquad y(L,t) = 0$$

and the *initial conditions*

$$y(x,0) = p(x) \qquad \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = q(x).$$

We try a substitution $y = X(x)T(t)$ in (2.1). This gives

$$c^{-2}\frac{\ddot{T}}{T} = \frac{X''}{X}.$$

Since the LHS depends only on t and the RHS only on x they must both be equal to a constant λ.

We have therefore split the PDE into two ODEs:

$$X'' - \lambda X = 0 \qquad \text{and} \qquad \ddot{T} - c^2\lambda T = 0.$$

We solve the x equation first:

$$X'' - \lambda X = 0 \qquad X(0) = X(L) = 0.$$

Since we don't know anything about λ we have to learn something...

- If $\lambda > 0$ the solution is $X = A\cosh\sqrt{\lambda}x + B\sinh\sqrt{\lambda}x$. If we apply the boundary values now we see that $A = B = 0$ — so this is not a useful solution.
- If $\lambda = 0$ the solution is $X = A + Bx$, and as before $A = B = 0$ on substituting the boundary values.

The only possibility now is $\lambda = -\nu^2$, which gives solutions

$$X = A_\nu \cos\nu x + B_\nu \sin\nu x.$$

Applying the boundary values gives $A = 0$ and $B_\nu \sin\nu L = 0$. If $B_\nu = 0$ then the entire solution is trivial, so the only useful solution has

$$\sin\nu L = 0 \Rightarrow \nu = \frac{n\pi}{L} \Rightarrow \lambda = -\frac{n^2\pi^2}{L^2}.$$

These special values of λ are *eigenvalues* and their *eigenfunctions* are

$$X_n = B_n \sin\frac{n\pi x}{L}.$$

These are the *normal modes*. Now all we need to do is to solve the t equation using these values for λ:

$$\ddot{T} + \frac{n^2\pi^2c^2}{L^2}T = 0.$$

This has a general solution

$$T_n = C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}.$$

Thus we have a *specific solution* of (2.1): $y_n = T_n X_n$. Since (2.1) is linear we can add solutions to get the general solution

$$y(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}. \tag{2.2}$$

This satisfies the boundary values by construction. The only thing left to do is to satisfy the initial conditions:

$$\begin{aligned} y(x,0) = p(x) &= \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \\ \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = q(x) &= \sum_{n=1}^{\infty} \frac{D_n n\pi c}{L} \sin \frac{n\pi x}{L}. \end{aligned}$$

C_n and D_n can now be found using the orthogonality relations for $\sin$. They turn out to be

$$C_n = \frac{2}{L} \int_0^L p(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x \qquad D_n = \frac{2}{n\pi c} \int_0^L q(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x.$$

2.3 Oscillation energy

A vibrating string has both KE and PE. The KE is

$$\frac{1}{2}\mu \int_0^L \dot{y}^2 \, \mathrm{d}x$$

and the PE is

$$T \int_0^L \left(\sqrt{1 + y'^2} - 1 \right) \, \mathrm{d}x \approx \frac{1}{2} T \int_0^L y'^2 \, \mathrm{d}x.$$

Since $c^2 = T\mu^{-1}$ the total sum is

$$E = \frac{1}{2}\mu \int_0^L \dot{y}^2 + (cy')^2 \, \mathrm{d}x,$$

which eventually evaluates as

$$\frac{1}{4}\mu \sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{L^2} \left(C_n^2 + D_n^2 \right) = \sum_{\text{normal modes}} \text{energy in mode.}$$

The energy is conserved in time — there is no dissipation. Further, there is no transfer of energy between modes.

2.4 Solution in characteristic co-ordinates

Consider the 1D wave equation (2.1)

$$\frac{\partial^2 y}{\partial x^2} - c^{-2}\frac{\partial^2 y}{\partial t^2} = 0$$

and make the change of variables $\xi = x + ct$, $\eta = x - ct$. Using the chain rule this becomes

$$4\frac{\partial^2 y}{\partial \xi \partial \eta} = 0,$$

with a general solution $y(\xi, \eta) = f(\xi) + g(\eta)$. Thus the general solution to (2.1) is

$$y(x,t) = f(x + ct) + g(x - ct).$$

This is a superposition of left and right moving waves.

Travelling waves (e.g. $g(x - ct)$) move with a constant speed c and retain their shape along characteristics (e.g. the line $x - ct =$ const).

2.5 Wave reflection and transmission

Suppose there is a density discontinuity in the string, say at $x = 0$. This becomes a discontinuity in c (although T is a constant). Let

$$c = \begin{cases} c_- & x < 0 \\ c_+ & x > 0. \end{cases}$$

Consider a given harmonic incident wave $A \exp \imath\omega \left(t - \frac{x}{c_-}\right)$. We want to find the reflected wave $B \exp \imath\omega \left(t + \frac{x}{c_-}\right)$ and the transmitted wave $C \exp \imath\omega \left(t - \frac{x}{c_+}\right)$.

The string does not break at $x = 0$, so that y is continuous for all t. This gives $A + B = D$.

We further want the forces to balance at $x = 0$:

$$T\frac{\partial y}{\partial x}\bigg|_{x=0^-} = T\frac{\partial y}{\partial x}\bigg|_{x=0^+},$$

and so $\frac{\partial y}{\partial x}$ is continuous for all time. This condition gives

$$-\frac{A}{c_-} + \frac{B}{c_-} = -\frac{D}{c_+}.$$

We can now solve to find

$$B = \frac{c_+ - c_-}{c_+ + c_-} A \qquad D = \frac{2c_+}{c_+ + c_-} A.$$

Note that the phase of the wave can (and generically does) change.

Chapter 3

Green's Functions

3.1 The Dirac delta function

Define a generalised function $\delta(x-\xi)$ with the properties $\delta(x-\xi) = 0$ for $x \neq \xi$ and

$$\int_{-\infty}^{\infty} \delta(x-\xi)\,\mathrm{d}x = 1.$$

These two properties imply

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)\,\mathrm{d}x = f(\xi).$$

Note that

- δ is *not* a function, but is classified as a distribution.[1]
- It is always employed in an integrand as a linear operator, where it is well defined.

3.1.1 Representations

We can represent the delta function as some sort of functional limit. A discontinuous representation is

$$\delta_\epsilon(x) = \begin{cases} 0 & x < -\frac{\epsilon}{2} \\ \epsilon^{-1} & -\frac{\epsilon}{2} \leq x \leq \frac{\epsilon}{2} \\ 0 & x > \frac{\epsilon}{2}, \end{cases}$$

and a continuous representation is

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-\frac{x^2}{\epsilon^2}}.$$

These are obviously both with $\epsilon \to 0$. Examples with $n \to \infty$ are

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{\imath kx}\,\mathrm{d}x$$

[1] See PDE's IIB for more details (than you could possibly want).

and $\delta_n(x) = \frac{n}{2}\operatorname{sech}^2 nx$.
The Heaviside step function is

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0, \end{cases}$$

and can be seen to be

$$H(x) = \int_{-\infty}^{x} \delta(\xi)\,\mathrm{d}\xi.$$

Thus (in some suitably refined sense) $H'(x) = \delta(x)$. We can also define the derivative of the delta function such that we can integrate it by parts:

$$\int_{-\infty}^{\infty} f(x)\delta'(x-\xi)\,\mathrm{d}x = [f(x)\delta(x-\xi)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x-\xi)\,\mathrm{d}x = -f'(\xi).$$

3.2 Second order linear ODEs

We wish to solve the general second order linear ODE:

$$\mathcal{L}y \equiv y'' + b(x)y' + c(x)y = f(x). \tag{3.1}$$

We know that the homogeneous equation (with $f \equiv 0$) has two linearly independent solutions y_1 and y_2, which give the homogeneous equation the *complementary function* solution $y_c = Ay_1 + By_2$. The inhomogeneous equation also has a particular solution y_p. The general solution of (3.1) is then $y_c + y_p$. Two boundary values (or initial conditions) are required to find A and B.

We hope to solve the boundary value problem. We will restrict to *homogeneous* boundary values: $y(a) = y(b) = 0$. More general values can be turned into homogeneous ones by judicious use of the complementary function.

3.3 Definition of Green's function

The Green's function $G(x, \xi)$ is the solution of

$$\mathcal{L}G(x,\xi) = \delta(x,\xi)$$

with $G \equiv 0$ at endpoints. By linearity we can now construct the solution of (3.1) for general f:

$$y(x) = \int f(\xi)G(x,\xi)\,\mathrm{d}x.$$

Now y clearly satisfies the homogeneous boundary values, and it is also easy to see that $\mathcal{L}y = f$.

3.3.1 Defining properties

$G(x, \xi)$ splits into two halves

$$G(x,\xi) = \begin{cases} G_1(x,\xi) & a < x < \xi \\ G_2(x,\xi) & \xi < x < b. \end{cases}$$

such that G solves the homogeneous equation for $x \neq \xi$, is continuous at $x = \xi$ and satisfies $[G']_{\xi^-}^{\xi^+} = 1$. Note that there are many different conventions for this jump condition.

3.4 Constructing $G(x, \xi)$: boundary value problems

There is a solution to the homogeneous problem $y_-(x)$ such that $y_-(a) = 0$. Then $G_1(x, \xi) = Cy_-(x)$. Similarly there is a solution $y_+(x)$ such that $y_+(b) = 0$ and so $G_1(x, \xi) = Dy_+(x)$. Now impose continuity at $x = \xi$ to give

$$Cy_-(\xi) = Dy_+(\xi).$$

The other equation comes from the jump condition:

$$Dy'_+(\xi) - Cy'_-(\xi) = 1.$$

We can solve these equations to give

$$C = \frac{y_+(\xi)}{W(\xi)} \qquad D = \frac{y_-(\xi)}{W(\xi)},$$

where $W(\xi)$ is the Wronskian:

$$W(\xi) = y_-(\xi)y'_+(\xi) - y_+(\xi)y'_-(\xi).$$

Thus

$$G(x,\xi) = \begin{cases} \frac{y_-(x)y_+(\xi)}{W(\xi)} & x < \xi \\ \frac{y_+(x)y_-(\xi)}{W(\xi)} & x > \xi, \end{cases}$$

and the solution of $\mathcal{L}y = f$, $y(a) = y(b) = 0$ is

$$y(x) = y_+(x)\int_a^x \frac{f(\xi)y_-(\xi)}{W(\xi)}\,\mathrm{d}\xi + y_-(x)\int_x^b \frac{f(\xi)y_+(\xi)}{W(\xi)}\,\mathrm{d}\xi.$$

3.4.1 Derivation of jump conditions

First suppose $G(x, \xi)$ is discontinuous at $x = \xi$, so that near $x = \xi$,

$$G(x,\xi) \propto H(x-\xi) \quad G'(x,\xi) \propto \delta(x-\xi) \quad G''(x,\xi) \propto \delta'(x-\xi).$$

Then the equation $\mathcal{L}G = \delta(x - \xi)$ becomes

$$\alpha\delta'(x-\xi) + \beta\delta(x-\xi) + \gamma H(x-\xi) = \delta(x-\xi),$$

which is certainly not possible. So $G(x,\xi)$ is continuous at $x = \xi$. The jump condition in G' can be derived by integrating $\mathcal{L}G = \delta(x-\xi)$ across $x = \xi$:

$$[G']_{\xi^-}^{\xi^+} + \underbrace{b(\xi)\,[G]_{\xi^-}^{\xi^+} + \int_{\xi^-}^{\xi^+} (c-b')G\,\mathrm{d}x}_{\to 0 \text{ as } \xi^-,\xi^+\to\xi} = 1.$$

Therefore

$$[G']_{\xi^-}^{\xi^+} = 1.$$

3.4.2 Example

Suppose we wish to solve

$$y'' = f(x) \qquad y(0) = y(L) = 0.$$

The homogeneous solutions are $y = Ax+B$ and so $G_1 = Cx$ and $G_2 = D(x-L)$. Applying the continuity condition

$$C\xi = D(\xi - L)$$

and then the jump condition

$$D - C = 1$$

gives $D = \frac{\xi}{L}$ and $C = \frac{\xi - L}{L}$.

3.5 Constructing $G(x,\xi)$: initial value problems

Greens' function methods can also solve initial value problems. Suppose we wish to solve $\mathcal{L}y = f$, $y(0) = y'(0) = 0$. Split G into G_1 and G_2 as before.

Since $G_1(a) = G_1'(a) = 0$ and $\mathcal{L}G_1 = 0$ then $G_1 \equiv 0$. Therefore $G_2(\xi) = 0$ and $G_2'(\xi) = 0$, so that

$$G(x,\xi) = \begin{cases} 0 & x < \xi \\ \frac{y(x)}{y'(\xi)} & x > \xi, \end{cases}$$

where $\mathcal{L}(y) = 0$ and $y(\xi) = 0$. The solution is then

$$y(x) = y(x)\int_a^x \frac{f(\xi)}{y'(\xi)}\,\mathrm{d}\xi.$$

We see that causality is built in to the solution.

3.5.1 Example

Solve $y'' - y = f(x)$, $x > 0$, $y(0) = y'(0) = 0$.

In $x < \xi$, $G(x,\xi) = 0$ and in $x > \xi$ we have

$$G(x,\xi) = Ae^{x} + Be^{-x}.$$

Continuity at $x = \xi$ gives $G(x,\xi) = C\sinh(x-\xi)$ in $x > \xi$. Now $y'(\xi) = C$ and so $C = 1$. Hence

$$y(x) = \int_0^x f(\xi) \sinh(x - \xi) \, \mathrm{d}\xi.$$

Chapter 4

Sturm-Liouville Theory

4.1 Self-adjoint form and boundary values

We wish to solve the general eigenvalue problem

$$\mathcal{L}y = y'' + b(x)y' + c(x)y = -\lambda d(x)y \tag{4.1}$$

with specified boundary conditions. This often occurs after separation of variables in a PDE. One classic example is the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi = \imath\hbar\frac{\partial\psi}{\partial t}.$$

We try a solution $\psi = U(\mathbf{x})e^{-\frac{\imath E t}{\hbar}}$. Substituting into the Schrödinger equation gives

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)U = EU.$$

E is the energy eigenvalue.[1]

The analysis greatly simplifies is $\mathcal{L}$ is in *self-adjoint form*: that is if (4.1) can be re-expressed in *Sturm-Liouville form*:

$$\mathcal{L}y = -(py')' + qy = \lambda wy, \tag{4.2}$$

where the *weighting function* $w(x)$ is assumed positive. We can easily put (4.1) in Sturm-Liouville form: multiply by $\exp\int^x b(\xi)\,\mathrm{d}\xi$.

Definition 4.1. *$\mathcal{L}$ is self-adjoint on the interval $a < x < b$ iff for all pairs of functions y_1, y_2 satisfying appropriate boundary values we have*

$$\int_a^b y_1\mathcal{L}y_2\,\mathrm{d}x = \int_a^b y_2\mathcal{L}y_1\,\mathrm{d}x. \tag{4.3}$$

If we substitute (4.2) into (4.3) we see that "appropriate boundary values" means

$$[-y_1py_2' + y_2py_1']_a^b = 0,$$

which includes $y(a) = y(b) = 0$, $y'(a) = y'(b) = 0$, $y + ky' = 0$, $y(a) = y(b)$, $p(a) = p(b) = 0$ or combinations of the above.

[1] See the Quantum Mechanics course for more details.

4.2 Eigenfunction expansions

Self-adjoint operators have three important properties.

4.2.1 Real eigenvalues

Suppose $\mathcal{L}y_n = \lambda_n y_n$, and so $\mathcal{L}y_n^* = \lambda_n^* y_n^*$. Then

$$\int_a^b y_n^* \lambda_n w y_n \,\mathrm{d}x - \int_a^b \lambda_n^* y_n y_n^* \,\mathrm{d}x = 0$$

and so $\lambda_n^* = \lambda_n$, since $\int w\,|y_n|^2 \neq 0$ for non-trivial w, y_n.

4.2.2 Orthogonal eigenfunctions

Suppose $\lambda_m \neq \lambda_n$. Then

$$(\lambda_n - \lambda_m)\int_a^b w y_m y_n \,\mathrm{d}x = 0$$

and so $\int w y_m y_n = 0$. y_n, y_m are thus orthogonal on $[a, b]$ wrt the weighting function $w(x)$.

4.2.3 Complete eigenfunctions

We can write sufficiently nice $f(x)$ as

$$f(x) = \sum_n a_n y_n(x),$$

with

$$\int_a^b f(x) y_n(x) \,\mathrm{d}x = a_n \int_a^b w y_n^2 \,\mathrm{d}x.$$

The eigenfunctions are sometimes normalised to unit modulus for convenience.

We also have Parseval's identity, which in this form is

$$\int_a^b \left(f - \sum_{n=1}^{\infty} a_n y_n \right)^2 w \,\mathrm{d}x = 0,$$

or

$$\int_a^b w f^2 \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b w y_n^2 \,\mathrm{d}x. \tag{4.4}$$

The expansions needed converge if the eigenfunctions are complete. If the eigenfunctions are not complete then the LHS of (4.4) is greater than its RHS. This is *Bessel's inequality*.

4.3 Example: Legendre polynomials

Consider Legendre's equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \tag{4.5}$$

which can be rewritten in Sturm-Liouville form as

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)y'\right) = \lambda y.$$

It is motivated by separation of variables in spherical polars. The boundary conditions are that y is finite at $x = \pm 1$. We try a power series solution about $x = 0$,

$$y = \sum_{n=0}^{\infty} c_n x^n,$$

which gives (prove this)

$$c_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} c_n.$$

Specifying c_0 and c_1 yields linearly independent solutions, one of which is odd and the other even.

As $n \to \infty$, $\frac{c_{n+2}}{c_n} \to 1$ and so we get a geometric series, which is divergent at $x = \pm 1$. One of the two series must terminate and so $\lambda = m(m+1)$ for $m \in \mathbb{N}$.

The eigenfunctions on $-1 \le x \le 1$ are the *Legendre polynomials* P_n. P_n is usually normalised so that $P_n(1) = 1$: with this normalisation we have

n	λ	P_n
0	0	1
1	2	x
2	6	$\frac{1}{2}(3x^2-1)$
3	12	$\frac{1}{2}(5x^3-3x)$

The orthogonality relation is

$$\int_{-1}^{1} P_n P_m \,\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}.$$

4.4 Inhomogeneous boundary value problem

$$(\mathcal{L} - \mu w)y = f(x).$$

Consider the above inhomogeneous ODE with homogeneous boundary values and a fixed μ (not an eigenvalue).

Now we can expand $f(x)$ in terms of eigenfunctions of $\mathcal{L}$:

$$f(x) = w(x)\sum_{n=1}^{\infty} a_n y_n,$$

where

$$a_n = \int_a^b f y_n \,\mathrm{d}x$$

and the eigenfunctions are normalised to $\int y_n^2 \,\mathrm{d}x = 1$. We seek a solution

$$f = \sum_n b_n y_n.$$

Substituting we find $b_n(\lambda_n - \mu) = a_n$ (by orthogonality) and so provided μ is not an eigenvalue,

$$y = \sum_n \frac{a_n}{\lambda_n - \mu} y_n(x) = \sum_n \frac{y_n(x)}{\lambda_n - \mu} \int_a^b f y_n \,\mathrm{d}x'.$$

If μ is an eigenvalue then this is a resonant frequency: the amplitude grows without limit and there is no solution consistent with the boundary values.

Chapter 5

Applications: Laplace's Equation

We seek to solve

$$\nabla^2 \phi = 0 \tag{5.1}$$

by the method of separation of variables.

ϕ can represent the electrostatic potential, gravitational potential, heat and so on. (5.1) is the homogeneous version of the *Poisson equation* $\nabla^2 \phi = \rho$.

Boundary values can be given on

- ϕ : Dirichlet boundary conditions
- $\mathbf{n} \cdot \nabla \phi$: von Neumann boundary conditions,

specified on a boundary surface in 3D, boundary curve in 2D or endpoints in 1D.

5.1 Cartesians

In Cartesians, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. We seek a solution $\phi = X(x)Y(y)Z(z)$ and get

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \lambda_l$$

Similarly $\frac{Y''}{Y} = \lambda_m$ and $\frac{Z''}{Z} = \lambda_n$, where $\lambda_l + \lambda_m + \lambda_n = 0$. We can then find eigenfunction solutions satisfying the given boundary values: $\phi_{lmn} = X_l Y_m Z_n$, so that then the general solution is

$$\phi = \sum_{lmn} c_{lmn} X_l Y_m Z_n.$$

Example: heat conduction

Solve the system

$$\begin{aligned} \nabla^2\phi &= 0 \quad \text{in } z > 0 \\ \phi &= 0 \quad x = 0,1 \text{ or } y = 0,1 \\ \phi &= 1 \quad \text{at } z = 0 \\ \phi &\to 0 \quad \text{as } z \to \infty. \end{aligned}$$

This models heat conduction on a semi-infinite square bar.

We separate variables to get $X_n = \sin l\pi x$ and $Y_m = \sin m\pi x$, with $\lambda_l = -l^2\pi^2$ and $\lambda_m = -m^2\pi^2$. Then we have

$$\frac{Z''}{Z} = \pi^2(l^2 + m^2),$$

and so $Z_{l,m} = e^{-\pi z\sqrt{l^2+m^2}}$ (to satisfy the bc at infinity). Therefore

$$\phi = \sum_{l,m} A_{l,m} \sin l\pi x \sin m\pi y \, e^{-\pi z\sqrt{l^2+m^2}}.$$

To find $A_{l,m}$ use the boundary condition at $z = 0$:

$$1 = \sum_{l,m} A_{l,m} \sin l\pi x \sin m\pi y.$$

Now

$$\int_0^1 \sin l\pi t \sin m\pi t \, \mathrm{d}t = \frac{1}{2}\delta_{lm}$$

and so

$$\int_0^1 \int_0^1 \sin l\pi x \sin m\pi y \, \mathrm{d}x\mathrm{d}y = \frac{A_{l,m}}{4}.$$

Thus

$$A_{l,m} = \begin{cases} \frac{16}{\pi^2 lm} & l,m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case we have *degenerate eigenvalues*: both X_1Y_2 and X_2Y_1 give the same constant in the z equation. Despite this, we can always choose orthogonal eigenfunctions.

5.2 Plane polars

In plane polars, Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial\phi}{\partial r^2} = 0. \tag{5.2}$$

We seek $\phi = R(r)\Theta(\theta)$, which gives

$$\frac{r(rR')'}{R} = \lambda \qquad \frac{\Theta''}{\Theta} = -\lambda.$$

Consider a drum surface with a distorted rim, with unit radius. The height of the surface is given by ϕ such that $\nabla^2\phi = 0$ and $\phi(1,\theta) = f(\theta)$.

Now the θ equation is

$$\Theta'' + \lambda\Theta = 0$$

and since Θ must be periodic, $\lambda = n^2$ for $n \in \mathbb{N}$.

The solution to this equation is

$$\Theta_n = a_n \cos n\theta + b_n \sin n\theta.$$

If $n = 0$ then the solution is $\Theta_0 = a_0 + b_0\theta : b_0 = 0$ from the periodic boundary conditions.

The r equation is

$$r(rR')' - n^2 R = 0,$$

which has solutions $R = c_n r^n + d_n r^{-n}$. Thus $d_n = 0$ to keep the solution finite in $r < 1$. When $n = 0$ the solution is $c_0 + d_0 \log r$ and so $d_0 = 0$. Thus

$$\phi(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n.$$

a_n and b_n can be found using $\phi(1,\theta) = f(\theta)$.

5.3 Spherical polars

The Laplace equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{5.3}$$

We seek separable solution $R(r)\Theta(\theta)\psi(\phi)$ and specialise to the axisymmetric case: $\psi = 1$. Then we have

$$(r^2R')' - \lambda R = 0 \qquad (\sin\theta\Theta')' + \lambda\sin\theta = 0.$$

Putting $x = \cos\theta$ in the θ equation gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) + \lambda\Theta = 0.$$

This is Legendre's equation (4.5), and so from the earlier analysis we know $\lambda_n = n(n+1)$. The radial equation becomes

$$(r^2R')' - n(n+1)R = 0,$$

Trying a solution r^m given $m = n$ or $m = -n-1$, so the eigenfunction expansion of Φ is

$$\Phi = \sum_n \left(A_n r^n + B_n r^{-n-1}\right) P_n(\cos\theta).$$

A_n and B_n can be determined from boundary conditions on a spherical surface.

5.3.1 ** The full glory of spherical polars **

If we drop the assumption of axisymmetry things become more complicated. The azimuthal eigenfunctions are $\psi_m = e^{\imath m\phi}$ and the polar eigenfunctions $P_l^m(\cos\theta)$ satisfy the *associated Legendre equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\Theta=0.$$

We combine the azimuthal and polar eigenfunctions to get the *spherical harmonics*:

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{(2l+1)!(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{\imath m\phi},$$

for $-l \le m \le l$. The radial equation is the same as before, giving

$$R_{lm}=a_{lm}r^l+b_{lm}r^{-l-1}.$$

Chapter 6

Calculus of Variations

6.1 The problem

Suppose we wish to minimise

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')\,\mathrm{d}x \tag{6.1}$$

over all functions y such that $y(x_1) = y_1$ and $y(x_2) = y_2$. This is clearly not just an ordinary calculus minimization, but something slightly harder...

6.2 Euler-Lagrange equations

We will do this, as in ordinary minimization problems, by finding a function such that the first order variation of J is zero. So, suppose $y(x)$ is the answer and perturb it slightly to $y(x) + \delta y(x)$, where $\delta y(x_1) = \delta y(x_2) = 0$. Then

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \text{ higher order.}$$

Hence

$$\begin{aligned} \delta J &= \int_{x_1}^{x_2} \delta y \frac{\partial F}{\partial y} + \delta y' \frac{\partial F}{\partial y'}\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \delta y \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \right)\,\mathrm{d}x + \underbrace{\left[\delta y \frac{\partial F}{\partial y'} \right]_{x_1}^{x_2}}_{=0}. \end{aligned}$$

Thus for the first order variation to be zero we require

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}, \tag{6.2}$$

since δy is arbitrary. This is an *Euler-Lagrange equation*.

One variant on this that is sometimes useful: (6.2) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial x}. \tag{6.3}$$

To prove this note that $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'}$.
There are three special cases:

- y' absent gives $\frac{\partial F}{\partial y} = 0$, which can be solved for y.
- y absent gives $\frac{\partial F}{\partial y'} = $ const.
- x absent gives $F - y' \frac{\partial F}{\partial y'} = $ const (use (6.3)).

6.3 Examples

Geodesics

In Euclidean $\mathbb{R}^2$ we have a metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ and we seek to minimise

$$\int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, \mathrm{d}x.$$

We can immediately apply the Euler-Lagrange equations, noting that y is absent and so

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{const},$$

which reduces to $y' = $ const and so the geodesics in $\mathbb{R}^2$ are straight lines (which is reassuring, if nothing else).

You can do something similar on the sphere, with

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$$

and show that the geodesics are great circles.

Brachistochrone

Consider a frictionless bead on a wire path $y(x)$ connecting two points A and B. What path gives the shortest travel time from A to B?

Assume A is at $y = 0$. The time of travel is then

$$T[y] = \int_A^B \frac{\mathrm{d}s}{V} = \int_A^B \frac{\mathrm{d}s}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{y}}\, \mathrm{d}x.$$

x is absent and the Euler-Lagrange equations eventually give

$$y(1 + y'^2) = \text{const},$$

or

$$x = \pm \int \left(\frac{y}{c - y} \right)^{\frac{1}{2}} \mathrm{d}y.$$

The substitution $y = c \sin^2 \frac{\theta}{2} = \frac{c}{2}(1 - \cos\theta)$ makes this integral doable and gives

$$x = \pm \frac{c}{2}(\theta - \sin\theta)^2.$$

6.4 Principle of Least Action

The action of a system is given by

$$S = \int \mathcal{L} \, \mathrm{d}t,$$

where $\mathcal{L} = \text{KE} - \text{PE}$. Trajectories minimize the action. Now suppose that $\text{KE} = \frac{1}{2}m\dot{x}^2$ and $\text{PE} = V(x)$. Then

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$$

and the Euler-Lagrange equations give

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = V',$$

which ought to be familiar...

Since t is absent, we know that $\mathcal{L} - \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}$ is constant — in fact it is the total energy.

Something similar is Fermat's principle, that light follows the path of minimum time.

Least action principles are important all over physics — see the General Relativity and Electrodynamics courses for more examples.

6.5 Generalisations

The trick with all of these is just to make the variation and see what happens, integrating by parts where necessary.

The generalisation to several dependent variables is easiest: extremise

$$J[\mathbf{y}] = \int_{x_1}^{x_2} F(x, \mathbf{y}, \mathbf{y}') \, \mathrm{d}x.$$

Performing the variation gives

$$\frac{\partial F}{\partial y_i} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y_i'}.$$

Generalisations to several dependent variables exist: but it's easiest just to do the variation explicitly.

The same is true of generalisations to more derivatives in F — just do the variation and integrate by parts.

6.6 Integral constraints

Suppose we wish to extremise $J = \int F(x, y, y') \, \mathrm{d}x$ subject to the constraint $K = \int G(x, y, y') \, \mathrm{d}x$ constant. This is done by using Lagrange multipliers: extremising

$$I = \int F(x, y, y') + \lambda G(x, y, y') \, \mathrm{d}x.$$

Examples are on the problem sheet.

Chapter 7

Cartesian Tensors in $\mathbb{R}^3$

Summation convention is used throughout this chapter unless explicitly stated otherwise.

7.1 Tensors?

A tensor is an object represented in a particular co-ordinate system by a set of functions called components such that the components in a new co-ordinate system are related to the components in the old co-ordinates in a prescribed way.

We will consider only orthogonal co-ordinate systems, and restrict the transformations to rotations and reflexions. More general transformations and co-ordinate systems are possible — see the General Relativity course for details.

Consider a vector $\mathbf{x}$ with components x_i in a given orthogonal basis:

$$\mathbf{x} = x_i \mathbf{e}_i.$$

Now consider new co-ordinates $\mathbf{e}'_i$, such that

$$\mathbf{e}'_i = (\mathbf{e}'_i \cdot \mathbf{e}_j)\, \mathbf{e}_j,$$

and denote $\mathbf{e}'_i \cdot \mathbf{e}_j \equiv l_{ij}$. Now

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = (l_{ik}\mathbf{e}_k) \cdot (l_{jm}\mathbf{e}_m) = l_{ik} l_{jm} \delta_{km} = l_{ik} l_{jk}.$$

Also,

$$\mathbf{e}_i = (\mathbf{e}_i \cdot \mathbf{e}'_p)\mathbf{e}'_p = l_{pi}\mathbf{e}'_p,$$

and so

$$\mathbf{x} = x_i \mathbf{e}_i = x_i l_{pi} \mathbf{e}'_p.$$

Hence $x'_p = x_i l_{pi}$.

7.2 Transformation laws

- *Scalars* remain invariant under a co-ordinate transformation. *Scalars* are zero-rank tensors.

- A *vector* $\mathbf{A}$ is a set of three functions A_i given in a particular co-ordinate system with the transformation property
$$A'_p = l_{pi}A_i,$$
so that A'_j is a vector in a co-ordinate system rotated by L. A vector is a 1^{st} rank tensor.

- A second rank tensor comprises 9 functions A_{ij} in a given co-ordinate system such that
$$A'_{pq} = l_{ip}l_{jq}A_{ij}.$$

- An n^{th} rank tensor comprises 3^n functions of position such that
$$A'_{pq\ldots r} = l_{pi}l_{qj}\ldots l_{rk}A_{ij\ldots k}.$$

We can see that $0 \leftrightarrow 0$, which means that tensor equations are preserved by change of co-ordinate system. To see this, suppose A and B are tensors with $A_{ij\ldots k} = B_{ij\ldots k}$ in one co-ordinate system. Then $A-B$ is a tensor — it's zero, and so $A'_{pq\ldots r}-B'_{pq\ldots r} = 0$ and hence $A'_{pq\ldots r} = B'_{pq\ldots r}$. This is why tensors are so useful.

7.3 Tensor algebra

Proof of all of these is obvious — just show that they obey the transformation law.

- If A is a n^{th} rank tensor then so is λA for scalar λ.
- If A and B are n^{th} rank tensors then so is $C = A + B$.
- If A is an n^{th} rank tensor and B is an m^{th} rank tensor then the outer product defined by
$$C_{ij\ldots kab\ldots c} = A_{ij\ldots k}B_{ab\ldots c}$$
is an $(n+m)^{\text{th}}$ rank tensor.
- If $A_{ijk\ldots l}$ is an n^{th} rank tensor then the contraction $A_{iik\ldots l}$ is an $(n-2)^{\text{th}}$ rank tensor.

7.4 Quotient Laws

Theorem 7.1 (Quotient Theorem). *If the inner product of some quantity A with an arbitrary vector $\mathbf{K}$ is an n^{th} rank tensor then A is an $(n+1)^{th}$ rank tensor.*

Proof. We know
$$A_{ij\ldots k}K_i = B_{j\ldots k}$$
In a new co-ordinate system

$$\begin{aligned} B'_{q\ldots r} &= A'_{pq\ldots r}K'_p \\ &= l_{qj}\ldots l_{rk}B_{j\ldots k} \\ &= l_{qj}\ldots l_{rk}A_{ij\ldots k}K_i \\ &= l_{qj}\ldots l_{rk}A_{ij\ldots k}l_{pi}K'_p, \end{aligned}$$

and so, since K is arbitrary, $A'_{pq\ldots r} = l_{pi}l_{qj}\ldots l_{rk}A_{ij\ldots k}$. □

This theorem generalises to any type of product — inner, outer or a mixture thereof. The proof is as above.

This can be used to identify the transformation properties of physical quantities, for instance in *Ohm's law*

$$J_i = \sigma_{ij} E_j,$$

where **J** is the current vector and **E** the electric field vector, then the conductivity σ must be a tensor.

7.5 Isotropic tensors

Isotropic tensors are invariant under all co-ordinate transformations:

$$A'_{pq\ldots r} = l_{pi} l_{qj} \ldots l_{rk} A_{ij\ldots k} = A_{pq\ldots r}.$$

Scalars are clearly isotropic. As for vectors, suppose $A'_p = l_{pi} A_i = A_p$. Then

$$(l_{pi} - \delta_{pi}) A_i = 0$$

for all l_{pi}, so $A_i = 0$.

Theorem 7.2. *The most general isotropic second rank tensor in* $\mathbb{R}^3$ *is* $\lambda\delta_{ij}$.

Proof. $\lambda\delta_{ij}$ is clearly isotropic, so we must prove that it the the most general isotropic second rank tensor in $\mathbb{R}^3$.

Let A_{ij} be isotropic, so that

$$A'_{pq} = l_{pi} l_{qj} A_{ij} = A_{pq}.$$

Rotate by 90° around the z-axis — i.e. take

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and then compare components. Do the same thing with the y axis. □

Theorem 7.3. *The only isotropic third rank tensor is the alternator* ϵ_{ijk} *(or the product of a scalar with the alternator).*

Proof. The same as before, more or less. □

Theorem 7.4. *The most general isotropic fourth rank tensor is*

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}.$$

7.5.1 Spherically symmetric integrals

Consider

$$A_{ij} = \int_{r<a} x_i x_j \, \mathrm{d}V.$$

It is clearly isotropic, so $A_{ij} = \lambda\delta_{ij}$. Now contract over i and j to get

$$3\lambda = \int_{r<a} r^2 \, \mathrm{d}V = \frac{4\pi a^5}{5}.$$

Therefore

$$\int_{r<a} x_i x_j \, \mathrm{d}V = \frac{4\pi a^5}{15}\delta_{ij}.$$

This is a surprisingly easy way of doing the above integral!

7.6 Symmetric and antisymmetric tensors

If $A_{ij\ldots k} = A_{ji\ldots k}$ then A is said to be *symmetric* in i and j. If $A_{ij\ldots k} = -A_{ji\ldots k}$ then A is *antisymmetric* or *skew symmetric* in i and j.

If this is true in one co-ordinate system then it is true in all co-ordinate systems (exercise).

Any second rank tensor can be decomposed into the sum of an symmetric tensor and an antisymmetric tensor:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}). \tag{7.1}$$

Symmetric second rank tensors can be diagonalised.

Antisymmetric second rank tensors in $\mathbb{R}^3$ have only three independent components:

$$A_{ij} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \epsilon_{ijk} v_k,$$

where $v_k = (c, -b, a)$. We can therefore continue the decomposition in (7.1) into

$$T_{ij} = \frac{1}{3} T_{kk}\delta_{ij} + \epsilon_{ijk} v_k + \tilde{S}_{ij},$$

the sum of a scalar part, a vector part and an irreducible tensor part.

7.7 Physical Applications

Tensors have a very wide range of physical applications. The relevant courses are:

- Dynamics, Principles of Dynamics: angular momentum tensor, moment of inertia tensor.
- Fluid Dynamics 2, Waves in Fluid and Solid Media, Theoretical Geophysics: stress tensor, strain tensor.
- General Relativity: metric tensor, Riemann tensor, Ricci tensor.
- Electrodynamics: Electromagnetic field tensor, stress-energy tensor.

References

- Arfken and Weber, *Mathematical Methods for Physicists*, Fourth ed., Academic Press, 1995.

  Quite a lot of people sing Arfken's praises; I am not one of them. Although it is useful I think it tries to do too much. If nothing else though it could be used to kill small mammals and it does have everything in this course in it. A good book to buy if you only want to buy one book in the next two years. Or if you have a problem with mice.

Related courses

Most of the applied courses over the next two years use this course to some extent. You have been warned!