Complex Methods
Course P3

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Small print The syllabus for the course is defined by the Faculty Board Schedules (which are minimal for lecturing and maximal for examining). I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. This document is written in LaTeX 2_ε and should be accessible from my home page. My e-mail address is `twk@dpmms.cam.ac.uk`.

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1 Complex differentiability is like real differentiability

The complex numbers have algebraic properties which are very similar to those of the real numbers (formally they are both fields) except that there is no order on the complex numbers. This similarity means that we can define differentiability in the complex case in exactly the same way as we did in the real case.

Definition 1.1. *A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z with derivative $f'(z)$ if*

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| \to 0$$

as $|h| \to 0$.

Exactly the same proofs as in the real case produce exactly the same elementary properties of differentiation.

Lemma 1.2. *(i) The constant function given by $f(z) = c$ for all $z \in \mathbb{C}$ is everywhere differentiable with derivative $f'(z) = 0$.*

(ii) The function given by $f(z) = z$ for all $z \in \mathbb{C}$ is everywhere differentiable with $f'(z) = 1$.

(iii) If $f,\ g : \mathbb{C} \to \mathbb{C}$ are both differentiable at z, then so is $f + g$ with $(f + g)'(z) = f'(z) + g'(z)$.

(iii) If $f,\ g : \mathbb{C} \to \mathbb{C}$ are both differentiable at z, then so is their product $f \times g$ with $(f \times g)'(z) = f'(z)g(z) + f(z)g'(z)$.

(iv) If $f : \mathbb{C} \to \mathbb{C}$ is nowhere zero and f is differentiable at z, then so is $1/f$ with $(1/f)'(z) = -f'(z)/(f(z))^2$.

(v) If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z and $g : \mathbb{C} \to \mathbb{C}$ is differentiable at $f(z)$ then the composition $g \circ f$ is differentiable at z with $(g \circ f)'(z) = f'(z)g'(f(z))$.

(vi) If $P(z) = \sum_{n=0}^{N} a_n z^n$, then P is everywhere differentiable with derivative given by $P'(z) = \sum_{n=1}^{N} n a_n z^{n-1}$.

The following extensive generalisation of part (iv) of Lemma 1.2 was proved in Analysis I (course C5).

Theorem 1.3. *Let $a_j \in \mathbb{C}$ $[0 \leq j]$. Then, either $\sum_{n=0}^{\infty} a_n z^n$ converges for all z and we write $R = \infty$, or there exists a real number $R \geq 0$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < R$ and diverges for all $|z| > R$. (R is called the* radius of convergence *of $\sum_{n=0}^{\infty} a_n z^n$.)*

If we write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *for* $|z| < R$*, then* f *is differentiable at all* z *with* $|z| < R$ *and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We shall see (for example in Theorems 5.4 and 5.5) that the study of power series and complex differentiable functions are closely linked.

In Analysis I you studied a particular power series of great importance to us.

Theorem 1.4. *(i)* $\sum_{n=0}^{\infty} z^n/n!$ *converges for all* z*. If we write*

$$\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

then exp *is everywhere differentiable with* $\exp' z = \exp z$*.*

(ii) $\exp z \exp w = \exp(z+w)$ *for all* $z,\ w \in \mathbb{C}$*.*

(iii) The equation $\exp z = 0$ *has no solution. If* $w \neq 0$ *the equation* $\exp z = w$ *has the solutions*

$$z = \log|w| + i\theta + 2n\pi i$$

with $n \in \mathbb{Z}$*, where* θ *is any particular real solution of*

$$\frac{w}{|w|} = \cos\theta + i\sin\theta.$$

(iv) If we write $e^z = \exp z$ *and*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},\ \ \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

then, when z *is real, we recover the traditional real functions* $\sin : \mathbb{R} \to \mathbb{R}$ *and* $\cos : \mathbb{R} \to \mathbb{R}$*.*

Combining the results of this section, we see that we have obtained a useful library of complex differentiable functions.

2 Complex differentiability is not like real differentiability

The first hint that complex differentiability is different from real differentiability is given by the following example.

Example 2.1. *The function $F : \mathbb{C} \to \mathbb{C}$ given by $F(z) = z^*$ is nowhere differentiable.*

To understand Example 2.1 it is helpful to view matters not algebraicly (as we did in Section 1) but geometrically. Observe that, if we ignore multiplication, $\mathbb{C}$ can be considered as the vector space $\mathbb{R}^2$. If we have a function $f : \mathbb{C} \to \mathbb{C}$ we can write

$$f(x + iy) = u(x, y) + iv(x, y)$$

with x, y, u and v real, obtaining the map from $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

Theorem 2.2. *If the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ is differentiable in the sense of the course Analysis II (Course P9), then the following statements are equivalent.*

(i) f is complex differentiable at z_0.

(ii) The map $h \mapsto f(z_0+h) - f(z_0)$ is locally the composition of a rotation and a dilation.

(iii) The Jacobian matrix of the map T satisfies

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with λ real and $\lambda \geq 0$.

(iv) The map T satisfies the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Thus $z \mapsto z^*$ is not complex differentiable because it is a reflection.

Most Cambridge examinees and a worryingly high proportion of Cambridge examiners believe that the Cauchy-Riemann relations are the best way of testing for complex differentiability. This is not the case in general. The methods of Section 1 usually furnish a more efficient tool. (One problem with the use of the Cauchy-Riemann equations is that, as is shown in Analysis II, the existence of partial derivatives does not imply differentiability.)

In any case it is the act of a lunatic (or a Cambridge examiner) to ask about complex differentiability at a single point. The subject of complex differentiability only becomes interesting when applied to functions differentiable on an open set.

Definition 2.3. *A set $\Omega \subseteq \mathbb{C}$ is said to be open if, given $w \in \Omega$, we can find a $\delta > 0$ such that $z \in \Omega$ whenever $|z - w| < \delta$.*

Thus, wherever we are in an open set, we can move some fixed distance (depending on the point chosen) in any direction whilst remaining within the set.

Definition 2.4. *Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that a function $f : \Omega \to \mathbb{C}$ is analytic on Ω if f is differentiable at every point of Ω.*

Since this is a non-rigorous treatment, it operates under the assumption that everything is well behaved. One of the surprises and one the great advantages of the rigorous treatment given in course C12 (Further Analysis) is that it reveals that *all analytic functions are well behaved analytic functions.*

The rigorous treatment reveals that the next lemma is true for all analytic functions although our 'proof' seems to require extra conditions.

Lemma 2.5. *Suppose $\Omega \subseteq \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic. Then, defining u and v, as usual we have u harmonic (that is, satisfying Laplace's equation $\nabla^2 u = 0$) on $\tilde{\Omega} = \{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$. The same is true of v.*

Lemma 2.5 has an important converse.

Lemma 2.6. *If u is harmonic, then it is locally the real part of an analytic function.*

Formally, we can restate Lemma 2.6 as follows.

Lemma 2.7. *Let $D = \{z \in \mathbb{C} : |z-a| < r\}$ and $\tilde{D} = \{(x, y) \in \mathbb{R}^2 : x+iy \in D\}$. If $u : \tilde{D} \to \mathbb{R}^2$ is harmonic, we can find a $v : \tilde{D} \to \mathbb{R}^2$ (unique up to the addition of a constant) such that, if we write $f(x + iy) = u(x, y) + iv(x, y)$, the function $f : D \to \mathbb{C}$ is an analytic.*

The result can clearly be extended to a result which we state informally (since we do not yet have the apparatus to state it, let alone prove it).

Lemma 2.8. *A harmonic function on a simply connected open set (that is one which consists of a single piece with no holes) is a real part of an analytic function.*

However there are genuine topological limitations.

Example 2.9. *(i) Let*

$$\Omega = \{z = re^{i\theta} \,:\, r > 0,\ 2\pi + \theta_0 > \theta > \theta_0\}$$

and $\tilde{\Omega} = \{(x, y) \,:\, x + iy \in \Omega\}$. *If* $u(x, y) = \log |x + iy| = \log(x^2 + y^2)^{1/2}$, *then* u *is the real part of an analytic function defined by* $\log z = \log r + i\theta$ *where* r *and* θ *form the unique solution of* $z = re^{i\theta}$ *with* $r > 0$ *and* $2\pi + \theta_0 > \theta > \theta_0$.

(ii) If we define a real valued function u *on* $\tilde{\Gamma} = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *by* $u(x, y) = \log |x + iy|$, *then* u *is harmonic but we cannot find a real valued function* v *on* $\tilde{\Gamma}$ *such that, if we write* $f(x + iy) = u(x, y) + iv(x, y)$, *the function* $f : \Gamma = \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *is analytic.*

Definition 2.10. *Let*

$$\Omega = \{z = re^{i\theta} \,:\, r > 0,\ 2\pi + \theta_0 > \theta > \theta_0\}.$$

If we define

$$\log z = \log r + i\theta,$$

where r *and* θ *form the unique solution of* $z = re^{i\theta}$ *with* $r > 0$ *and* $2\pi + \theta_0 > \theta > \theta_0$, *then* log *is called a branch of the logarithm function.*

Lemma 2.11. *With the notation of Definition 2.10, we have the following results.*

(i) $\log : \Omega \to \mathbb{C}$ *is analytic.*

(ii) $\log(\Omega) = \{w \,:\, 2\pi + \theta_0 > \Im w > \theta_0\} = \Lambda$, *say.*

(iii) $\exp(\log z) = z$ *for all* $z \in \Omega$.

(iv) If $\Im w \notin 2\pi\mathbb{Z} + \theta_0$, *then* $\log(\exp w) = w$.

(v) $\log'(z) = z^{-1}$ *for all* $z \in \Omega$.

(vi) If $z_1,\ z_2,\ z_1z_2 \in \Omega$, *then* $\log z_1z_2 = \log z_1 + \log z_2 + 2n\pi i$ *for some* $n \in \mathbb{Z}$.

Lemma 2.12. *There does not exist a continuous function* $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *such that* $\exp(L(z)) = z$.

Exercise 2.13. *We use the notation of Definition 2.10. Show that we cannot choose* θ_0 *so that* $\log z_1z_2 = \log z_1 + \log z_2$ *for all* $z_1,\ z_2,\ z_1z_2 \in \Omega$.

In Analysis I, you saw that the easiest way to define x^α when x and α are real and $x > 0$ is to write $x^\alpha = \exp(\alpha \log x)$.

Definition 2.14. *We use the notation of Definition 2.10. Suppose* $\alpha \in \mathbb{C}$. *We define the map* $z \mapsto z^\alpha$ *on* Ω *by* $z^\alpha = \exp(\alpha \log z)$. *We call the resulting function a branch of* z^α.

Lemma 2.15. *If we define $p_\alpha : \Omega \to \mathbb{C}$ by $p_\alpha(z) = z^\alpha$ as in Definition 2.14, then p_α is analytic on Ω. We have*

$$p'_\alpha(z) = \alpha p_{\alpha-1}(z).$$

If α is real, $r > 0$ and $2\pi + \theta_0 > \theta > \theta_0$ then $p_\alpha(z) = r^\alpha \exp i\alpha\theta$ where r^α has its traditional meaning.

Except in the simplest circumstances, it is probably best to deal with z^α by rewriting it as $\exp(\alpha \log z)$.

If $0 > \theta_0 > -2\pi$, it is traditional to refer to the function defined by

$$\log re^{i\theta} = \log r + i\theta$$

for $r > 0$ and $2\pi + \theta_0 > \theta > \theta_0$ as the principal branch of the logarithm (with a similar convention for the associated powers). This has the same effect and utility as my referring to myself as the King of Siam.

3 Conformal mapping

We start with our definition of a conformal map.

Definition 3.1. *Let Ω and Γ be open subsets of $\mathbb{C}$. We say that $f : \Omega \to \Gamma$ is a conformal map if f is bijective and analytic and f' never vanishes.*

In more advanced work it is shown that, if f is bijective and analytic, then f' never vanishes. The phrase 'and f' never vanishes' can then be omitted from the definition.

Lemma 3.2. *Let Ω and Γ be open subsets of $\mathbb{C}$. If $f : \Omega \to \Gamma$ is conformal, then $f^{-1} : \Gamma \to \Omega$ is analytic. We have*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))},$$

so f^{-1} is also conformal.

Exercise 3.3. *We say that open subsets Ω and Γ of $\mathbb{C}$ are conformally equivalent if there exists a conformal map $f : \Omega \to \Gamma$. Show that conformal equivalence is an equivalence relation.*

The reader is warned that some mathematicians use definitions of conformal mapping which are not equivalent to ours. (The most common change is to drop the condition that f is bijective but to continue to insist that f' is never zero.) Sometimes people use conformal simply to mean angle preserving, so you must be prepared to be asked 'Show that an analytic map with non-zero derivative is conformal'.

So far as 1B examinations are concerned, we are chiefly interested in the following conformal maps.

(i) $z \mapsto z + a$. Translation. Takes $\mathbb{C}$ to $\mathbb{C}$.

(ii) $z \mapsto e^{i\theta}z$ where θ is real. Rotation. Takes $\mathbb{C}$ to $\mathbb{C}$.

(iii) $z \mapsto \lambda z$ where λ is real and $\lambda > 0$. Dilation (scaling). Takes $\mathbb{C}$ to $\mathbb{C}$.

(iv) $z \mapsto z^{-1}$. Inversion in unit circle followed by reflection in real axis. Takes circles and straight lines 'not through the origin' to circles and circles and lines 'through the origin' to straight lines. Takes $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$. [Note, in this course we are not interested in the 'point at infinity'.]

(v) $z \mapsto z^{\alpha}$ with α real and $\alpha > 0$. [N.B. You must specify a branch!] Takes (appropriate) sectors (of the form $\{re^{i\theta} : r > 0,\ \theta_1 > \theta > \theta_2\}$) to sectors with base angle multiplied by α.

(vi) $z \mapsto \exp z$. Takes (appropriate) planks

$$\{z : \theta_1 > \Im z > \theta_2\}$$

to sectors. The map $z \mapsto \log z$ [N.B. You must specify a branch!] does the reverse.

We observe that maps of the type (i) to (iv) are Möbius and together generate the Möbius group. Möbius maps were extensively discussed in the first year course 'Algebra and Geometry'. Observe also that we do not really need maps of type (v) explicitly, since we can obtain them using maps of the type (iii) and (vi).

The author strongly recommends constructing conformal maps in a large number of simple steps, as the composition of the simple maps given above, rather than trying to do everything at once.

Example 3.4. *Find a conformal map taking*

$$\Omega = \{z : \Im z > 0,\ \Re z > 0,\ |z| < 1\}$$

to the unit disc $D = \{z : |z| < 1\}$.

Explain why the map $z \mapsto z^4$ *does not work.*

It should be noted that conformal mapping problems like Example 3.4 do not have unique solutions since there are non-trivial conformal maps of the disc into itself (for example rotation).

In the early days of aviation, conformal mappings (of a very slightly more complicated kind) were used to find the flow of air past the wings of aeroplanes. The method depended on the following result.

Lemma 3.5. *Let Ω and Γ be open subsets of $\mathbb{C}$ and $f : \Omega \to \Gamma$ a conformal map. Set*

$$\tilde{\Omega} = \{(x, y) : x + iy \in \Omega\}, \ \tilde{\Gamma} = \{(x, y) : x + iy \in \Gamma\}$$

and let $\begin{pmatrix} u \\ v \end{pmatrix} : \tilde{\Omega} \to \tilde{\Gamma}$ be the mapping given by $f(x + iy) = u(x, y) + iv(x, y)$. Then, if $\phi : \tilde{\Gamma} \to \mathbb{R}$ is harmonic, so is $\psi : \tilde{\Omega} \to \mathbb{R}$ where $\psi(x, y) = \phi(u(x, y), v(x, y))$.

It must be admitted that the use of Lemma 3.5 and the general practice of conformal mapping at 1B level and substantially above it depends on the fact that, for the kind of Ω and Γ considered, the conformal map $f : \Omega \to \Gamma$ does, indeed, behave well near the boundaries. The reader is warned that, should she ever attend an advanced pure course on conformal maps or try to use theorems which merely guarantee the existence of such a map f without actually giving an explicit construction, this assumption can no longer be relied on[1].

4 Contour integration and Cauchy's theorem

It is natural to define the integral of a function $F : \mathbb{R} \to \mathbb{C}$ by

$$\int_a^b F(t)\, dt = \int_a^b \Re F(t)\, dt + i \int_a^b \Im F(t)\, dt.$$

In the course C9 (Analysis) it is shown that this definition produces an integral with all the properties we want. In addition, the following useful lemma is proved.

Lemma 4.1. *If $F : [a, b] \to \mathbb{C}$ is continuous then*

$$\left| \int_a^b F(t)\, dt \right| \leq (b - a) \sup_{t \in [a,b]} |F(t)|.$$

We summarise this result in a slogan

[1]

In the midst of the word he was trying to say,
In the midst of his laughter and glee,
He had softly and silently vanished away –
For the Snark *was* a Boojum, you see.

modulus integral $\leq$ length $\times$ supremum.

Next we wish to define the contour integral.

$$\int_C f(z)\,dz$$

where C is a path in $\mathbb{C}$. Roughly speaking

$$\int_C f(z)\,dz \approx \sum_{j=1}^{N} f(z_j)(z_j - z_{j-1})$$

where the polygonal path joining z_0, z_1, … z_N is a 'good approximation to C'. We formalise this idea as follows. (A function $g : \mathbb{R} \to \mathbb{C}$ is said to be differentiable if $\Re g$ and $\Im g$ are. We write $g'(t) = (\Re g)'(t) + i(\Im g)'(t)$.)

Definition 4.2. *If $\gamma : [a, b] \to \mathbb{C}$ is a sufficiently smooth*[2] *function describing the path C and $f : \mathbb{C} \to \mathbb{C}$ is continuous, we define*

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)dt.$$

If a path C is made up of a path C_1 followed by a path C_2 followed by a path C_3 … followed by a path C_n with each path satisfying the conditions of our definition, then we take

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} f(z)\,dz.$$

It is, more or less, clear that our definitions are unambiguous but a rigorous development would need to prove this. If a contour begins and ends at the same point we call it at closed contour. In this case, some older texts use the pleasant notation

$$\oint_C f(z)\,dz = \int_C f(z)\,dz.$$

Lemma 4.1 now takes the following form.

Lemma 4.3. *Under the conditions above,*

$$\left|\int_C f(z)\,dz\right| \leq \textit{length } C \times \sup_{z\in C} |f(z)|.$$

[2] Continuously differentiable will certainly do.

The next result is very important.

Lemma 4.4. *Suppose $a,\ w \in \mathbb{C}$ and $r \in \mathbb{R}$ with $r > 0$. Let C be the path $w + r\exp i\theta$ described as θ runs from 0 to 2π (less formally, the circle radius r and centre w described once anticlockwise). Then*

$$\int_C \frac{1}{z-a}\,dz = 2\pi i \qquad \text{if } |a-w| < r,$$
$$\int_C \frac{1}{z-a}\,dz = 0 \qquad \text{if } |a-w| > r.$$

Note that this illustrates an important point

Change of contour is not change of variable.

The next result has very little to do with the course but I could not resist including it.

Lemma 4.5. *If C is a closed contour which does not cross over itself and is described once anticlockwise then*

$$\int_C z^*\,dz = 2i \times \text{Area enclosed by } C.$$

We now come to our master theorem.

Theorem 4.6 (Cauchy's theorem). *Let Ω be an open, simply connected (that is all in a single piece and with no holes) set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an analytic function. Then*

$$\int_C f(z)\,dz = 0$$

Note Observe that Lemma 4.4 shows that the 'no holes' condition can not be dropped. Given a particular Ω it is usually trivial to check that it has no holes but a rigorous development of complex analysis for general Ω is somewhat delicate. With our sturdy English common sense we have banished the study of holes[3] to the higher reaches of pure mathematics but, in the US, some textbooks of mathematical methods for engineers devote quite a lot of time to it.

To save ink in future, we shall call Ω a simply connected domain if it is an open simply connected (that is one piece without holes) subset of $\mathbb{C}$. You should note that this notation is not universal.

Here is a nice application of Cauchy's theorem which foreshadows much of the course.

[3] Called homology by its practitioners.

Lemma 4.7. *If λ is real, then*

$$\int_{-\infty}^{\infty} e^{-i\lambda x} e^{-x^2/2}\, dx = (2\pi)^{1/2} e^{-\lambda^2/2}.$$

In particular,

$$\int_{-\infty}^{\infty} \cos(\lambda x) e^{-x^2/2}\, dx = (2\pi)^{1/2} e^{-\lambda^2/2}.$$

We remind the reader that 'Change of contour is not change of variable'.

5 Applications of Cauchy's theorem

Cauchy's theorem has far reaching implications. Our first result depends on the introduction of a well understood singularity.

Theorem 5.1 (Cauchy's formula). *Let Ω be a simply connected domain and $f : \Omega \to \mathbb{C}$ be analytic. If C is a closed contour in Ω which does not cross over itself and is described once anticlockwise, and a lies inside C, then*

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$$

Given a particular C and a particular a, it is usually trivial to check that C does not cross over itself and is described once anticlockwise, and that a lies inside C. However a rigorous development of these notions is somewhat delicate (to repeat the refrain of our song). We shall say that a C which does not cross over itself and is described once anticlockwise is a simple closed contour.

Differentiating under the integral (not hard to justify with the ideas of Analysis II) with respect to a we get the following result.

Theorem 5.2 (Cauchy's formula, extended version). *Let Ω be a simply connected domain and $f : \Omega \to \mathbb{C}$ be analytic. If C is a simple closed contour in Ω, and a lies inside C then f is n times differentiable with*

$$n! \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = 2\pi i f^{(n)}(a).$$

It is worth emphasising part of the result just given.

Theorem 5.3. *Let Ω be a open subset of $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is once complex differentiable then it is infinitely complex differentiable.*

This is a truly remarkable result, it is surely worth going to course C12 simply to see it proved rigorously!

Another remarkable result is the following.

Theorem 5.4 (Taylor's theorem). *Let Ω be a open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an analytic function. Suppose the disc*

$$D(b, \rho) = \{z : |z - b| < \rho\}$$

(with $\rho > 0$) is a subset of Ω. Then we can find a_n such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - b)^n$$

for all $z \in D(b, \rho)$. If $0 < r < \rho$ then

$$a_n = \frac{1}{2\pi i} \int_{C(r)} \frac{f(z)}{(z - b)^{n+1}}\, dz,$$

where $C(r)$ is the circular contour centre b and radius r described once anti-clockwise.

Note (as is shown in Exercise 10.20, also due to Cauchy) that there exist infinitely differentiable functions $E : \mathbb{R} \to \mathbb{R}$ which have no Taylor expansion. Taylor's theorem has the following important corollary.

Theorem 5.5. *Let Ω be a open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is analytic if and only if it can be expanded locally as a power series. (That is, given $w \in \Omega$, we can find a $\rho > 0$ such that $D(w, \rho) \subseteq \Omega$ and $a_n \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n$ for all $z \in D(w, \rho)$.)*

Exercise 5.6. *Deduce Theorem 5.2, in the case that C is a circle, from Theorem 5.4 and results on power series.*

Taylor's theorem for analytic functions has a striking and useful generalisation.

Theorem 5.7 (Laurent's expansion). *Let Ω be a open subset of $\mathbb{C}$ and let $b \in \Omega$. Suppose that $f : \Omega \setminus \{b\} \to \mathbb{C}$ is an analytic function and the disc*

$$D(w, \rho) = \{z : |z - b| < \rho\}$$

(with $\rho > 0$) is a subset of Ω. Then we can find a_n such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - b)^n$$

for all $z \in D(b,\rho)$. If $0 < r < \rho$ and $C(r)$ is the circular contour centre b and radius r described once anticlockwise, then

$$a_n = \frac{1}{2\pi i}\int_{C(r)} \frac{f(z)}{(z-b)^{n+1}}\,dz$$

for all $n \in \mathbb{Z}$.

Definition 5.8. *In Theorem 5.7, we call b an* isolated singularity*.*

We call a_{-1} the residue *at w.*

If $a_n = 0$ for all $n \leq -1$, we say that w is a removable singularity*.*

If there exists an $N \geq 1$ such that $a_{-N} \neq 0$ but $a_{-n} = 0$ for all $n > N$, we say that w is a pole *or more specifically that w is a* pole of order N*. Sometimes a pole of order 1 is called a* simple pole[4]*.*

If there does not exist an N with $a_{-n} = 0$ for all $n \geq N$, we call w an essential singularity*.*

The next lemma is really just a commentary on Definition 5.8

Lemma 5.9. *We continue with the notation of Theorem 5.7 and Definition 5.8.*

(i) The point b is a removable singularity if and only if we can find an analytic function $\tilde{f} : \Omega \to \mathbb{C}$ such that $f(z) = \tilde{f}(z)$ for all $z \in \Omega \setminus \{b\}$.

(ii) The point b is a pole of order exactly N if and only if we can find an analytic function $h : \Omega \to \mathbb{C}$ with $h(b) \neq 0$ such that $f(z) = (z-b)^{-N}h(z)$ for all $z \in \Omega \setminus \{b\}$.

Thus the behaviour of an analytic function in the neighbourhood of a removable singularity or a pole is no more difficult to understand than the behaviour of an analytic function away from singularities.

We shall see in the next section that there are particular reasons for wishing to calculate the residue.

Lemma 5.10. *(i) If $f(z) = g(z)(z-a)^{-1}$ with g analytic in a disc centre a and, then, if $g(a) \neq 0$, f has simple pole at a with residue $g(a)$ and, if $g(a) = 0$, f has a removable singularity at a.*

(i) If $f(z) = g(z)/h(z)$ with g and h analytic in a disc centre a and $h(a) = 0$, $h'(a) \neq 0$ then, if $g(a) \neq 0$, f has simple pole at a with residue $g(a)/h'(a)$ and, if $g(a) = 0$, f has a removable singularity at a.

If the pole is not simple, then power series expansion is often, though not always, the simplest way of proceeding.

[4]Complex analysts are much attached to the word 'simple'. Comment is unnecessary.

Example 5.11. *(The short question on paper I, 1998.) Find the residues of the following functions at $z = 0$, using the principle branch of* $\log$ *in (iii).*

$$\text{(i) } \cot z, \quad \text{(ii) } \frac{\sin z - z}{z^4}, \quad \text{(iii) } \frac{\log(\cos z)}{z(1-\cos z)},$$
$$\text{(iv) } \frac{\cos z}{z^2} \quad \text{and} \quad \text{(v) } z^3 \exp\left(\frac{1}{z}\right).$$

6 Calculus of residues

The reason for devoting special attention to the coefficient of $(z-w)^{-1}$ in the Laurent expansion (recall that we called it the 'residue') is revealed in the next theorem[5].

Theorem 6.1 (Cauchy's residue theorem). *Let Ω be a simply connected domain and $w_1, w_2, \ldots, w_n$ distinct points in Ω. Let*

$$f : \Omega \setminus \{w_1,\ w_2,\ \ldots,\ w_n\} \to \mathbb{C}$$

be analytic and let the residue at w_j be τ_j $[1 \le j \le n]$. If C is a simple closed contour enclosing a region which contains $\{w_1,\ w_2,\ \ldots,\ w_n\}$ [N.B. we do not allow the w_j to lie on C], then

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \tau_j.$$

Exercise 6.2. *Show that Theorem 5.2 is a special case of Theorem 6.1.*

In the same spirit, deduce Theorem 5.2 from Theorem 6.1 and the statement that every analytic function satisfies Taylor's theorem

$$f(w+h) = \sum_{j=0}^{\infty} \frac{f^{(j)}(w)}{j!} h^j$$

for $|h|$ sufficiently small.

Here are some typical applications of Cauchy's residue theorem. I have tried to place them in increasing order of complexity.

Example 6.3. *Show that*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{2^{1/2}}.$$

Deduce the value of $\int_0^\infty \frac{1}{1+x^4}\,dx$.

[5] If you invent a new and useful branch of mathematics, then you too can have all the theorems named after you.

Example 6.4. *Show that, if λ is real,*

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+x^2}\,dx = \pi e^{-|\lambda|}.$$

Our next integral requires a preliminary lemma.

Lemma 6.5 (Jordan's lemma). *Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous on the region $|z| > R_0$ and satisfies the inequality $|f(z)| < K/|z|$ for $|z| > R_0$. If $C(r)$ is the contour consisting of the semicircle $re^{i\theta}$ described as θ runs from 0 to π then, provided λ is real and strictly positive,*

$$\int_{C(r)} f(z)e^{i\lambda z}\,dz \to 0$$

as $r \to \infty$.

In the opinion of the writer, it is slightly unsporting to use Jordan's lemma when simpler estimates will do. It should also be noted that, if we *genuinely* need to use Jordan's lemma in the evaluation of a real integral, then that integral may only exist for certain definitions of the integral.

Example 6.6. *Show that*

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Exercise 6.7. *If $t \in \mathbb{R}$, let us write*

$$F(t) = \int_0^{\infty} \frac{\sin tx}{x}\,dx.$$

Show, using the result of Example 6.6 and change of variable, that

$$\begin{aligned} F(t) &= \frac{\pi}{2} && \text{for } t > 0,\\ F(0) &= 0, &&\\ F(t) &= -\frac{\pi}{2} && \text{for } t < 0. \end{aligned}$$

Example 6.8. *Show that, if α is real and $-1 < \alpha < 1$, then*

$$\int_0^{\infty} \frac{x^{\alpha}}{1+x^2}\,dx = \frac{\pi}{2\cos(\alpha\pi/2)}.$$

What happens if α lies outside the range $(-1, 1)$?

Our final example uses a slightly different idea.

Example 6.9. *Show that, if a is real and $a > 0$,*

$$\int_0^\pi \frac{a}{a^2 + \sin^2\theta}\, d\theta = \frac{\pi}{(1+a^2)^{1/2}}.$$

There is a mixture of good and bad news about contour integration.

(1) Most examples (particularly at 1B level) are based on combining a limited number of tricks. If you are stuck, try to identify parts of the problem which you have met before.

(2) The only way, for most people, to become fluent in contour integration is to do lots of examples yourself.

(3) Almost every book on complex analysis in your college library[6] will contain a chapter with a large collection of worked examples for you to take as model.

7 Fourier transforms

Many systems in nature, engineering and mathematics are linear and allow us to build complex solutions as linear combinations of simpler solutions. Thus, for example, light and sound may be considered as a mixture of simple, single frequency waves.

Mathematically we start by considering a single frequency wave

$$e^{i\omega t},$$

we then consider a sum of a finite number of such simple waves

$$\sum_{j=1}^{n} a_j e^{i\omega_j t}$$

with $a_j \in \mathbb{C}$ and are then driven to consider the integral analogue

$$\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega.$$

To emphasise the connection with Fourier series (see the course Mathematical Methods, C10) we use the following definition.

[6] Often an architectural gem and well worth visiting for its own sake.

Definition 7.1. *If $f : \mathbb{R} \to \mathbb{C}$ is reasonably well behaved, we define*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t}\, dt,$$

and call the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ the Fourier transform.

This is a methods course, so we shall not go into what is meant by good behaviour. However, the condition that f, f' and f'' are continuous and $t^2 f(t),\ t^2 f'(t),\ t^2 f''(t) \to 0$ as $|t| \to \infty$ are amply sufficient for our purpose (much less is required, but there always has to be some control over behaviour towards infinity).

The following results form part of the grammar of Fourier transforms.

Lemma 7.2. *(i) If $a \in \mathbb{R}$, let us write $f_a(t) = f(t - a)$. Then*

$$\hat{f}_a(\lambda) = e^{-ia\lambda} \hat{f}(\lambda).$$

(Translation on one side gives phase change on other.)

(ii) If $K \in \mathbb{R}$ and $K > 0$, let us write $f_K(t) = f(Kt)$. Then

$$\hat{f}_K(\lambda) = K^{-1} \hat{f}(\lambda/K).$$

(Narrowing on one side gives broadening on the other.)

(iii) $\hat{f}(\lambda)^* = (f^*)\hat{}(-\lambda)$.

(iv) $(\hat{f})'(\lambda) = -i\hat{F}(\lambda)$ where $F(t) = tf(t)$.

(v) $(f')\hat{}(\lambda) = i\lambda\hat{f}(\lambda)$.

The next result is both elegant and important.

Lemma 7.3. *We have*

$$\int_{-\infty}^{\infty} f(t)\hat{g}(t)\, dt = \int_{-\infty}^{\infty} \hat{f}(\lambda) g(\lambda)\, d\lambda.$$

Taking $g(\lambda) = \exp(-(K^{-1}\lambda)^2/2)$ and allowing $K \to \infty$, we obtain the key inversion formula.

Theorem 7.4 (Inversion formula). *We have $f\hat{}\hat{}(t) = 2\pi f(-t)$.*

In other words,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega.$$

Thus we can break down any (well behaved) function into its constituent frequencies and then reconstruct it.

The inversion formula gives a uniqueness result which is often more useful than the inversion formula itself.

Theorem 7.5 (Uniqueness). *If $\hat{f} = \hat{g}$ then $f = g$.*

Combining the inversion formula with Lemma 7.3, we get the following formula which is much loved by 1B examiners and of considerable theoretical importance.

Lemma 7.6 (Parseval's formula). [7] *We have*

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\lambda.$$

Fourier transforms are closely linked with the important operation of convolution.

Definition 7.7. *If $f,\ g : \mathbb{R} \to \mathbb{C}$ are well behaved, we define their* convolution *$f * g : i\mathbb{R} \to \mathbb{C}$ by*

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s) \, ds.$$

Lemma 7.8. *We have $\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$.*

For many mathematicians and engineers, Fourier transforms are important because they convert convolution into multiplication and convolution is important because it is transformed by Fourier transforms into multiplication. We shall see that convolutions occur naturally in the study of differential equations. It also occurs in probability theory where the sum $X + Y$ of two independent random variables X and Y with probability densities f_X and f_Y is $f_{X+Y} = f_X * f_Y$. In the next section we outline the connection of convolution with signal processing.

8 Signals and such like

Suppose we have a black box $\mathcal{K}$. If we feed in a signal $f : \mathbb{R} \to \mathbb{C}$ we will get out a transformed signal $\mathcal{K}f : \mathbb{R} \to \mathbb{C}$. Simple black boxes will have the following properties

(1) *Time invariance* If $\mathcal{T}_a f(t) = f(t-a)$, then $\mathcal{K}(\mathcal{T}_a f)(t) = (\mathcal{K}f)(t-a)$. In other words, $\mathcal{K}\mathcal{T}_a = \mathcal{T}_a\mathcal{K}$.

(2) *Causality* If $f(t) = 0$ for $t < 0$, then $(\mathcal{K}f)(t) = 0$ for $t < 0$. (The response to a signal cannot precede the signal.)

[7] The opera has an 'f' and goes on for longer.

(3) *Stability* Roughly speaking, the black box should consume rather than produce energy. Roughly speaking, again, if there exists a R such that $f(t) = 0$ for $|t| \geq R$, then we should have $(\mathcal{K}f)(t) \to 0$ as $t \to \infty$. If conditions like this do not apply, both our mathematics and our black box have a tendency to explode. (Unstable systems may be investigated using a close relative of the Fourier transform called the Laplace transform.)

(4) *Linearity* In order for the methods of this course to work, our black box must be linear, that is

$$\mathcal{K}(af + bg) = a\mathcal{K}(f) + b\mathcal{K}(g).$$

(Engineers sometimes spend a lot of effort converting non-linear systems to linear for precisely this reason.)

As our first example of such a system, let us consider the differential equation

$$F''(t) + (a + b)F'(t) + abF(t) = f(t) \qquad \bigstar$$

(where $a,\ b > 0$), subject to the boundary condition $F(t),\ F'(t) \to 0$ as $t \to -\infty$. We take $\mathcal{K}f = F$.

Before we can solve the system using Fourier transforms we need a preliminary definition and lemma.

Definition 8.1. *The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is given by*

$$\begin{aligned} H(t) &= 0 \qquad \textit{for } t < 0, \\ H(t) &= 1 \qquad \textit{for } t \geq 0. \end{aligned}$$

Lemma 8.2. *Suppose that $\Re\alpha < 0$. Then, if we set $e_\alpha(t) = e^{\alpha t}H(t)$, we obtain*

$$\hat{e}_\alpha(\lambda) = \frac{1}{i\lambda - \alpha}.$$

(Some applied mathematicians would leave out the condition $\Re\alpha < 0$ in the lemma just given and most would write $\hat{H}(\lambda) = 1/(i\lambda)$. The study of Laplace transforms reveals why this reckless behaviour does not lead to disaster.)

Lemma 8.3. *The solution $F = \mathcal{K}f$ of*

$$F''(t) + (a + b)F'(t) + abF(t) = f(t) \qquad \bigstar$$

(where $a,\ b > 0$), subject to the boundary condition $F(t),\ F'(t) \to 0$ as $t \to -\infty$, is given by

$$\mathcal{K}f = K \star f \ \textit{ where } K(t) = \frac{e^{-bt} - e^{-at}}{a - b}H(t).$$

Observe that $K(t) = 0$ for $t \leq 0$ and so, if $f(t) = 0$ for $t \leq 0$, we have

$$\mathcal{K}f(t) = K \star f(t) = 0 \qquad \text{for } t \leq 0,$$
$$\mathcal{K}f(t) = K \star f(t) = \int_0^t f(s)K(t-s)\,ds \qquad \text{for } t > 0.$$

Thus $\mathcal{K}$ is indeed causal.

There is another way of analysising black boxes. Let g_n be a sequence of functions such that

(i) $g_n(t) \geq 0$ for all t,

(ii) $\int_{-\infty}^{\infty} g_n(t)\,dt = 1$,

(iii) $g_n(t) = 0$ for $|t| > 1/n$.

In some sense, the g_n 'converge' towards the 'idealised impulse function' δ whose defining property runs as follows.

Definition 8.4. *If $f : \mathbb{R} \to \mathbb{R}$ is a well behaved function then*

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0).$$

If the black box is well behaved we expect $\mathcal{K}f_n$ to converge to some function E. We write

$$\mathcal{K}\delta = E$$

and say that the response of the black box to the delta function is the elementary solution E. Note that, since our black box is causal, $K(t) = 0$ for $t < 0$.

If f is a ordinary function, we define its translate by some real number a to be f_a where $f_a(t) = f(t-a)$. In the same way, we define the translate by a of the delta function by a to be δ_a where $\delta_a(t) = \delta(t-a)$ or, more formally, by

$$\int_{-\infty}^{\infty} f(t)\delta_a(t)\,dt = \int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a).$$

Since our black box is time invariant, we have

$$\mathcal{K}\delta_a = E_a$$

and, since it is linear,

$$\mathcal{K}\sum_{j=1}^{n} \lambda_j \delta_{a_j}(t) = \sum_{j=1}^{n} \lambda_j E_{a_j}(t).$$

In particular, if F is a well behaved function,

$$\begin{aligned}\mathcal{K}\sum_{j=-MN}^{MN} N^{-1}F(j/N)\delta_{j/N}(t) &= \sum_{j=-MN}^{MN} N^{-1}F(j/N)E_{j/N}(t)\\ &= \sum_{j=-MN}^{MN} N^{-1}F(j/N)E(t-j/N).\end{aligned}$$

Crossing our fingers and allowing M and N to tend to infinity, we obtain

$$\mathcal{K}F(t) = \int_{-\infty}^{\infty} F(s)E(t-s)\,ds,$$

so

$$\mathcal{K}F = F * E.$$

Thus the response of the black box to a signal F is obtained by convolving F with the response of the black box to the delta function. (This is why the acoustics of concert halls are tested by letting off starting pistols.) We now understand the importance of convolution, delta functions and elementary solutions in signal processing and the study of partial differential equations.

To see what happens in our specific example, we use Fourier transform methods find the elementary solution of equation ★.

Lemma 8.5. *The solution* $E = \mathcal{K}\delta$ *of*

$$E''(t) + (a+b)E'(t) + abE(t) = \delta(t) \qquad \bigstar$$

(where, $a,\ b > 0$*), subject to the boundary condition* $E(t),\ E'(t) \to 0$ *as* $t \to -\infty$*, is given by*

$$E(t) = \frac{e^{-bt} - e^{-at}}{a-b}H(t).$$

Observe that Lemma 8.5 implies Lemma 8.3 and vice versa.

9 Miscellany

The previous sections form a complete course and I shall be happy simply to cover it. If there is more time I will talk about some secondary topics.

Example 9.1. *(Question 7, Paper II, 1996.)*

(i) Find the poles of $\cot z$*, and the residues at them. Show that the first three terms of the Laurent expansion of* $\cot z$ *in* $0 < |z| < \pi$ *are*

$$\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45}.$$

(ii) Let Γ_N *be the rectangular contour with vertices at* $\pm(N + \frac{1}{2}) \pm iN$*, where* N *is any positive integer. Show that, on this contour,* $|\cot \pi z| \leq \coth \pi N$*. Hence, show that, for any integer* $r \geq 2$*,*

$$\int_{\Gamma_N} \frac{\cot \pi z}{z^r}\, dz \to 0 \text{ as } N \to \infty.$$

Hence, using Part (a), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

The interested student can push matters a little further.

Exercise 9.2. *Show that, if* k *is a strictly positive integer,*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = A_k \pi^{2k}$$

where A_k *is rational.*

Over two centuries have passed since Euler obtained the correct formula for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ but, apart from a recent result of Apéry to the effect that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational, we know nothing about $\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}}$.

The next result is not part of the course but crops up from time to time as a problem in the examinations. It is also quite useful to have the general idea of this result before embarking on the more cautious modern treatment in more advanced courses like C12.

Theorem 9.3. *Let* Ω *be a simply connected domain and* $f : \Omega \to \mathbb{C}$ *be analytic. Suppose that* C *is a simple closed contour in* Ω*. If* f *has no zeros on* C *and finitely many zeros within* C*, then the change in argument of* f *round* C

$$[\arg f]_C = 2\pi N$$

where N *is the number of zeros of* f *within* C*, multiple zeros being counted multiply.*

Finally, I include a note on the derivative δ' of the δ function. Personally, I consider this a bridge too far for this level, but some examiners can not refrain from introducing it.

Suppose f, $g : \mathbb{R} \to \mathbb{C}$ are well behaved (in particular f, f', g, g' decrease rapidly towards infinity). Then integration by parts gives

$$\int_{-\infty}^{\infty} f(t)g'(t)\,dt = -\int_{-\infty}^{\infty} f'(t)g(t)\,dt.$$

If f and all its derivatives are well behaved (with rapid decrease towards infinity) we are tempted to relax the conditions on g. If $g = H$, the Heaviside function, this leads to the *formal* manipulations

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)H'(t)\,dt &= -\int_{-\infty}^{\infty} f'(t)H(t)\,dt\\ &= -\int_{0}^{\infty} f'(t)\,dt\\ &= f(0)\end{aligned}$$

and to the satisfactory conclusion that, in some sense, $H' = \delta$.

If $g = \delta$ we get

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt = -\int_{-\infty}^{\infty} f'(t)\delta(t)\,dt = -f'(0).$$

The minus sign confuses many students (and, possibly, some of their elders and betters) but leaving out the minus sign leads to all sorts of inconsistencies.

10 Exercises

These exercises are divided into three groups. I suggest that you work through the questions in Part A in order, getting as far as you can in your alloted supervisions. Part B consists of questions for those who get through Part A quickly. They are not more difficult and will give you further practice. Part C consists of a few questions which are a bit skew to the course but which are quite interesting. The notation (Qx, Paper X, 19AB) tells you that the question is based on question x on Paper X in 19AB, but I have often made slight changes.

Part A

Q 10.1. Prove Lemma 1.2.

Q 10.2. Prove Theorem 1.4.

Q 10.3. (This is Exercise 2.13.) We use the notation of Definition 2.10. Show that we can not choose θ_0 so that $\log z_1 z_2 = \log z_1 + \log z_2$ for all z_1, z_2, $z_1 z_2 \in \Omega$.

Q 10.4. (i) Write out the standard properties of powers x^α when x and α are real and $x > 0$. (For example $(xy)^\alpha = x^\alpha y^\alpha$.) Investigate the extent to which they remain true in the complex case.

(ii) Show that $z^{1/3}$ has three possible branches on $\mathbb{C} \setminus \{x : x \text{ real and } x \geq 0\}$. Show that the same is true for $z^{2/3}$.

For each real α determine the number of branches (possibly infinite) of z^α.

Q 10.5. (This is Exercise 3.3.) We say that open subsets Ω and Γ of $\mathbb{C}$ are conformally equivalent if there exists a conformal map $f : \Omega \to \Gamma$. Show that conformal equivalence is an equivalence relation.

Q 10.6. Let

$$\Lambda = \{w \,|\, 2\pi + \theta_0 > \Im w > \theta_0\},\ \Omega = \{z = re^{i\theta} \,:\, r > 0,\ 2\pi + \theta_0 > \theta > \theta_0\}.$$

Let $f(z) = \exp z$ for $z \in \Lambda$. Show that $f : \Lambda \to \Omega$ is conformal and use Lemma 3.2 to establish the existence of a function log with properties given in Lemma 2.11.

Q 10.7. (Q7(b), Paper I, 1993) For each of the following conformal maps f_j and simply connected domains D_j find the image of the domain under the map (as usual $z = x + iy$).

$$\begin{array}{rll} \text{(i)} & f_1(z) = 1/(1+z), & D_1 = \{x + iy \,:\, x^2 + y^2 < 1,\ y > 0\} \\ \text{(ii)} & f_2(z) = z^2, & D_2 = \{x + iy \,:\, x > 0,\ y < 0\} \\ \text{(iii)} & f_3(z) = \log z, & D_3 = \{x + iy \,:\, y > 0\} \end{array}$$

(You should make it clear which branch of log you choose for f_3.)

Hence, or otherwise, show that

$$g(z) = \frac{1}{\pi} \log \left(-\frac{1}{4} \left(\frac{1-z}{1+z} \right)^2 \right)$$

is a conformal map of $\{x + iy \,:\, x^2 + y^2 < 1,\ y > 0\}$ onto the infinite strip

$$\{x + iy \,:\, 0 < y < 1\}.$$

Q 10.8. (Q7, Paper I, 2000) Let ϕ be a function of $u(x, y)$ and $v(x, y)$ which can also be regarded as a function of x and y. [I repeat the examiner's wording without necessarily approving it.] Starting from the formula

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x},$$

obtain the formula

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 \phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial \phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial \phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.$$

By using the Cauchy-Riemann equations and associated results, deduce that, if $w = u + iv$ is an analytic function of $z = x + iy$, then

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = |w'(z)|^2 \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right).$$

Explain briefly the relevance of this result to the solution of Laplace's equation via conformal mapping.

Q 10.9. (Q7(b), Paper I, 1994) Find a conformal mapping f that sends the unit disc $\mathcal{D} = \{z : |z| < 1\}$ onto the strip $\{z : -\pi/2 < \Im(z) < \pi/2\}$ for which $f(0) = 0$ and $f'(0)$ is real and positive.
[Cambridge exams are often a conspiracy between examiner and examinee. If you choose the 'obvious' conformal maps then you will either get the answer at once or obtain one which is easily converted into the required one. If you get a f that sends the unit disc onto the strip $\{z : -\pi/2 < \Im(z) < \pi/2\}$ but which you can not bring to the right form do not worry unduly but do not go on to the rest of the question.]

Find an analytic function h on $\mathcal{D}$ with the property that $|h(re^{i\theta})| \to e^{\pi/2}$ as $r \nearrow 1$ for $0 < \theta < \pi$ and $|h(re^{i\theta})| \to e^{-\pi/2}$ as $r \nearrow 1$ for $-\pi < \theta < 0$.

Q 10.10. (Q7(a), Paper II, 1993) Suppose f has a pole of order k at $z = 0$. Show that the residue of f at 0 is

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z^k f(z))\bigg|_{z=0}.$$

Let r and s be analytic functions such that $r(0) \neq 0$ and $s(0) = s'(0) = 0$, $s''(0) \neq 0$. Show that the residue of $r(z)/s(z)$ at $z = 0$ is

$$\frac{6r'(0)s''(0) - 2r(0)s'''(0)}{3(s''(0))^2}.$$

[My reason for including this is not to provide a formula for you to learn but to show that, once we move from simple poles, we can not expect simple 'one size fits all' methods for finding residues.]

Q 10.11. (Q7, Paper II, 2000) Evaluate the integrals $\oint_C f(z)\,dz$, where C is the unit circle centred at the origin and $f(z)$ is given by the following

$$\text{(a)}\ \frac{\sin z}{z}, \quad \text{(b)}\ \frac{\sin z}{z^2},\ \text{(c)}\ \frac{\cosh z - 1}{z^3},$$
$$\text{(d)}\ \frac{1}{z^2 \sin z},\ \text{(e)}\ \frac{1}{\cos 2z},\ \text{(f)}\ e^{1/z}.$$

Q 10.12. Use a result about $\int_{-\infty}^{\infty} e^{i\lambda x}/(1+x^2)\,dx$ already obtained in the course to show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

for $m > 0$.

Hence evaluate

$$\int_0^\infty \frac{\sin^2 x}{1+x^2}\,dx.$$

Q 10.13. (Q8(b), Paper IV, 1994) Consider the integral

$$I(a) = \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)^2}$$

where $0 < |a| < 1$. By means of the substitution $z = e^{i\theta}$, express $I(a)$ as an integral around the contour $|z| = 1$ and hence show that

$$I(a) = \frac{2\pi}{(1-a^2)^{3/2}}.$$

[The examiner added that no credit would be given for answers obtained by real methods.]

Q 10.14. (Q16, Paper II, 1997) By integrating a branch of $(\log z)/(1+z^4)$ about a suitable contour, show that

$$\int_0^\infty \frac{\log x}{1+x^4}\,dx = -\frac{\pi^2}{8\sqrt{2}}\ ,$$

and evaluate

$$\int_0^\infty \frac{1}{1+x^4}\,dx.$$

Q 10.15. (A golden oldie, last set as Q16, Paper I, 1998) Let

$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{(x+1)^3}\,dx,$$

where α is real. Use real methods to find the range of α for which the integral converges. Use real methods to evaluate $I(0)$ and $I(1)$.

Now consider the integral of $z^\alpha/(z+1)^3$ around a contour consisting of two circles of radius R and ϵ and straight lines on both sides of a cut along the positive real axis. What restrictions must be placed on α for the contributions from the circles to become negligible as $r \to \infty$ and $\epsilon \to 0$? Under such conditions, show that

$$I(\alpha) = \frac{\pi\alpha(1-\alpha)}{2\sin\pi\alpha}.$$

Show that I is continuous at 0 and 1.

Q 10.16. (Q7(a), Paper II, 1994) The function $h(t)$ vanishes for $t < 0$. The integral

$$P(t) = \int_{-\infty}^t h(t-\tau)f(\tau)\,d\tau$$

has the property that

$$\frac{df}{dt} = \int_{-\infty}^t h(t-\tau)P(\tau)\,d\tau$$

for all well behaved f. Find $\hat{h}(\omega)^2$.

Q 10.17. (Q7, Paper II, 1998) Suppose that

$$\hat{f}(\omega) = \frac{e^{i\omega} - e^{-i\omega}}{i\omega}$$

Find f by the following two methods.

(i) By using formulae for such things as the Fourier transforms of derivatives, translates and Heaviside type functions together with the uniqueness of Fourier transforms.

(ii) Directly from the inversion formula.

[*Hint: You will need to distinguish between,* $t < -1$, $-1 < t < 1$ *and* $t > 1$.]

Q 10.18. (Q16, Paper II, 1998) Use Fourier transform methods to solve the following integral equation for $f(t)$,

$$f(t) + \int_0^\infty e^{-s} f(t-s)\, ds = \begin{cases} e^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Evaluate the convolution integral for your solution and hence confirm that $f(t)$ solves the integral equation in the form stated above.

Q 10.19. (Q8(a), Paper IV, 1994) The analytic function $f(z)$ has P poles and Z zeros. All the poles and zeros lie strictly within the smooth, non-self-intersecting curve C. Using Cauchy's integral formulas, show that, if all the poles and zeros are simple,

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi(Z - P).$$

Explain how your result must be modified if the poles and zeros are not simple and prove the modified result.

Restate your result in terms of the argument of f.

Part B

Q 10.20. Cauchy gave the following example of a well behaved real function with no useful Taylor expansion about 0. It is important that you work through it at some stage in your mathematical life.

Let $E : \mathbb{R} \to \mathbb{R}$ be defined by $E(t) = \exp(-1/t^2)$ for $t \neq 0$ and $E(0)$. Use induction to show that E is infinitely differentiable with

$$E^{(n)}(t) = Q_n(1/t)E(t) \qquad \text{for } t \neq 0,$$
$$E^{(n)}(0) = 0.$$

For which values of t is it true that

$$E(t) = \sum_{n=0}^{\infty} \frac{E^n(0)t^n}{n!}?$$

Why does this not contradict Theorem 5.4?

Q 10.21. (Q7, Paper I, 1999) For each of the following five functions, state the region of the complex plane in which it is complex differentiable. State also the region in which it has partial derivatives and the region in which they satisfy the Cauchy-Riemann conditions.

$$
\begin{aligned}
&f_1(z) = |z|,\\
&f_2(z) = e^{-z},\\
&f_3(z) = z^*,\\
&f_4(z) = (z-1)^3,\\
&f_5(z) = |z|^2
\end{aligned}
$$

and f_6 given by

$$
f_6(x+iy) = \begin{cases} \frac{xy}{(x^2+y^2)^{1/2}} & \text{if } x+iy \neq 0,\\ 0 & \text{if } x+iy = 0. \end{cases}
$$

[*Note that, in one case, the region of complex differentiability does not coincide with that of the validity of the Cauchy-Riemann equations.*]

Q 10.22. (Q17, Paper IV, 1999) Write down the Cauchy-Riemann equations for the real and imaginary parts of the analytic function $w(z) = u(x,y) + iv(x,y)$, where $z = x+iy$. Show that $\nabla^2 u = \nabla^2 v = 0$ (i.e. that u and v are harmonic functions). Prove also that the curves of constant u in the x, y plane intersect those of constant v orthogonally.

Find analytic functions $w_1(z)$ and $w_2(z)$ that are real for real z and for which the following functions are their respective real parts:

(i) $u_1(x,y) = e^x \cos y$,

(ii) $u_2(x,y) = \dfrac{x(x^2+y^2+1)}{2(x^2+y^2)}$.

From your answet to (ii) find a non-zero harmonic function that vanishes on the circle $x^2+y^2 = 1$ and on the line $y = 0$.

In case (i), find the images in the z-plane of the circles $|w_1| = \rho$, for constant ρ.

In case (ii), find the images in the w_2-plane of the circles $|z| = r$, for constant $r > 1$.

Q 10.23. (Q7(b), Paper I, 1996) In four of the following five cases there exists a bijective analytic map $f : U \to V$. In one case there is a topological reason why no such map is possible. Find a suitable f in the four cases and briefly explain the fifth.

(i) $U = \{z \in \mathbb{C} : \Im(z) > 0\}$,	$V = \{z \in \mathbb{C} : \Im(z) > 0\}$
(ii) $U = \{z \in \mathbb{C} : \lvert z\rvert < 1\}$,	$V = \{z \in \mathbb{C} : \Re(z) > 0,\ \Im(z) > 0\}$
(iii) $U = \{z \in \mathbb{C} : 2 > \lvert z\rvert > 1\}$,	$V = \{z \in \mathbb{C} : \lvert z\rvert < 1\}$
(iv) $U = \mathbb{C} \setminus \{z \in \mathbb{R} : z \geq 0\}$,	$V = \mathbb{C} \setminus \{z \in \mathbb{R} : \lvert z\rvert \geq 1\}$
(v) $U = \{z \in \mathbb{C} : 0 < \Im(z) < 1\}$,	$V = \mathbb{C} \setminus \{z \in \mathbb{R} : z \geq 0\}$.

[*In case (iv) you may find it useful to consider the effect of a translation followed by the map $z \mapsto 1/z$.*]

Q 10.24. (Q7(a), Paper I, 1995) Find the residue at each of the poles of the function

$$f(z) = \frac{1}{z^2(1+z^4)}$$

in the complex plane.

Q 10.25. (Q7, Paper II, 1999) You are asked to find the Laurent expansion about $z = 0$ for each of the following three functions.

$$f_1(z) = e^{1/z},$$
$$f_2(z) = z^{-1/2},$$
$$f_3(z) = \frac{\sinh z}{z^3}.$$

In one case, you reply that you can not supply such an expansion. Why?

In the other two cases, where there is a Laurent expansion, state the nature of the singularity at $z = 0$ and find its residue.

Show that there is a function f analytic on $\mathbb{C}$ except, possibly, at finitely many points such that

$$f(z) = \sum_{n=1}^{\infty} z^{-n}$$

for $|z| > 1$. Find any singularities of $f(z)$ in the region $|z| \leq 1$ and find the residues at those singularities.

Q 10.26. (Q7(a), Paper II, 1995) What are the poles and associated residues of $f(z) = (\cosh z)^{-1}$ in the complex z-plane?

By considering a rectangular contour, or otherwise, evaluate the Fourier transform

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh x}\, dx.$$

Q 10.27. Show that, for $a > b > 0$, we have

$$I(a,b) = \int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a^2-b^2)}\left(\frac{1}{be^b} - \frac{1}{ae^a}\right).$$

Find $I(a,a)$ for $a > 0$ and check that $I(a,b) \to I(a,a)$ as $b \to a$.
Find $I(a,b)$ for all real non-zero values of a and b.

Q 10.28. (Q7(b), Paper II, 1993) By integrating around an appropriate closed curve in the complex plane cut along one half of the real axis, show that

$$I(a) = \int_0^\infty \frac{x^{a-1}}{1+x+x^2}\,dx = \frac{2\pi}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right)\csc(\pi a)$$

if $0 < a < 2$ and $a \neq 1$.
Evaluate $I(1)$ and show that $I(a) \to I(1)$ as $a \to 1$.

Q 10.29. (Q8, Paper IV, 1999) By interpreting the angle θ as the argument of a complex variable z, convert the real integral

$$I(\alpha) = \int_0^{2\pi} (1+\alpha\cos\theta)^{-1}\,d\theta \qquad (|\alpha| < 1)$$

into a contour integral in the z plane and hence evaluate it using the calculus of residues.

Q 10.30. (Q16, Paper I, 2000) By using a rectangular contour with corners at $\pm R$, $\pm R + i$ and taking the limit as $R \to \infty$, or otherwise, show that if a is real and $|a| < \pi$, then

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\,dx = \frac{1}{2}\sec\left(\frac{a}{2}\right).$$

Q 10.31. (Q16, Paper II, 1999) A real function $y(t)$ satisfies the differential equation

$$\ddot{y} + 4\dot{y} + 3y = f(t),$$

where $f(t)$ vanishes as $|t| \to \infty$ and has Fourier transform $F(\omega)$. Assuming that $y(t)$ and $\dot{y}(t)$ vanish as $|t| \to \infty$, find the Fourier transform $Y(\omega)$ of $y(t)$ in terms of in terms of $F(\omega)$. Hence show that $y(t)$ vanishes for $t < 0$ if $f(t)$ vanishes for $t < 0$. Comment on the significance of this fact.

Find the Fourier transform of the function

$$f(t) = H(t)e^{-t},$$

where $H(t)$ is the unit step function

$$H(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Hence determine the function $y(t)$.

Check, by direct substitution, that y is, indeed, the required solution. [*The examiner said that you could use the Fourier inversion theorem and Jordan's lemma without proof. In spite of this invitation, I suggest you might be better off using the uniqueness of Fourier transforms.*]

Q 10.32. (Q17, Paper IV, 1997) Use Fourier transforms to find g in the equation

$$5e^{-|t|} - 8e^{-2|t|} + 3e^{-3|t|} = \int_{-\infty}^{\infty} g(\tau)e^{-|t-\tau|}\,d\tau.$$

Q 10.33. (Q17, Paper IV, 2000) Assuming suitable decay of the function $w(x)$ as $x \to \pm\infty$, express the Fourier transforms of $w'(x)$ and $xw(x)$ in terms of the Fourier transform of $w(x)$.

Find the form of the Fourier transform of $w(x)$, if

$$x(w''(x) - w(x)) + w'(x) = 0. \qquad (*)$$

By using the inversion formula and a suitable change of variables, or otherwise, deduce that

$$w(x) = \frac{1}{\pi}\int_0^{\infty} \cos(x \sinh u)\,du$$

is a solution of $(*)$.

Part C

Q 10.34. (Q16, Paper II, 2000) [This is not really very hard but is a bit out of the ordinary and a bit beyond the syllabus.]

The complex plane is cut along the real axis from $z = -1$ to $z = 1$, and the branch of $f(z) = (z^2 - 1)^{1/2}$ is chosen so that $f(z)$ is real and positive when z is real and $z > 1$. Obtain expressions for $f(z)$ just above and just below the cut and also when $|z| \gg 1$.

Show that

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \pi(\sqrt{2} - 1),$$

where the square root gives positive values for $-1 < x < 1$.

Q 10.35. (Q8(b), Paper IV, 1996) [This is very much on the hard side, involves fairly messy calculations and is beyond the syllabus so you may consider it as starred. Unfortunately the examiner did not.]

In real life, Fourier integral techniques are most useful not for solving differential equations but for solving partial differential equations in the manner shown below. As a warm up, explain why the general solution of the partial differential equation

$$\frac{\partial F}{\partial x}(x,y) = 0$$

is $F(x,y) = A(y)$ and the general solution of

$$\frac{\partial^2 F}{\partial x^2}(x,y) + F(x,y) = 0$$

is $F(x,y) = A(y)\sin x + B(y)\cos x$.

A slab of material occupies the region $\{(x,y) : 0 \leq y \leq 1\}$ and moves with constant velocity U in the x-direction. The temperature $T(x,y)$ in the slab satisfies the 'advection diffusion equation'

$$U\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

and the boundary conditions are such that

$$T(x,0) = T_0 e^{-x^2},$$
$$T(x,1) = 0.$$

Let

$$\hat{T}(k,y) = \int_{-\infty}^{\infty} T(x,y)e^{-ikx}\,dx$$

(i.e. let $\hat{T}$ be the Fourier transform of T with respect to the first variable). Explain why

$$T(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T}(k,y)e^{ikx}\,dk.$$

By using differentiation under the integral and the uniqueness of the Fourier transform, obtain a partial differential equation for $\hat{T}$ only involving partial differentiation with respect to y.

Solve this differential equation, obtaining a result involving two unknown functions of k, call them $A(k)$ and $B(k)$. By setting $y = 0$ and $y = 1$ obtain $A(k)$ and $B(k)$ in terms of $\hat{T}(k,0)$ and $\hat{T}(k,1)$. Hence, find the function $\hat{T}(k,y)$. By exploiting symmetry, verify that $T(x,y)$ is real.

Q 10.36. (Q7(b), Paper II, 1995) [The first paragraph is routine. The second paragraph is heavily starred.] A linear system is such that that an input $g(t)$ is related to an output $f(t)$ by

$$f(t) = \int_{-\infty}^{t} K(t-t')g(t')\,dt'.$$

Let $\theta(t)$ be the step function defined by $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t \geq 0$. Suppose that $g(t) = \theta(t)e^{-\gamma t}$ where $\gamma > 0$ and that $f(t) = \theta(t)e^{-\gamma t}(1 - e^{-t})$. Find $K(t)$ for $t > 0$ assuming that $K(t) = 0$ for $t < 0$. Explain why you can drop the assumption $K(t) = 0$ for $t < 0$. Is it possible to find $K(t)$ for $t < 0$ from the information given? Why?

When $g(t) = \delta'(t)$, use the expression previously found for $K(t)$ to calculate $f(t)$ both (i) by direct evaluation of the integral and (ii) by calculating the Fourier transform of g in this case and hence finding the Fourier transform of f.
[Most students obtain different answers for (i) and (ii). If this happens to you, the object of the game is to discover what has gone wrong.]