CHAPTER 0 INTRODUCTION

Standard notation

$\mathbb{R}^n$ is the set of n-tples of real numbers with the usual Euclidean metric.

The n-sphere $S^n \subset \mathbb{R}^{n+1}$ is given by $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

The n-ball $B^n \subset \mathbb{R}^n$ is given by $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

The unit interval $I \subset \mathbb{R}$ is given by $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

The torus is $S^1 \times S^1$, the annulus is $S^1 \times I$.

Recall If (X, d) is a metric space a subset $U \subset X$ is called *open* if for each $x \in U$ there exists $\delta > 0$ such that $d(x, \tilde{x}) < \delta$ implies that $\tilde{x} \in U$.

Recall A *topological space* is a set X together with a collection of subsets of X called open sets such that

(i) $\emptyset$ and X are open,

(ii) any union of open sets is open,

(iii) if U_1 and U_2 are open then so is $U_1 \cap U_2$.

Definition. If X and Y are topological spaces a function $f : X \to Y$ is *continuous* if for every open $V \subset Y$ the set $f^{-1}(V)$ is open in X. A continuous function will be called a 'map'.

Definition. Topological spaces X and Y are *homeomorphic* (topologically 'the same') if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$, where 1_X and 1_Y are the identity maps. Note that functions are 'on the left'. These maps f and g are called homeomorphisms (each is a bijection).

Products. If X and Y are topological spaces, $X \times Y$ is the set of all pairs $\{(x, y)\}$ with the topology given by defining a set to be open if it can be expressed as some union of sets of the form $U \times V$, where U is open in X and V is open in Y.

Quotients. Suppose $\sim$ is an equivalence relation on the points of a topological space X. Let $X/\sim$ be the set of equivalence classes and let $q : X \to X/\sim$ be the quotient map ($q(x) = [x]$). A set $V \subset X/\sim$ is defined to be open if and only if $q^{-1}V$ is open in X.

Example. On S^n define $x \sim x'$ if and only if $x = \pm x'$. The quotient $S^n/\sim$ is called real projective n-space and denoted $\mathbb{R}P^n$.

When the idea of identifying (or 'gluing together') points of a space X is needed, define $\sim$ by $x \sim x'$ if and only if $x = x'$ or x and x' are required to be identified.

CHAPTER 1 HOMOTOPY AND THE FUNDAMENTAL GROUP

Definition. A *homotopy* between maps $f, g : X \to Y$ is a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Write $f \underset{F}{\simeq} g$ and let $F_t(x) = F(x, t)$.
If $A \subset X$ and $F(a, t) = f(a) = g(a)$ for all $(a, t) \in A \times I$ then the homotopy is *relative* to A.

1.1 Lemma. *Homotopy relative to A is an equivalence relation on the class of all maps $X \to Y$.*

1.2 Lemma. *Suppose that $f_0 \underset{F}{\simeq} f_1$ and $g_0 \underset{G}{\simeq} g_1$ where $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, then $g_0 f_0 \simeq g_1 f_1$.*

1.3 Lemma. *(Linear homotopy) Suppose $f, g : X \to Y \subset \mathbb{R}^n$ and that for all $x \in X$ the straight line segment joining $f(x)$ to $g(x)$ is contained in Y, then $f \simeq g$. If $f|A = g|A$ for some $A \subset X$ then $f \simeq g$ relative to A.*

Definition. Spaces X and Y are *homotopy equivalent* or *of the same homotopy type*, written $X \simeq Y$, if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$. The map f is a *homotopy equivalence* with g its *homotopy inverse* (and vice versa).

1.4 Lemma. *Homotopy equivalence is an equivalence relation on the class of all topological spaces.*

Definition. A space X is *contractible* if it is homotopy equivalent to a single point.

Definition. Suppose $r : X \to X$, $r(X) \subset A \subset X$ and $r|A = 1_A$. Then r is a *retraction* of X onto A and A is a *retract* of X. If $r \simeq 1_X$ relative to A then r is a (strong) *deformation retraction* of X to A. This implies that $X \simeq A$ and formalises the idea of X being squeezable to the subspace A.

Definition. A *path* in X from x_0 to x_1 is a map $u : I \to X$ such that $u(0) = x_0$ and $u(1) = x_1$. If $x_0 = x_1$ the path u is a *loop based at* x_0. The space X is *path connected* if, for any $x_0, x_1 \in X$, there exists a path from x_0 to x_1.

Definition. Suppose, for $i = 1, 2, \ldots, n$, that u_i is a path in X from x_{i-1} to x_i. Define the *product path* $u_1 \cdot u_2 \cdot \ldots \cdot u_n$ to be the path given by

$$(u_1 \cdot u_2 \cdot \ldots \cdot u_n)(s) = u_i(ns - i + 1) \quad \text{whenever} \quad \frac{i-1}{n} \leq s \leq \frac{i}{n}.$$

Definition. The *inverse* u^{-1} of a path u is defined by $u^{-1}(s) = u(1 - s)$. Note that $(u_1 \cdot u_2)^{-1} = u_2^{-1} \cdot u_1^{-1}$.

1.5 Lemma. *(i) Suppose that $u_i \underset{F_i}{\simeq} v_i$ relative to $\{0, 1\}$ where u_i and v_i are, for $i = 1, 2, \ldots, n$, paths in X from x_{i-1} to x_i, then $u_1 \cdot u_2 \cdot \ldots \cdot u_n \simeq v_1 \cdot v_2 \cdot \ldots \cdot v_n$ relative to $\{0, 1\}$.*
(ii) If $u \simeq v$ relative to $\{0, 1\}$ then $u^{-1} \simeq v^{-1}$ relative to $\{0, 1\}$.

1.6 Lemma. *(i) If u_i is, for $i = 1, 2, \ldots, n$, a path in X from x_{i-1} to x_i then $(u_1 \cdot u_2 \cdot \ldots \cdot u_r) \cdot (u_{r+1} \cdot u_{r+2} \cdot \ldots \cdot u_n) \simeq u_1 \cdot u_2 \cdot \ldots \cdot u_n$ relative to $\{0, 1\}$.*
(ii) If u is a path from x_0 to x_1 and e_0 and e_1 are the constant paths at x_0 to x_1 respectively, then $e_0 \cdot u \simeq u$ relative to $\{0, 1\}$ and $u \cdot e_1 \simeq u$ relative to $\{0, 1\}$.
(iii) $u \cdot u^{-1} \simeq e_0$ relative to $\{0, 1\}$ and $u^{-1} \cdot u \simeq e_1$ relative to $\{0, 1\}$.

1.7 Theorem. *The set of homotopy classes relative to $\{0, 1\}$ of loops based at $x_0 \in X$, together with a product defined by $[u][v] = [u \cdot v]$ (where $[u]$ is the homotopy class relative to $\{0, 1\}$ of loop u in X based at x_0), forms a group called the fundamental group of X with base point x_0 and denoted $\pi_1(X, x_0)$.*

1.8 Theorem. *A map $f : X, x_0 \to Y, y_0$ induces a group homomorphism $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that*

(i) *if $f \simeq f'$ relative to x_0 then $f_\star \simeq f'_\star$,*

(ii) *$(1_X)_\star$ is the identity homomorphism,*

(iii) *if $g : Y, y_0 \to Z, z_0$, then $(gf)_\star = g_\star f_\star : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.*

1.9 Theorem. *A path u in X from x_0 to x_1 induces an isomorphism $u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that*

(i) *$u \simeq \hat{u}$ relative to $\{0, 1\}$ implies $u_\# = \hat{u}_\#$,*

(ii) *$(e_0)_\#$ is the identity homomorphism,*

(iii) *if v is a path from x_1 to x_2 then $(u \cdot v)_\# = v_\# u_\#$,*

(iv) *if $f : X, x_0, x_1 \to Y, y_0, y_1$, then $(fu)_\# f_\star = f_\star u_\# : \pi_1(X, x_0) \to \pi_1(Y, y_1)$.*

1.10 Theorem. *Suppose that $f \underset{F}{\simeq} g : X \to Y$, that $x_0 \in X$ and that v is the path in Y from $f(x_0)$ to $g(x_0)$ defined by $v(t) = F(x_0, t)$. Then*

$$v_\# f_\star = g_\star \ : \ \pi_1(X, x_0) \to \pi_1(Y, g(x_0)).$$

Corollary Let $f : X, x_0 \to Y, y_0$ be a homotopy equivalence then $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups.

Definition. A space X is *simply connected* if it is path connected and $\pi_1(X, x_0)$ is the trivial group for some (and hence every) base point $x_0 \in X$.

CHAPTER 2 COVERING SPACES

In what follows X is a path connected topological space.

Definition. A *covering space* of X is a non-empty path connected space $\tilde{X}$ for which there is a (covering) map $p : \tilde{X} \to X$ such that for each $x \in X$ there exists an open neighbourhood V of x such that $p^{-1}V$ is a disjoint union of open sets in $\tilde{X}$ each of which is mapped homeomorphically by p onto V.
The map p is called the projection of the covering space $\tilde{X}$ to the base space X .

Examples.

(i) $p : \mathbb{R} \longrightarrow S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$ given by $p(t) = \exp(2\pi i t)$.

(ii) $p : S^1 \longrightarrow S^1$ given by $p(z) = z^n$.

(iii) $p : S^n \longrightarrow \mathbb{R}P^n \equiv S^n/(x \sim \pm x)$ where p is the quotient map.

(iv) $\overline{p} : S^3 \longrightarrow L_{p,q}$ where, for p and q coprime integers , $L_{p,q}$ is the 'lens space' defined as the quotient of S^3 by a certain action of the cyclic group C_p with generator g . Regarding S^3 as $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, the action is defined by $g(z_1, z_2) = \big(z_1 \exp(2\pi i/p),\ z_2 \exp(-2\pi i q/p)\big)$ and $\overline{p}$ is the quotient map.

2.1 Lemma (path lifting property). *Let $p : \tilde{X} \to X$ be a covering map. Suppose that $u : I \to X$ is a path in X and $\tilde{x}_0 \in \tilde{X}$ is such that $p(\tilde{x}_0) = u(0)$. Then there exists a unique path $\tilde{u} : I \to \tilde{X}$ such that $\tilde{u}(0) = \tilde{x}_0$ and $p\tilde{u} = u$.*

2.2 Lemma (homotopy lifting property). *Let $p : \tilde{X} \to X$ be a covering map. Suppose that $F : I \times I \to X$ and $\tilde{F} : I \times \{0\} \to \tilde{X}$ are such that $F(s, 0) = p\tilde{F}(s, 0)$ for all $s \in I$. Then there exists a unique extension of $\tilde{F}$ over the whole of $I \times I$ such that $p\tilde{F} = F$.*

2.3 Theorem. *Suppose that a group G acts as a group of homeomorphisms on a simply connected space Y. Suppose that each y belonging to Y has an open neighbourhood U such that $U \cap gU = \emptyset$ for all $g \in G - \{1\}$. Then $\pi_1(Y/G)$ is isomorphic to G.*

2.4 Lemma*. *Suppose that $p : \tilde{X} \to X$ is a covering map and for some $\tilde{x}_0 \in \tilde{X}$, $p(\tilde{x}_0) = x_0 \in X$. The group homomorphism $p_\star : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective and there is a bijection between the points of $p^{-1}x_0$ and the right cosets of $p_\star\pi_1(\tilde{X}, \tilde{x}_0)$ in $\pi_1(X, x_0)$.*

2.5 Proposition*. *Let $p : \tilde{X} \to X$ be a covering map and $p(\tilde{x}_0) = x_0$. Suppose Y is a path-connected, locally path-connected, space and $y_0 \in Y$. For any map $f : (Y, y_0) \to (X, x_0)$ there exists a map $g : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that $pg = f$ if and only if*

$$f_\star\, \pi_1(Y, y_0) \subset p_\star\, \pi_1(\tilde{X}, \tilde{x}_0)\ .$$

When such a g exists it is unique.

2.6 Theorem*. *If X is a path connected, locally contractible space, then X has a unique (up to equivalence) simply connected covering $p : \widehat{X} \to X$ and the group $\pi_1(X, x_0)$ acts on $\hat{X}$ with X as quotient.*

Definition. The above $\widehat{X}$ is called the *universal cover* of X.

CHAPTER 3 SIMPLICIAL COMPLEXES

Definition. The points $a_0, a_1, \ldots, a_n$ in $\mathbb{R}^N$ are (affinely) *independent* if $\{(a_i - a_0) : i = 1, 2, \ldots, n\}$ are independent vectors in $\mathbb{R}^N$. Thus $a_0, a_1, \ldots, a_n$ are independent if and only if $\sum_0^n \lambda_i a_i = 0$ with $\sum_0^n \lambda_i = 0$ implies that $\lambda_i = 0$ for each i.

Definition. Independent points $a_0, a_1, \ldots, a_n$ in $\mathbb{R}^N$ are the *vertices* of an *n-dimensional simplex* σ in $\mathbb{R}^N$ where

$$\sigma = \{\sum_0^n \lambda_i a_i : 0 \le \lambda_i \in \mathbb{R},\ \sum_0^n \lambda_i = 1\}\ .$$

The $\{\lambda_i\}$ are the *barycentric coordinates* of the point $\sum_0^n \lambda_i a_i$.

Write $\sigma = (a_0 a_1 \ldots a_n)$.

A simplex τ is a *face* of σ, written $\tau \le \sigma$, if $\{\text{vertices } \tau\} \subseteq \{\text{vertices } \sigma\}$ and τ is a proper face if $\tau \neq \sigma$. Note that $\emptyset \le \sigma$ for any simplex σ. By definition σ°, the *interior* of σ, is the set $\sigma - \bigcup\{\text{proper faces of}\sigma\}$.

$$\sigma^\circ = \{\sum_0^n \lambda_i a_i : 0 < \lambda_i \in \mathbb{R},\ \sum_0^n \lambda_i = 1\}\ .$$

The *barycentre* $\hat{\sigma}$ of σ is $\frac{1}{n+1}(a_0 + a_1 + \ldots + a_n) \in \sigma^\circ$.

Definition. A (finite) *simplicial complex* K is a finite collection of simplexes in some $\mathbb{R}^N$ such that

(i) if $\sigma \in K$ and $\tau \leq \sigma$ then $\tau \in K$,

(ii) if $\sigma \in K$ and $\tau \in K$ then $\sigma \cap \tau \leq \sigma$.

A *subcomplex* of K is a subcollection of the simplexes of K that satisfies (i) (and hence also (ii)).

A simplex σ together with all of its faces is an obvious example of a simplicial complex; this will often also be denoted by σ.

The *underlying* polyhedron $|K|$ of K is the union of all simplexes in K.

The *dimension* $\dim K$ of K is the maximal dimension of a simplex in K.

Definition. Let K and L be simplicial complexes. A *simplicial map* $f : K \to L$ is a function $f : \{\text{vertices of } K\} \to \{\text{vertices of } L\}$ such that for every simplex $(a_0a_1 \ldots a_n) \in K$, the points $\{f(a_0), f(a_1), \ldots, f(a_n)\}$ are the vertices of some simplex in L (though maybe $f(a_i) = f(a_j)$).
Extending f by defining $f \sum \lambda_i a_i = \sum \lambda_i f a_i$ gives a continuous function $f : |K| \to |L|$.
Of course, this $f : |K| \to |L|$ might be an injection (whereupon it is often called an embedding), a surjection, or a bijection which is often referred to as a simplicial isomorphism.

3.1 Lemma. *$|K|$ is the disjoint union $\bigsqcup_{\sigma \in K} \sigma^\circ$.*

Definition. If a is a vertex (that is a 0-simplex) in K then star $(a, K) = \bigcup_{a \leq \sigma \in K} \sigma^\circ$.

3.2 Lemma. *The sets $\{\text{star } (a, K) : a \text{ a vertex of } K\}$ form an open cover of $|K|$.*

Definition. A simplicial map $f : K \to L$ is a simplicial approximation to a (continuous) map $\phi : |K| \to |L|$ if, for every vertex $a \in K$,

$$\phi(\text{star } (a, K)) \;\subset\; \text{star } (f(a), L).$$

Note that the composition of approximations is an approximation to the composition of maps.

3.3 Lemma. *Suppose that $f : K \to L$ is a simplicial approximation to a map $\phi : |K| \to |L|$. Let $A = \{x \in |K| : f(x) = \phi(x)\}$. The $\phi \simeq f$ relative to A.*

3.4 Lemma. *Suppose that $\phi : |K| \to |L|$ and for each vertex $a_i \in K$ there exists a vertex $b_i \in L$ such that $\phi(\text{star } (a_i, K)) \subset \text{star } (b_i, L)$ then there is a simplicial approximation f to ϕ such that $f(a_i) = b_i$.*

Definition. The *first derived subdivision* $K^{(1)}$ of a simplicial complex K is defined by

$$K^{(1)} = \left\{(\hat{\sigma}_0\hat{\sigma}_1 \ldots \hat{\sigma}_r) : \sigma_0 < \sigma_1 < \ldots < \sigma_r \in K\right\}$$

where barycentre $\hat{\sigma}_i$ is the barycentre of σ_i. The r^{th} derived subdivision $K^{(r)}$ is defined inductively by $K^{(r)} = (K^{(r-1)})^{(1)}$.

Definition. The *mesh* of a simplicial complex K is defined to be the maximum of the diameters of all the star $(a, K))$ where a is a vertex of K.

3.5 Lemma. *Let K be a simplicial complex. Given $\epsilon > 0$ there exists an r such that* mesh $K^{(r)} < \epsilon$.

3.6 Theorem. *Let K and L be simplicial complexes and $\phi : |K| \to |L|$ be a (continuous) map. For any r sufficiently large, there exists a simplicial approximation $f : K^{(r)} \to L$ to $\phi : |K^{(r)}| \to |L|$.*

Definition. Two simplicial maps $f, g : K \to L$ are *contiguous* if for every $\sigma \in K$ there exists a $\tau \in L$ such that both $f\sigma \leq \tau$ and $g\sigma \leq \tau$. (This implies that $f \simeq g$.)

3.7 Lemma. *If $f, g : |K^{(r)}| \to |L|$ are both simplicial approximations to $\phi : |K^{(r)}| \to |L|$ then f and g are contiguous.*

3.8 Lemma. *Let K and L be simplicial complexes. There exists $\delta > 0$ such that, if two maps $\phi, \psi : |K| \to |L|$ are such that $d(\phi(x), \psi(x)) < \delta$ for all $x \in |K|$, then for some r there is a simplicial map $f : K^{(r)} \to L$ that is an approximation to both ϕ and ψ.*

CHAPTER 4 HOMOLOGY GROUPS OF SIMPLICIAL COMPLEXES

Definition. An ordered simplex is a simplex together with an ordering assigned to its vertices. Write an ordered simplex σ as $\sigma = (a_0 a_1 \ldots a_n)$ when the ordering is $a_0 < a_1 < \ldots a_n$.

Definition. The n^{th} *chain group* $C_n(K)$ of a finite simplicial complex K is , for $n \geq 0$, the free abelian group generated by (symbols in one to one correspondence with) all ordered n-simplexes in K with all possible orderings quotiented by the group generated by

$$\big\{(a_0 a_1 \ldots a_n) - \epsilon_\pi (a_{\pi 0} a_{\pi 1} \ldots a_{\pi n}) : \pi \in \Sigma_{n+1},\ (a_0 a_1 \ldots a_n) \in K\big\}.$$

Here Σ_{n+1} is the permutation group of $\{0, 1, \ldots, n\}$. By convention $C_n(K) = 0$ if $n < 0$ or $n > \dim K$.

Definition. The *boundary homomorphism* $d_n : C_n(K) \to C_{n-1}(K)$ is the homomorphism defined on generators by

$$d_n(a_0 a_1 \ldots a_n) = \sum_{i=0}^{n} (-1)^i (a_0 a_1 \ldots a_{i-1} a_{i+1} \ldots a_n).$$

Notation: $(a_0 a_1 \ldots a_{i-1} a_{i+1} \ldots a_n) = (a_0 a_1 \ldots \overset{i}{\uparrow} \ldots a_n)$.

4.1 Lemma. *The boundary homomorphism $d_n : C_n(K) \to C_{n-1}(K)$ is well defined.*

4.2 Lemma. *$d^2 = 0$, that is, $C_n(K) \xrightarrow{d_n} C_{n-1}(K) \xrightarrow{d_{n-1}} C_{n-2}(K)$ is the zero homomorphism.*

Note. A collection of groups and homomorphisms $\{C_n, d_n\}$ such that $d_{n-1} d_n = 0$ is called a *chain complex*.

Definition. In $C_n(K)$, the *n-boundary chains* $B_n(K)$ are the image of $d_{n+1} : C_{n+1}(K) \to C_n(K)$, the *$n$-cycles* $Z_n(K)$ are the kernel of $d_n : C_n(K) \to C_{n-1}(K)$ and the *n^{th}-homology group* $H_n(K)$ is the quotient $Z_n(K)/B_n(K)$.

Definition. Suppose that $f : K \to L$ is a *simplicial map*. Define the induced chain homomorphism $f_n : C_n(K) \to C_n(L)$ by $f_n(a_0 a_1 \ldots a_n) = (fa_0 fa_1 \ldots fa_n)$ if $\{fa_0, fa_1, \ldots, fa_n\}$ are all distinct and $f_n(a_0 a_1 \ldots a_n) = 0$ otherwise.

4.3 Lemma. *If $f: K \to L$ is a simplicial map then*

(i) $f_{n-1}d_n^K = d_n^L f_n$

(ii) *if f is the identity so is f_n ,*

(iii) $(gf)_n = g_n f_n$.

Corollary. f induces $\{f_n\}$ which induces $f_*: H_n(K) \to H_n(L)$ such that $1_* = 1$ and $(fg)_* = f_* g_*$.

Definition. A collection of homomorphisms $\{f_n : C_n(K) \to C_n(L)\}$ between the groups of two chain complexes is a *chain map* if $f_{n-1}d_n^K = d_n^L f_n$. A *chain homotopy* between chain maps $\{f_n : C_n(K) \to C_n(L)\}$ and $\{g_n : C_n(K) \to C_n(L)\}$ is a collection of homomorphisms $\{h_n : C_n(K) \to C_{n+1}(L)\}$ such that

$$f_n - g_n = d_{n+1}^L h_n + h_{n-1} d_n^K .$$

4.4 Lemma. *If $\{f_n : C_n(K) \to C_n(L)\}$ and $\{g_n : C_n(K) \to C_n(L)\}$ are chain homotopic then $f_* = g_* : H_n(K) \to H_n(L)$ for all n.*

Definition. Suppose that K is a simplicial complex in $\mathbb{R}^N$ and $v \in \mathbb{R}^{N+1} - \mathbb{R}^N$, then the cone vK with vertex v and base K is the simplicial complex $\{v\} \cup K \cup \{(va_0a_1 \ldots a_n) : (a_0a_1 \ldots a_n) \in K\}$.

4.5 Lemma. $H_n(vK) \cong H_n(v) \cong \begin{cases} \mathbb{Z} & \textit{if n = 0,} \\ 0 & \textit{otherwise.} \end{cases}$

Corollary. Let σ be the simplicial complex consisting of just one $(n+1)$-simplex and all its faces and let $\partial\sigma$ be the subcomplex consisting of its *proper* faces only. Then

$$H_r(\sigma) \cong \begin{cases} \mathbb{Z} & \text{if r = 0,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and, if } n \geq 1, \quad H_r(\partial\sigma) \cong \begin{cases} \mathbb{Z} & \text{if r = 0 or n,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition. A sequence $\ldots \longrightarrow G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} G_{n-2} \longrightarrow \ldots$ of groups and homomorphisms is called *exact* if, for all n, kernel $f_{n-1} =$ image f_n.

4.6 Theorem (Mayer Vietoris). *Let L and M be subcomplexes of a simplicial complex K such that $K = L \cup M$. Then there is an exact sequence*

$$\ldots \longrightarrow H_n(L \cap M) \xrightarrow{\alpha_*} H_n(L) \oplus H_n(M) \xrightarrow{\beta_*} H_n(K) \xrightarrow{\Delta_n} H_{n-1}(L \cap M) \xrightarrow{\alpha_*} \ldots$$

in which $\alpha_*(x) = (i_*^1(x), i_*^2(x))$ and $\beta_*(y,z) = j_*^1(y) - j_*^2(z)$, where i^1 and i^2 are the inclusion maps of $L \cap M$ into L and M respectively and j^1 and j^2 are the inclusion maps of L and M respectively into K.

CHAPTER 5 INVARIANCE OF HOMOLOGY GROUPS

5.1 Lemma. *Suppose α : {vertices of $K^{(1)}$} $\to$ {vertices of K} is such that $\alpha(\hat{\sigma})$ is a vertex of σ for every $\sigma \in K$. Then α is a simplicial map $\alpha : K^{(1)} \to K$ that is a simplicial approximation to the identity map $|K^{(1)}| \to |K|$.*

5.2 Lemma. *There is a chain map $\{\theta_n : C_n(K) \to C_n(K^{(1)})\}$ so that if σ is an n-simplex of K then $\theta_n(\sigma) = \sum_{\tau^\circ \subset \sigma^\circ} \pm\tau$.*

5.3 Lemma. *Suppose that* $\{f_n : C_n(K) \to C_n(L)\}$ *and* $\{g_n : C_n(K) \to C_n(L)\}$*are chain maps such that*

(i) f_0 *and* g_0 *map generators (vertices) to generators,*

(ii) for every n*-simplex* $\sigma \in K$ *there is a cone* Λ_σ*, a subcomplex of* L*, such that* $f_n\sigma \in C_n(\Lambda_\sigma)$ *and* $g_n\sigma \in C_n(\Lambda_\sigma)$ *and*

(iii) if $\tau < \sigma$ *then* $\Lambda_\tau \subset \Lambda_\sigma$.
Then $f_* = g_* : H_r(K) \to H_r(L)$ *for all* r.

Corollary. With notation from 5.1 and 5.2 above, $\alpha_* : H_n(K^{(1)}) \to H_n(K)$ and $\theta_* : H_n(K) \to H_n(K^{(1)})$ are mutually inverse isomorphisms.

5.4 Theorem. *A (continuous) map* $\phi : |K| \to |L|$ *induces for each* n *a well defined homomorphism* $\phi_* : H_n(K) \to H_n(L)$ *such that* $1_* = 1$ *and* $(\psi\phi)_* = \psi_*\phi_*$.

Corollary. If ϕ is a homeomorphism then ϕ_* is an isomorphism.

5.5 Theorem. *If* $\phi \underset{F}{\simeq} \psi : |K| \to |L|$ *then* $\phi_* = \psi_*$.

5.6 Theorem (The Brouwer fixed point theorem). *Any (continuous) map* $\phi : B^n \to B^n$ *has a fixed point.*

CHAPTER 6 CLASSIFICATION OF SURFACES

Definition. An n-manifold without boundary is a (Hausdorff and second countable) topological space M with the property that for each $x \in M$ there is an open set U, with $x \in U \subset M$, such that U is homeomorphic to $\mathbb{R}^n$.

Definition. Suppose M_1 and M_2 are connected n-manifolds and B_1 and B_2 are n-balls with $B_1 \subset M_1$ and $B_2 \subset M_2$. The manifold $\big(M_1 - \operatorname{int} B_1\big) \cup_h \big(M_2 - \operatorname{int} B_2\big)$, where h is a homeomorphism from the boundary of B_1 to the boundary of B_2, is called the connected sum $M_1 \# M_2$.

6.1 Theorem*. *Let* M *be a compact connected 2-manifold without boundary then* M *is homeomorphic to one and only one of*

(a) $M_0 = S^2$

(b) $M_g = T \# T \# \ldots \# T$*, where* T *is the torus* $S^1 \times S^1$ *and there are* g *summands*

(c) $N_h = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$*, where* $\mathbb{R}P^2$ *is the real projective plane and there are* h *summands.*

Note. The Mayer-Vietoris Theorem implies that

$$H_1(M_g) \cong \bigoplus_{2g \text{ copies}} \mathbb{Z}\,, \qquad H_1(N_h) \cong \mathbb{Z}/2 \oplus \bigoplus_{(h-1) \text{ copies}} \mathbb{Z}\,.$$