EXCURSIONS OF BIRTH AND DEATH PROCESSES, ORTHOGONAL POLYNOMIALS, AND CONTINUED FRACTIONS

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Abstract. On the basis of Karlin and McGregor result, which states that the transition probability functions of a birth and death process can be expressed via the introduction of an orthogonal polynomial system and a spectral measure, we investigate in this paper how the Laplace transforms and the distributions of different transient characteristics related to excursions of a birth and death process can be expressed by means of the basic orthogonal polynomial system and the spectral measure. This allows us in particular to give a probabilistic interpretation of the series introduced by Stieltjes to study the convergence of the fundamental continued fraction associated with the system. Throughout the paper, we pay special attention to the case when the birth and death process is ergodic. Under the assumption that the spectrum of the spectral measure is discrete, we show how the distributions of different random variables associated with excursions depend on the fundamental continued fraction, the orthogonal polynomial system and the spectral measure.

Keywords. Continued fraction, birth and death process, spectral measure, orthogonal polynomial.

1. Introduction

The connection between probability theory, continued fractions, and orthogonal polynomials systems (OPS) is usually addressed in the literature via the very fundamental result of Karlin and McGregor [12], which states that the transition probability functions of a birth and death process can be expressed by means of some orthogonal polynomials associated with some spectral measure. This is equivalent to claim that the backward Chapman-Kolmogorov equations can be solved via the introduction of an OPS and a spectral measure.

Specifically, consider a birth and death processes $\{\Lambda_t\}$ with state space $\{0,1,2,3,\ldots\}$ and defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$; the transition rates of the process $\{\Lambda_t\}$ are denoted by

$$q_{m,m+1}=\lambda_m>0,\ q_{m,m-1}=\mu_m,\ q_{m,m}=-\left(\lambda_m+\mu_m\right)\ \text{for } m\geq 0; \tag{1.1}$$

$q_{m,n}=0$ otherwise. (The rate μ_0 is equal to 0 and $\mu_m>0$ for $m>0$.) Karlin and McGregor result asserts that there exist an OPS $\{Q_n(x)\}$ and a regular positive spectral measure $\mu(dx)$ of total mass one and not supported by a finite set of points so that

$$\text{for } m,n\geq 0,\ p_{m,n}(t)\stackrel{\text{def}}{=}\mathbb{P}\{\Lambda_t=n\mid\Lambda_0=m\}=\pi_n\int_0^\infty e^{-tx}Q_n(x)Q_m(x)\mu(dx) \tag{1.2}$$

and

$$\int_0^\infty Q_m(x)Q_n(x)\mu(dx)=0 \text{ for } m\neq n,\ \text{and}\ \int_0^\infty Q_m^2(x)\mu(dx)=\frac{1}{\pi_m}, \tag{1.3}$$

where the polynomials $\{Q_n(x)\}$ form an OPS with respect to the spectral measure $\mu(dx)$ and satisfy the three-term recurrence relation

$$\begin{cases} Q_0(x)\equiv 1,\quad Q_{-1}(x)\equiv 0,\\ \lambda_nQ_{n+1}(x)+(x-\lambda_n-\mu_n)Q_n(x)+\mu_nQ_{n-1}(x)=0,\ n\geq 0, \end{cases} \tag{1.4}$$

and the quantities π_m are defined by

$$\pi_0 = 1 \text{ and } \pi_m = \frac{\lambda_0 \dots \lambda_{m-1}}{\mu_1 \dots \mu_m} \text{ for } m \geq 1. \tag{1.5}$$

The OPS $\{Q_n(x)\}$ will be referred to in the following as the fundamental OPS associated with the birth and death process $\{\Lambda_t\}$.

The result of Karlin McGregor can notably be used to classify birth and death processes (especially linear-growth birth an death processes), where the concepts of recurrence and transience are studied in a purely analytic way instead of a probabilistic one [13]. The result is also of great interest for the community dealing with the properties of orthogonal polynomials occurring in the modelling of physical systems (see [10] for instance).

A complementary approach to the connection between probability theory and OPS [5] consists in using the continued fraction defined by

$$\text{for } \Re(z) > 0, \ \tilde{p}_0(z) = \cfrac{1}{z+\lambda_0 - \cfrac{\lambda_0\mu_1}{z+\lambda_1+\mu_1 - \cfrac{\lambda_1\mu_2}{z+\lambda_2+\mu_2 - \ddots}}}, \tag{1.6}$$

and representing the Laplace transform of $p_{0,0}(t) = \mathbb{P}\{\Lambda_t = 0 \mid \Lambda_0 = 0\}$ (i.e., $\tilde{p}_0(z) = \int_0^\infty e^{-zt} p_{0,0}(t)dt$). The continued fraction and the associated OPS can then be used to study the recurrence and the transience of the birth and death process.

We adopt in this paper a slightly different approach to the aforementioned connection. We specifically show how the theory of continued fractions naturally arises when studying some transient characteristics associated with the birth and death process $\{\Lambda_t\}$, namely the time to an excursion above a given threshold starting from a given state, and the duration of an excursion above a given threshold. In analogy with the $M/M/\infty$ system studied in [8], special attention will be paid in this paper to the case when the birth and death process $\{\Lambda_t\}$ is an ergodic Markov chain, which is a natural assumption in a probabilistic setting. This assumption entails in particular that [4]

$$\sum_{m=0}^{\infty} \pi_m < \infty. \tag{C$_1$}$$

Moreover, we shall examine the case when the birth and death process satisfies Aldous' local linearization property [2]. (The exact meaning of this property will be detailed in the subsequent sections.) Under condition (C_1), the stationary distribution $\{p_m\}$ of the process $\{\Lambda_t\}$ is given by

$$p_0 = \frac{1}{\sum_{m=0}^{\infty} \pi_m} > 0 \text{ and for } m \geq 1, \ p_m = \frac{\pi_m}{\sum_{m=0}^{\infty} \pi_m} > 0. \tag{1.7}$$

The contribution of this paper can be summarized as follows:

- we give a probabilistic interpretation of the series introduced by Stieltjes to study the convergence of the continued fraction $\tilde{p}_0(z)$;
- we show how the Laplace transforms of different transient characteristics associated with excursions of the birth and death process can be expressed in terms of the fundamental OPS and the continued fraction $\tilde{p}_0$;
- under the assumption that the spectrum of the measure $\mu(dx)$ is discrete, we show how the distributions of different transient characteristics depend on the continued fraction $\tilde{p}_0(z)$, the basic OPS and the spectral measure $\mu(dx)$.

2. Basic Definitions and Results

We recall in this section some basic concepts and results, which hold for the continued fraction $\tilde{p}_0(z)$. The successive denominators $\{P_n^-(z)\}$ of the continued fraction $\tilde{p}_0(z)$ satisfy the Wallis recurrence relations [9]

$$(2.1)\qquad \begin{cases} P_0^-(z) \equiv 1, \quad P_{-1}^-(z) \equiv 0, \\ \\ P_{n+1}^-(z) + (-z - \lambda_n - \mu_n) P_n^-(z) + \lambda_{n-1}\mu_n P_{n-1}^-(z) = 0, \ n \geq 0, \end{cases}$$

and it is easily checked that

$$(2.2)\qquad \begin{cases} P_0^-(-z) \equiv 1, \quad P_{-1}^-(-z) \equiv 0, \\ \\ P_n^-(-z) = \lambda_0 \ldots \lambda_{n-1} Q_n(z). \end{cases}$$

Hence, the polynomials $\{P_n^-(-z)\}$ form an OPS with respect to the measure $\mu(dx)$.

The J (Jacobi) fraction $\tilde{p}_0(z)$ can be viewed as the even part of a RITZ^{-1} fraction of the form

$$\tilde{P}(z) = \frac{\alpha_1|}{|z} + \frac{\alpha_2|}{|1} + \frac{\alpha_3|}{|z} + \frac{\alpha_4|}{|1} + \cdots,$$

whose even part $\tilde{P}^e(z)$ is given by

$$\tilde{P}^e(z) = \frac{\alpha_1|}{|z+\alpha_2} - \frac{\alpha_2\alpha_3|}{|z+\alpha_3+\alpha_4} - \frac{\alpha_4\alpha_5|}{|z+\alpha_5+\alpha_6} - \frac{\alpha_6\alpha_7|}{|z+\alpha_7+\alpha_8} - \cdots$$

In the present case, we clearly have

$$(2.3)\qquad \begin{cases} \alpha_1 = 1 \text{ and } \alpha_2 = \lambda_0, \\ \\ \alpha_{2k}\alpha_{2k+1} = \lambda_{k-1}\mu_k \text{ and } \alpha_{2k+1} + \alpha_{2(k+1)} = \mu_k + \lambda_k \text{ for } k \geq 1, \end{cases}$$

which immediately yields [5]

$$(2.4)\qquad \alpha_1 = 1, \ \alpha_{2k} = \lambda_{k-1}, \text{ and } \alpha_{2k+1} = \mu_k \text{ for } k \geq 1.$$

Since $\alpha_k > 0$ for all k, $\tilde{P}(z)$ is an S fraction. Note that the method of directly proving that the coefficients of the RITZ^{-1} fraction are all positive can be used instead of checking Stieltjes criterion as in [5].

The measure $\mu(dx)$ is one solution of the Stieltjes problem corresponding to the polynomials $\{P_m^-(-z)\}$, i.e., a measure satisfying eq. (1.3) (see [12]). Note that without further assumptions, the measure $\mu(dx)$ is a priori not the unique solution of the Stieltjes moment problem.

The successive numerators $P_m^+(z)$ of the continued fraction $\tilde{p}_0(z)$ satisfy the Wallis recurrence relations

$$(2.5)\qquad \begin{cases} P_0^+(z) \equiv 0, \quad \dfrac{1}{\lambda_0} P_1^+(z) \equiv \dfrac{1}{\lambda_0}, \\ \\ \lambda_{n+1} \dfrac{P_{n+2}^+(z)}{\lambda_0 \ldots \lambda_{n+1}} + (-z - \lambda_{n+1} - \mu_{n+1}) \dfrac{P_{n+1}^+(z)}{\lambda_0 \ldots \lambda_n} + \mu_{n+1} \dfrac{P_n^+(z)}{\lambda_0 \ldots \lambda_{n-1}} = 0, \ n \geq 0. \end{cases}$$

At this stage, let us introduce the concept of polynomials associated with the basic OPS $\{Q_n(x)\}$ [6]. Specifically, the associated polynomials $\{Q_n(x;\gamma)\}$, $\gamma = 0, 1, 2, \ldots$ are defined by the recursion

$$(2.6)\qquad \begin{cases} Q_{-1}(z;\gamma) \equiv 0, \quad Q_0(z;\gamma) \equiv 1, \\ \\ \lambda_{n+\gamma} Q_{n+1}(z;\gamma) + (z - \lambda_{n+\gamma} - \mu_{n+\gamma}) Q_n(z;\gamma) + \mu_{n+\gamma} Q_{n-1}(z;\gamma) = 0, \ n \geq 0. \end{cases}$$

When $\gamma = 1$, these polynomials are also referred to as polynomials of the second kind in [3]. It is known in the literature that the associated polynomials $\{Q_n(x;\gamma)\}$ form an OPS.

With regard to the associated polynomials, the initial value of the coefficients $\mu_{n+\gamma}$ (i.e., μ_γ) may change the orthogonal polynomial class and the spectral measure. This point is discussed for example in [10] and a natural choice for the initial value of μ_γ is zero. In this paper, however, since we deal with excrusions of the birth and death process, which can then be viewed as a transient (absorbed) process, we shall consider the case $\mu_\gamma > 0$.

By identification, it is easily checked that

$$P_n^+(z) = \lambda_1 \dots \lambda_{n-1} Q_{n-1}(-z;1) \tag{2.7}$$

with the convention $\mu_2 \dots \mu_n = 1$ for $n = 0, 1$. It follows that the [n-1/n] Padé approximant $[\tilde{p}_0(z)]_n$ of the continued fraction $\tilde{p}_0(z)$ is given by

$$[\tilde{p}_0(z)]_n \stackrel{\text{def}}{=} \frac{P_n^+(z)}{P_n^-(z)} = \frac{1}{\lambda_0} \frac{Q_{n-1}(-z;1)}{Q_n(-z;0)}, \tag{2.8}$$

where by definition $Q_n(x;0) = Q_n(x)$.

Regarding the convergence of the continued fraction $\tilde{p}_0(z)$, the properties of the spectrum and the uniqueness of the solution of the Stieltjes moment problem, let us introduce as in [5] the sequence $\{a_n\}$ defined by

$$\alpha_1 = \frac{1}{a_1} \text{ and } \alpha_n = \frac{1}{a_{n-1} a_n} \text{ for } n > 1. \tag{2.9}$$

In the present case, it can be shown that

$$a_{2m} = \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} = \frac{1}{\lambda_{m-1} \pi_{m-1}} \text{ for } m \geq 1, \tag{2.10}$$

$$a_{2m+1} = \frac{\lambda_0 \dots \lambda_{m-1}}{\mu_1 \dots \mu_m} = \pi_m \text{ for } m \geq 0. \tag{2.11}$$

On the basis of Stieltjes' work, the authors of [5] recall that by considering the series

$$S = \sum_{n=1}^{\infty} a_n, \tag{2.12}$$

$$B = \sum_{n=1}^{\infty} (a_1 + \dots + a_{2n-1}) a_{2n}, \tag{2.13}$$

$$C = \sum_{n=1}^{\infty} (a_2 + \dots + a_{2n}) a_{2n+1}, \tag{2.14}$$

we have the following convergence results and properties for the poles of the continued fraction:

Result 1.: The zeros of the successive even denominators $\tilde{P}_{2n}^-(z)$ (which coincide with the polynomials $P_n^-(z)$) and odd denominators $\tilde{P}_{2n+1}^-(z)$ of the continued fraction $\tilde{P}(z)$ are real, negative and simple. (This entails that the zeros of the polynomials $Q_n(x)$ are real, positive and simple).

Result 2.: If $S < \infty$, then the odd and even parts of the continued fraction $\tilde{P}(z)$ converge, possibly to different functions, which are meromorphic functions with poles on the negative real axis; the poles s_k, $k = 1, 2, \dots$ of $\tilde{p}_0$ satisfy the property

$$\sum_{k=2}^{\infty} \frac{1}{s_k} < \infty. \tag{2.15}$$

Result 3.: If $S = \infty$, then the odd and even parts of the continued fraction $\tilde{P}(z)$ converge to the same function, which is analytic in the whole complex plane deprived of the negative real axis; the successive numerators and denominators do not have in general finite limits; however there are two (and only two) cases where the even and the odd polynomials have finite limits

Result 3.1.: if $B < \infty$, $P_n^-(z)$ converges to a finite limit $\tilde{P}^-(z)$ and the poles of $\tilde{P}(z)$ satisfy inequality (2.15);

Result 3.2.: if $C < \infty$, the odd polynomials $\tilde{P}_{2n+1}^-(z)$ of the continued fraction $\tilde{P}(z)$ have a finite limit and the poles of $\tilde{P}(z)$ satisfy inequality (2.15).

3. Probabilistic Interpretation of the Series S, B, and C

We consider the random variable θ_m representing the duration of an excursion by the process $\{\Lambda_t\}$ above the level $m-1$, $m \geq 1$, which is defined by

$$\theta_m = \inf\{t > 0 : \Lambda_t = m-1 \mid \Lambda_0 = m\}. \tag{3.1}$$

Moreover, let τ_m be the first passage time from state $m-1$ to state m. This random variable is precisely defined by

$$\tau_m = \inf\{t > 0 : \Lambda_t \geq m \mid \Lambda_0 = m-1\}. \tag{3.2}$$

On the basis of the above definitions, let $\mathcal{U}_m$ and $\mathcal{D}_m$ denote the upcrossing time from state 0 to state m and the downcrossing time from state m to state 0, respectively. We clearly have

$$\mathcal{U}_m = \sum_{n=1}^{m} \tau_n \text{ and } \mathcal{D}_m = \sum_{n=1}^{m} \theta_n. \tag{3.3}$$

Finally, let

$$\mathcal{C}_m = \mathcal{U}_m + \mathcal{D}_m \tag{3.4}$$

The random variable $\mathcal{C}_m$ represents the time between two visits by the process $\{\Lambda_t\}$ at state 0, knowing that the process $\{\Lambda_t\}$ hits the state m.

Lemma 1. *Under condition* (C$_1$)*, the respective mean values $\overline{\theta}_m$ and $\overline{\tau}_m$ of the random variables θ_m and τ_m are given by*

$$\overline{\theta}_m = \frac{1}{\lambda_{m-1}\pi_{m-1}} \sum_{n=m}^{\infty} \pi_n, \tag{3.5}$$

$$\overline{\tau}_m = \frac{1}{\lambda_{m-1}\pi_{m-1}} \sum_{n=0}^{m-1} \pi_n. \tag{3.6}$$

Proof. From the strong Markov property satisfied by the process $\{\Lambda_t\}$ and the memoryless property of the exponential distribution, the successive excursion times by process $\{\Lambda_t\}$ above the level $m-1$ are i.i.d. Hence, we can write

$$\sum_{n=m}^{\infty} \int_0^t \mathbf{1}_{\{\Lambda_s = n\}} ds = \sum_{\ell=1}^{\mathcal{N}_t^m - 1} \theta_\ell^m + \varepsilon_t$$

where the random variables θ_ℓ^m are i.i.d. with the same law as θ_m, and $\{\mathcal{N}_t^m\}$ is the point process counting the excursions by the process $\{\Lambda_t\}$ above the level $m-1$. ε_t is the excursion time of the last excursion up to time t. Owing to the ergodicity property satisfied by the process $\{\Lambda_t\}$,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{n=m}^{\infty} \int_0^t \mathbf{1}_{\{\Lambda_s = n\}} ds = \sum_{n=m}^{\infty} p_m \text{ a.s.}$$

From the strong law of large numbers,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \theta_k^m = \bar{\theta}_m \text{ a.s.}$$

Finally, since $\{\mathcal{N}_t^m - \lambda_{m-1}\int_0^t \mathbf{1}_{\{\Lambda_s=m-1\}}ds\}$ is a martingale whose quadratic variation process is simply $\{\lambda_{m-1}\int_0^t \mathbf{1}_{\{\Lambda_s=m-1\}}ds\}$, satisfying

$$\lim_{t\to\infty}\frac{1}{t}\lambda_{m-1}\int_0^t \mathbf{1}_{\{\Lambda_s=m-1\}}ds = \lambda_{m-1}p_{m-1} \text{ a.s.,}$$

we deduce from the strong law of large numbers for martingales that

$$\lim_{t\to\infty}\frac{\mathcal{N}_t^m}{t} = \lambda_{m-1}p_{m-1} \text{ a.s.}$$

Obviously, $\varepsilon_t/t \to 0$ a.s. and eq. (3.5) follows.

The mean value of the random variable τ_m can be computed as follows. Using once again the strong Markov property of the process $\{\Lambda_t\}$, we can write

$$\text{(3.7)} \qquad \begin{cases} \tau_m \stackrel{d}{=} \mathcal{E}_{\mu_{m-1}+\lambda_{m-1}} \text{ with probability } \frac{\lambda_{m-1}}{\lambda_{m-1}+\mu_{m-1}} \\ \\ \tau_m \stackrel{d}{=} \mathcal{E}_{\mu_{m-1}+\lambda_{m-1}} + \tau_m' + \tau_{m-1} \text{ with probability } \frac{\mu_{m-1}}{\lambda_{m-1}+\mu_{m-1}} \end{cases} \quad \text{for } m \geq 1,$$

where $\mathcal{E}_{\mu_{m-1}+\lambda_{m-1}}$ is a random variable exponentially distributed with parameter $\lambda_{m-1}+\mu_{m-1}$, the random variables τ_m, τ_m' and τ_{m-1} are independent, and τ_m and τ_m' are identically distributed. Taking expectations, we get

$$\lambda_{m-1}\bar{\tau}_m = 1 + \mu_{m-1}\bar{\tau}_{m-1},$$

and straightforward manipulations yield the desired result, given that $\bar{\tau}_1 = 1/\lambda_0$. This completes the proof. □

From the definition of the random variables $\mathcal{U}_m$, $\mathcal{D}_m$ and $\mathcal{C}_m$ for $m \geq 1$, we have the following probabilistic interpretation of the series S, B, and C.

Theorem 1. *Under condition* (C_1)*, the series S, B, and C are related to the respective mean values $\bar{\mathcal{U}}_m$, $\bar{\mathcal{D}}_m$ and $\bar{\mathcal{C}}_m$ of the random variables $\mathcal{U}_m$, $\mathcal{D}_m$ and $\mathcal{C}_m$ for $m \geq 1$ as follows:*

$$\text{(3.8)} \qquad B = \lim_{m\to\infty}\bar{\mathcal{U}}_m,$$

$$\text{(3.9)} \qquad C = \lim_{m\to\infty}\bar{\mathcal{D}}_m,$$

$$\text{(3.10)} \qquad S = \frac{1}{p_0} + \lim_{m\to\infty}\bar{\mathcal{C}}_m.$$

The above result shows that the series S, B, and C can be expressed in terms of the mean values of some transient characteristics related to the excursions of the birth and death process.

As already shown in the paper by Karlin and Mc Gregor [11] (see also [4]), the ergodicity assumption (C_1) implies that $S = \infty$. Indeed, on the one hand, under the assumption that the Markov process $\{\Lambda_t\}$ is ergodic, we have $\lim_{t\to\infty} p_{0,0}(t) = p_0 > 0$ and hence, $\tilde{p}_0(0) = \infty$. On the other hand, from the theory of continued fractions [9, eq. (12.1-23)], we have

$$\text{(3.11)} \qquad \tilde{p}_0(z) = \sum_{m=1}^{\infty}\frac{\lambda_0\ldots\lambda_{m-2}\mu_1\ldots\mu_{m-1}}{P_{m-1}^-(z)P_m^-(z)}.$$

It is easily checked that $P_m^-(0) = \lambda_0\ldots\lambda_{m-1}$ and it then follows that

$$\text{(3.12)} \qquad \sum_{m=1}^{\infty}\frac{\mu_1\ldots\mu_{m-1}}{\lambda_0\ldots\lambda_{m-1}} = \infty.$$

As a consequence, the series $S = \infty$, which entails, under condition (C_1), $B = \infty$. In this case, the solution of the Stieltjes moment problem is unique [12, Chap. IV, Theorem 14]. The odd and even parts of the S-fraction $\tilde{P}(z)$ converge to the same limit, which is an analytic function in the complex plane deprived of the non positive real axis. Note that the above result entails that the measure

$\mu(dx)$ has the mass $1/\sum_{m=0}^{\infty}\pi_m = p_0 > 0$ located at the origin [12]. Furthermore, from [9], $\tilde{p}_0$ can be expressed as the Stieltjes transform of the measure $\mu(dx)$, that is, for $z \in \mathbb{C}\backslash(-\infty, 0)$,

$$\tilde{p}_0(z) = \int_0^\infty \frac{1}{z+x}\mu(dx). \tag{3.13}$$

The condition (C_1) is however not sufficient to determine the convergence of the series C. Indeed, take for instance $\lambda_n = u$ for some real $u > 0$ and $\mu_n = n$. Then, the condition (C_1) is satisfied and the series $S = B = C = \infty$. Now, if we consider the case $\lambda_n = u$ and $\mu_n = n^2$, then condition (C_1) is satisfied and the series $S = B = \infty$, but $C < \infty$.

The condition (C_1) is also not sufficient to determine the asymptotic behavior of the mean values $\bar{\theta}_m$. Indeed, in the case $\lambda_n = u$ and $\mu_n = n$ for some real $u > 0$, $\bar{\theta}_m \to 0$ as $m \to \infty$. In the case $\lambda_n = u$ and $\mu_n = 1$ for some real $u \in (0,1)$, $\bar{\theta}_m = 1/(1-u)$ for all $m \geq 1$.

To progress in the investigations of the properties of the birth and death process $\{\Lambda_t\}$, we are led to make further assumptions on the process $\{\Lambda_t\}$. In the literature, it is usual to take some specific birth and death rates or more generally to suppose some asymptotic behavior for the birth and death rates. For instance, for linear growth birth and death processes, a classical assumption consists in supposing the asymptotic behavior $\lambda_n = O(n^\alpha)$ and $\mu_n = O(n^\alpha)$ for some real α; in this case, the process is said to be asymptotically proportional.

In this paper, we adopt a slightly different approach. Specifically, we assume in the following that

$$C = \infty \tag{C_2}$$

but with

$$\overline{\theta}_m \to 0 \text{ as } m \to \infty. \tag{C_3}$$

The above assumptions are notably verified by the occupation process of an $M/M/\infty$ queue.

The introduction of the two above assumptions is motivated by the Aldous local linearization property satisfied by some birth and death processes arising in the modelling of physical systems. Specifically, we say that a birth and death process $\{\Lambda_t\}$ satisfies the Aldous local linearization property if

$$\{\Lambda^m_{k_m t}\} \xrightarrow{d} \{n_t\} \text{ as } m \to \infty, \tag{3.14}$$

where $\{\Lambda^m_t\}$ is the excursion process above the level m, defined by

$$\Lambda^m_t = \Lambda_{t\wedge\theta_m} - m \text{ given that } \Lambda_0 = m+1, \tag{3.15}$$

$\{n_t\}$ is a birth and death process taking values in $\{0, 1, 2, \ldots\}$, with initial state 1, and absorbed at state 0 at time ζ, and $\{k_m\}$ is a sequence of real numbers such that $k_m \to \infty$ as $m \to \infty$. The Aldous local linearization property is satisfied for instance for the birth and death process with rates $\lambda_m = \rho(m+1)^\alpha$ and $\mu_m = m^\alpha$ for positive constants ρ and α; the limiting process $\{n_t\}$ is in this case the occupation process of an $M/M/1$ queue with input rate ρ and unit service rate. Note that the conditions (C_2) and (C_3) are satisfied once $\sum_k 1/k_m = \infty$ and $\mathbb{E}[\zeta] < \infty$. For the above example, the conditions (C_1) is satisfied if $\rho \in (0,1)$ and the conditions (C_2) and (C_3) hold if in addition $\alpha \leq 1$. In that case, the limiting $M/M/1$ queue is stable.

4. Laplace Transforms of Transient Characteristics

The fundamental OPS $\{Q_n(x)\}$ has been introduced so far in connection with the continued fraction $\tilde{p}_0(z)$. We now show how this polynomial system arises in the computation of the Laplace transforms of some transient characteristics associated with the birth and death process. We suppose that the process $\{\Lambda_t\}$ satisfies the assumptions (C_1), (C_2), and (C_3) and we consider in a first step the first passage time $\tau_{m,n}$ from state m to state n, $n > m$. Note that $\tau_m \stackrel{\text{def}}{=} \tau_{m-1,m}$, where τ_m is defined by eq. (3.2). The analysis of the Laplace transform in the $M/M/\infty$ case has been carried out in [15].

Proposition 1. *The Laplace transform $\tilde{\tau}_{m,n}(z)$ of the passage time $\tau_{m,n}$ from state m to state n, $n > m$ is given by*

$$\text{for } \Re(z) \geq 0,\ \tilde{\tau}_{m,n}(z) = \frac{Q_m(-z)}{Q_n(-z)}. \tag{4.1}$$

Proof. By taking Laplace transforms in eq. (3.7), we obtain, owing to the independence between the different random variables,

$$(z + \lambda_m + \mu_m)\tilde{\tau}_{m+1}(z) = \lambda_m + \mu_m \tilde{\tau}_m(z)\tilde{\tau}_{m+1}(z) \tag{4.2}$$

Let

$$T_0(z) = 1, \text{ and for } m \geq 1,\ T_m(z) = \left(\prod_{\ell=1}^{m} \tilde{\tau}_\ell(z)\right)^{-1}, \tag{4.3}$$

where $\tilde{\tau}_m$ is the Laplace transform of τ_m. From the recurrence relations (3.7), we have for $m \geq 1$

$$\lambda_m T_{m+1}(z) - (z + \lambda_m + \mu_m)T_m(z) + \mu_m T_{m-1}(z) = 0 \tag{4.4}$$

Since

$$T_1(z) = \frac{1}{\tilde{\tau}_0(z)} = \frac{z + \lambda_0}{\lambda_0},$$

eq. (4.4) is valid for $m = 0$ by setting $T_{-1}(z) = 0$. We immediately deduce that

$$T_0(-z) = 1 \text{ and for } m \geq 1,\ T_m(z) = Q_m(-z). \tag{4.5}$$

By the strong Markov property,

$$\tilde{\tau}_{m,n}(z) = \tilde{\tau}_{m+1}(z)\ldots\tilde{\tau}_n(z) = \frac{T_m(z)}{T_n(z)}$$

and the relation (4.1) follows. This completes the proof. □

As an easy consequence of the above result, we have the following corollary.

Corollary 1. *The Laplace transform of the random variable $\mathcal{U}_m$ is given by*

$$\text{for } \Re(z) \geq 0,\ \tilde{\mathcal{U}}_m(z) = \frac{1}{Q_m(-z)}. \tag{4.6}$$

We now consider the duration θ_m of an excursion by the process $\{\Lambda_t\}$ above the level $m-1$, $m \geq 1$. We give in a first step the formal continued fraction representation of its Laplace transform; the convergence of the continued fraction will be addressed in the next section. In fact, we extend to the general birth and death process $\{\Lambda_t\}$ the analysis carried out in [7] for the occupation process of an $M/M/\infty$ system.

Proposition 2. *The Laplace transform $\tilde{\theta}_m$ of the random variable θ_m for $m \geq 1$ can formally be represented by a continued fraction as*

$$\tilde{\theta}_m(z) \quad = \quad -\frac{1}{\lambda_{m-1}} \operatorname*{K}_{n=1}^{\infty} \frac{-\lambda_{m+n-2}\mu_{m+n-1}}{z + \lambda_{m+n-1} + \mu_{m+n-1}} \ \text{for } \Re(z) \geq 0. \tag{4.7}$$

Proof. From the strong Markov property satisfied by the process $\{\Lambda_t\}$, we can write

$$\theta_m \stackrel{d}{=} \mathcal{E}_{\lambda_m+\mu_m}$$

with probability $\mu_m/(\mu_m + \lambda_m)$ and

$$\theta_m \stackrel{d}{=} \mathcal{E}_{\lambda_m+\mu_m} + \theta_{m+1} + \theta'_m$$

with probability $\lambda_m/(\lambda_m+\mu_m)$, where $\mathcal{E}_{\lambda_m+\mu_m}$ denotes an exponentially distributed random variable with parameter $\lambda_m + \mu_m$, and where θ_m and θ'_m are i.i.d., the random variables $\mathcal{E}_{\lambda_m+\mu_m}$, θ_m, θ_{m+1}, and θ'_m being independent. Denoting by $\tilde{\theta}_m$ the Laplace transform of the random variable θ_m, the above relations yield

$$(z + \lambda_m + \mu_m)\tilde{\theta}_m(z) = \mu_m + \lambda_m \tilde{\theta}_{m+1}(z)\tilde{\theta}_m(z) \tag{4.8}$$

from which we deduce

$$\tilde{\theta}_m(z) = \frac{\mu_m}{z + \lambda_m + \mu_m - \lambda_m \tilde{\theta}_{m+1}(z)}. \tag{4.9}$$

From condition (C_3), $\overline{\theta}_m \to 0$ as $m \to \infty$ and then since $\theta_m \geq 0$, $\theta_m \overset{d}{\to} 0$ as $m \to \infty$, which entails that $\tilde{\theta}_m(z) \to 1$ as $m \to \infty$. Hence, we can formally write

$$\tilde{\theta}_m(z) = \cfrac{\mu_m}{z + \mu_m + \lambda_m + \cfrac{-\lambda_m \mu_{m+1}}{z + \mu_{m+1} + \lambda_{m+1} + \cfrac{-\lambda_{m+1}\mu_{m+2}}{z + \lambda_{m+2} + \mu_{m+2} + \ddots}}}$$

and we formally obtain eq. (4.7). □

Let $\theta_n^-(z;m)$ be the nth denominator of $\tilde{\theta}_m(z)$, which satisfy the Wallis recurrence relations: $\theta_{-1}^-(z;m) = 0$, $\theta_0^-(z;m) = 1$ and for $n \geq 0$,

$$\lambda_{m+n}\left(\frac{\theta_{n+1}^-(z;m)}{\lambda_m \ldots \lambda_{m+n}}\right) - (z + \lambda_{m+n} + \mu_{m+n})\left(\frac{\theta_n^-(z;m)}{\lambda_m \ldots \lambda_{m+n-1}}\right) + \mu_{m+n}(1 - \delta_{0,n})\left(\frac{\theta_{n-1}^-(z;m)}{\lambda_m \ldots \lambda_{m+n-2}}\right) = 0 \tag{4.10}$$

from which we deduce that

$$\theta_n^-(z;m) = \lambda_m \ldots \lambda_{m+n-1} Q_n(-z;m), \tag{4.11}$$

where $\{Q_n(z;m)\}$ are the polynomials associated with the fundamental OPS $\{Q_n(z)\}$.

The numerators $\theta_n^+(z;m)$ of the continued fraction $\tilde{\theta}_m(z)$ satisfy the recursion $\theta_0^+(z;m) = 0$, $\theta_1^+(z;m) = \mu_m$ and for $n \geq 0$,

$$\lambda_{m+n+1}\left(\frac{\theta_{n+2}^+(z;m)}{\lambda_m \ldots \lambda_{m+n+1}}\right) - (z + \lambda_{m+n+1} + \mu_{m+n+1})\left(\frac{\theta_{n+1}^+(z;m)}{\lambda_m \ldots \lambda_{m+n}}\right) + \mu_{m+n+1}\left(\frac{\theta_n^+(z;m)}{\lambda_m \ldots \lambda_{m+n-1}}\right) = 0.$$

By identification, we have

$$\theta_n^+(z;m) = \mu_m \lambda_{m+1} \ldots \lambda_{m+n-1} Q_{n-1}(-z, m+1). \tag{4.12}$$

It follows that the [n-1/n] Padé approximant $[\tilde{\theta}_m(z)]_n$ of the continued fraction $\tilde{\theta}_m(z)$ is

$$\frac{\mu_m}{\lambda_m}\frac{Q_{n-1}(-z;m+1)}{Q_n(-z;m)}.$$

The above results show how the fundamental OPS, via the associated polynomials, arise in the computation of the Laplace transform of the random variable θ_m. The analysis can be led further and we can show that the Laplace transform under consideration can directly be expressed in terms of the fundamental OPS and of the continued fraction $\tilde{p}_0(z)$. For this purpose, we assume that all the continued fraction appearing below are converging for $z \in \mathbb{C}\backslash(-\infty, 0)$; the convergence issues will be addressed in the next section.

Proposition 3. *For $m \geq 1$, $\tilde{\theta}_m(z)$ satisfies*

$$\forall z \in \mathbb{C}\backslash(-\infty, 0),\ \tilde{\theta}_m(z) = \frac{Q_m(-z)}{Q_{m-1}(-z)} - \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1} Q_{m-1}^2(-z)\left(\tilde{p}_0(z) - [\tilde{p}_0(z)]_{m-1}\right)}, \tag{4.13}$$

where $\{Q_m(x)\}$ is the fundamental OPS defined by the recurrence relations (1.4) and where $[\tilde{p}_0(z)]_{m-1}$ denotes the mth approximant of the continued fraction $\tilde{p}_0(z)$.

Proof. We prove the result by mathematical induction. For $m = 1$, using the relations (4.7) and (1.6), it is easily checked that

$$\tilde{p}_0(z) = \frac{1}{z + \lambda_0 - \lambda_0 \tilde{\theta}_1(z)}$$

and whence

$$\tilde{\theta}_1(z) = \frac{z + \lambda_0}{\lambda_0} - \frac{1}{\lambda_0 \tilde{p}_0(z)}.$$

This expression has the desired form with the convention $[\tilde{p}_0(z)]_0 = 0$. The relation (4.13) is hence valid for $m = 1$.

Assume now that the relation (4.13) is valid at rank m. Using the recurrence relation (4.9) for $n = m$, it follows after some algebra that $\tilde{\theta}_m(z)$ can be expressed as

$$\tilde{\theta}_{m+1}(z) = B_m - \frac{N_m}{\tilde{p}_0(z) - R_m} \tag{4.14}$$

with

$$B_m = \frac{z + \lambda_m + \mu_m}{\lambda_m} - \frac{\mu_m}{\lambda_m}\frac{Q_{m-1}(-z)}{Q_m(-z)}, \tag{4.15}$$

$$N_m = \frac{\lambda_0 \dots \lambda_m}{\mu_1 \dots \mu_m Q_m(-z)^2}, \tag{4.16}$$

$$R_m = [\tilde{p}_0(z)]_{m-1} + \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} \frac{1}{Q_{m-1}(-z) Q_m(-z)}. \tag{4.17}$$

Using the recurrence relations satisfied by the polynomials $Q_m(z)$ immediately yields

$$B_m = \frac{Q_{m+1}(-z)}{Q_m(-z)}.$$

It is known [9, eq. (12.1-22)] that the difference between two consecutive approximants of the continued fraction $\tilde{p}_0(z)$ is given by

$$\begin{aligned}
[\tilde{p}_0(z)]_m - [\tilde{p}_0(z)]_{m-1} &= \frac{\lambda_0 \mu_1 \lambda_2 \mu_1 \dots \lambda_{m-2} \mu_{m-1}}{\lambda_0 \dots \lambda_{m-2} Q_{m-1}(-z) \lambda_0 \dots \lambda_{m-1} Q_m(-z)} \\
&= \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} \frac{1}{Q_{m-1}(-z) Q_m(-z)},
\end{aligned}$$

where we have used the relation between the denominators of the continued fraction $\tilde{p}_0(z)$ and the polynomials $Q_n(z)$. It follows that

$$R_m = [\tilde{p}_0(z)]_m.$$

Bringing the above results all together, we see that if eq. (4.13) is valid at rank m, it is valid at rank $m + 1$. Since this relation holds for $m = 1$, the proof is done. $\square$

5. Convergence and Properties of the Continued Fraction

To alleviate the notation, we make use in this section of the scale function σ associated with the birth and death process $\{\Lambda_t\}$. This function is precisely defined so that for $0 \leq j < i_0 \leq i$

$$\mathbb{P}_{i_0}\{T_i < T_j\} = \frac{\sigma(i_0) - \sigma(j)}{\sigma(i) - \sigma(j)}, \tag{5.1}$$

where $T_i = \inf\{t > 0 : \Lambda_t = i\}$. A possible choice for the scale function is

$$\sigma(0) = 0, \ \sigma(1) = \frac{1}{\lambda_0}, \ \sigma(m) = \sum_{\ell=1}^{m-1} \frac{\mu_1 \dots \mu_l}{\lambda_0 \dots \lambda_\ell} \text{ for } m > 1, \tag{5.2}$$

and we immediately see that

$$S = \frac{1}{p_0} + p_0 \lim_{m \to \infty} \sigma(m). \tag{5.3}$$

To show that the J-fraction $\tilde{\theta}_m(z)$ is convergent for $\Re(z) \geq 0$, we study the series S_m associated with this continued fraction in the same manner as the series S is associated with the continued fraction $\tilde{p}_0(z)$. Let us first compute the RITZ^{-1} fraction $\tilde{\Theta}_m(z)$ which even part is the J-fraction $\tilde{\theta}_m(z)$.

$\tilde{\Theta}_m(z)$ is of the form

$$\tilde{\Theta}_m(z) = \frac{\alpha_1^m|}{|z} + \frac{\alpha_2^m|}{|1} + \frac{\alpha_3^m|}{|z} + \frac{\alpha_4^m|}{|1} + \cdots,$$

whose even part $\tilde{\Theta}^e_m(z)$ is given by

$$\tilde{\Theta}^e_m(z) = \frac{\alpha_1^m|}{|z + \alpha_2^m} - \frac{\alpha_2^m \alpha_3^m|}{|z + \alpha_3^m + \alpha_4^m} - \frac{\alpha_4^m \alpha_5^m|}{|z + \alpha_5^m + \alpha_6^m} - \frac{\alpha_6^m \alpha_7^m|}{|z + \alpha_7^m + \alpha_8^m} - \cdots$$

In the present case, we clearly have

$$\begin{cases} \alpha_1^m = \mu_m \text{ and } \alpha_2^m = \lambda_m + \mu_m, \\ \alpha_{2k}^m \alpha_{2k+1}^m = \lambda_{m+k-1}\mu_{m+k} \text{ and } \alpha_{2k+1}^m + \alpha_{2(k+1)}^m = \lambda_{m+k} + \mu_{m+k} \text{ for } k \geq 1. \end{cases} \tag{5.4}$$

We show in a first step that the coefficients α_k^m are positive.

Lemma 2. *The coefficients α_k^m are given by $\alpha_1^m = \mu_m$ and for $k \geq 1$,*

$$\alpha_{2k}^m = \lambda_{m+k-1} \frac{\sigma(m+k) - \sigma(m-1)}{\sigma(m+k-1) - \sigma(m-1)} \tag{5.5}$$

$$\alpha_{2k+1}^m = \mu_{m+k} \frac{\sigma(m+k-1) - \sigma(m-1)}{\sigma(m+k) - \sigma(m-1)}, \tag{5.6}$$

where σ is the scale function defined by eq. (5.2).

Proof. From the recurrence relations (5.4), we can write for $k \geq 1$

$$\alpha_{2(k+1)}^m \alpha_{2k}^m - (\lambda_{m+k} + \mu_{m+k})\alpha_{2k}^m + \lambda_{m+k-1}\mu_{m+k} = 0. \tag{5.7}$$

Define then

$$A_{-1}^m = 0, \ A_0^m = 1 \text{ and } A_k^m = \frac{1}{\lambda_m \dots \lambda_{m+k-1}} \prod_{\ell=1}^{k} \alpha_{2\ell}^m \text{ for } k \geq 1.$$

From eq. (5.7), we obtain

$$\lambda_{m+k}(A_{k+1}^m - A_k^m) = \mu_{m+k}(A_k^m - A_{k-1}^m), \tag{5.8}$$

and as a consequence,

$$A_k^m = \sum_{\ell=0}^{k} \frac{\mu_m \dots \mu_{m+\ell-1}}{\lambda_m \dots \lambda_{m+\ell-1}} = \frac{\sigma(m+k) - \sigma(m-1)}{\sigma(m) - \sigma(m-1)}$$

for $k \geq 1$. Since $\alpha_{2k}^m = \lambda_{m+k-1} A_k^m / A_{k-1}^m$, eq. (5.5) follows. Relation (5.6) is obtained by using the recurrence relations (5.4). $\square$

The above result shows that the coefficients α_k^m are positive and then that $\tilde{\Theta}_m(z)$ is a formal S-fraction. Define the sequence $\{a_k^m\}$ by

$$a_1^m = \frac{1}{\alpha_1^m} \text{ and } \alpha_k^m = \frac{1}{a_{k-1}^m a_k^m} \text{ for } k \geq 2. \tag{5.9}$$

It is easily checked that

$$a_{2k}^m = \frac{\alpha_1^m \dots \alpha_{2k-1}^m}{\alpha_2^m \dots \alpha_{2k}^m} \text{ and } a_{2k+1}^m = \frac{\alpha_2^m \dots \alpha_{2k}^m}{\alpha_1^m \dots \alpha_{2k+1}^m} \text{ for } k \geq 1. \tag{5.10}$$

Since

$$\alpha_2^m \dots \alpha_{2k}^m = \lambda_m \dots \lambda_{m+k-1} A_k^m \text{ and } \alpha_1^m \dots \alpha_{2k+1}^m = \mu_m \dots \mu_{m+k} \frac{1}{A_k^m},$$

we obtain for $k \geq 0$

$$a_{2k}^m = \frac{\mu_m \dots \mu_{m+k-1}}{\lambda_m \dots \lambda_{m+k-1}} \frac{1}{A_{k-1}^m A_k^m} \text{ and } a_{2k+1}^m = \frac{\lambda_m \dots \lambda_{m+k-1}}{\mu_m \dots \mu_{m+k}} [A_k^m]^2 . \tag{5.11}$$

Before proceeding further, let us show how the series $\sum_k a_{2k}$ and $\sum_k a_{2k+1}$ occur in the work of Karlin and McGregor. For this purpose, assume that $\Lambda_0 = i + m$ with $i \geq 0$ and let $\{\Lambda_t^m\}$ be the excursion process defined by eq. (3.15). Define then the absorbing time ζ_m as

$$\zeta_m = \inf\{t > 0 : \Lambda_t = m - 1\}. \tag{5.12}$$

The process $\{\Lambda_t^m\}$ is a birth and death process taking values in $0, 1, 2, \dots$. The evolution of the process $\{\Lambda_t^m\}$ in this state space is governed by the infinitesimal generator

$$\begin{pmatrix} -(\lambda_m + \mu_m) & \lambda_m & 0 & 0 \\ \mu_{m+1} & -(\lambda_{m+1} + \mu_{m+1}) & \lambda_{m+1} & 0 \\ 0 & \mu_{m+2} & -(\lambda_{m+2} + \mu_{m+2}) & \lambda_{m+2} \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{5.13}$$

Note that unlike the process $\{\Lambda_t\}$, the process $\{\Lambda_t^m\}$ is absorbed at state -1. It turns out that the OPS associated with the birth and death process $\{\Lambda_t^m\}$ consists of the associated polynomials $\{Q_n(x; m)\}$ (defined with the initial condition corresponding to $\mu_n > 0$.).

In [12], it is shown that the uniqueness of the solution of the Stieltjes moment problem associated with the polynomials $\{Q_n(x; m)\}$ depends on the convergence of the series $\sum_k \pi_k^m Q_k^2(0; m)$ with the quantities π_k^m defined by

$$\pi_k^m = \frac{\lambda_m \dots \lambda_{m+k-1}}{\mu_{m+1} \dots \mu_{m+k}}. \tag{5.14}$$

Since $A_k^m = Q_k(0; m)$, it is easily checked that

$$\mu_m \sum_{k=0}^{\infty} a_{2k+1}^m = \sum_{k=0}^{\infty} \pi_k^m Q_k^2(0; m) \text{ and } \mu_m \sum_{k=1}^{\infty} a_{2k}^m = \sum_{k=0}^{\infty} \frac{1}{\lambda_{m+k} \pi_k^m Q_{k+1}(0; m) Q_k(0; m)} \tag{5.15}$$

Moreover, we can write A_k^m as

$$A_k^m = \lambda_{m-1}\pi_{m-1}\left[\sum_{\ell=1}^{m+k}\frac{1}{\lambda_{\ell-1}\pi_{\ell-1}} - \sum_{n=1}^{m-1}\frac{1}{\lambda_{n-1}\pi_{n-1}}\right], \tag{5.16}$$

and then, using the fact that $A_k^m \geq 1$, straightforward manipulations yield

$$\begin{aligned}\sum_{k=0}^{\infty} a_{2k+1}^m &\geq \sum_{k=0}^{\infty}\frac{\lambda_m \ldots \lambda_{m+k-1}}{\mu_m \ldots \mu_{m+k}} A_k^m \\ &\geq C - \sum_{k=1}^{m-1}\pi_k \sum_{\ell=1}^{k}\frac{1}{\lambda_{\ell-1}\pi_{\ell-1}} - \left(\sum_{k=m}^{\infty}\pi_k\right)\left(\sum_{n=1}^{m-1}\frac{1}{\lambda_{n-1}\pi_{n-1}}\right).\end{aligned}$$

Under the condition (C_3), $\sum_{k=0}^{\infty} a_{2k+1}^m = \infty$ and then, the solution of the Stieltjes moment problem associated with the polynomials $\{Q_n(x;m)\}$ is unique. Hence, there exists a unique regular positive spectral measure $\hat{\theta}_m(dx)$ of total mass one for the polynomials $\{Q_n(x;m)\}$, which satisfies

$$\int_0^{\infty} Q_i(x;m)Q_j(x;m)\hat{\theta}_m(dx) = \frac{1}{\pi_i^m}\delta_{i,j}.$$

Coming back to the convergence of the continued fraction $\tilde{\Theta}_m(z)$, we immediately deduce from the above results that $S_m \stackrel{\text{def}}{=} \sum_k a_k^m = \infty$. This shows that the continued fraction $\tilde{\Theta}_m(z)$ converges to an analytic function over the complex plane deprived of the negative real axis and that the even and odd parts of this continued fraction converge to the same limit. Moreover, from [5], we know that the poles of $Q_n(x;m)$ are real, simple, and positive. This allows us to state the following result.

Proposition 4. *Under conditions* (C_1), (C_2), *and* (C_3), *the continued fraction $\tilde{\theta}_m(z)$ defined by eq. (4.7) converges to an analytic function over the whole complex plane deprived of the negative real axis.*

From [9, Theorem 12.9e], $\tilde{\Theta}_m(z)$ is the Stieltjes transform of some spectral measure, say $\hat{\theta}'_m(dx)$, that is,

$$\tilde{\Theta}_m(z) \equiv \tilde{\theta}_m(z) = \int_0^{\infty}\frac{1}{z+s}d\hat{\theta}'_m(s). \tag{5.17}$$

From [9, Corollary 12.11c], we know that the polynomials $\{\theta_n^-(x)\}$ are orthogonal with respect to the measure $\hat{\theta}'_m(dx)$ and satisfy

$$\int_0^{\infty}\theta_n^-(x)\theta_\ell^-(x)\hat{\theta}'_m(dx) = \alpha_1^m \ldots \alpha_{2n+1}^m\delta_{n,\ell}.$$

Owing to the relationship between the polynomials $\{\theta_n^-(x)\}$ and $\{Q_n(x;m)\}$ and to the uniqueness of the measure $\hat{\theta}_m(dx)$, it follows from the above relation that $\hat{\theta}'_m(dx) = \mu_m\hat{\theta}_m(dx)$. In particular, the total mass of the measure $\hat{\theta}'_m(dx)$ is equal to μ_m. Note moreover that since $\tilde{\theta}_m(z)$ is the Laplace transform of a proper random variable, $\tilde{\theta}_m(0) = 1$ and hence,

$$\mu_m\int_0^{\infty}\frac{1}{s}\hat{\theta}_m(ds) = 1,$$

which is the relation given in [12, Lemma 6].

In the notation of the paper by Karlin and McGregor [12], the above analysis shows that when a birth and death process is absorbed at state -1 and when the solution of the S moment problem is unique, the Stieltjes transform of the spectral measure is related to the Laplace transform of the random time to absorption. To some extent, this analysis generalizes that of Karlin and Mc Gregor, who showed the above result only for $m = 1$. Here, we show that the spectral measure of the associated polynomials of rank m are related to the excursion time of the birth and death process above the level $m-1$.

6. Properties of the Spectrum

In this section, we assume that the spectrum of the measure $\mu(dx)$ is discrete and we show how the different spectral measures $\hat{\theta}_m(dx)$ for $m \geq 1$ can be deduced from the measure $\mu(dx)$. For this purpose, let $0 < \sigma_1 < \sigma_2 < \dots$ denote the atoms of the measure $\mu(dx)$. $-\sigma_n$ for $n \geq 1$ are the (simple) poles of the Laplace transform $\tilde{p}_0$. Let r_n be the residue of $\tilde{p}_0$ at pole $-\sigma_n$. We have

$$\tilde{p}_0(z) = \sum_{n=1}^{\infty} \frac{r_n}{z+\sigma_n}, \tag{6.1}$$

so that

$$\mathbb{P}\{V > x\} = \sum_{n=1}^{\infty} \frac{r_n}{\sigma_n} e^{-\sigma_n x} \tag{6.2}$$

and

$$\mu(dx) = \sum_{n=1}^{\infty} \frac{r_n}{\sigma_n} \delta_{\sigma_n}(dx), \tag{6.3}$$

where $\delta_{\sigma_n}(dx)$ is the Dirac mass at point σ_n. Note that we have the normalization condition

$$\sum_{n=1}^{\infty} \frac{r_n}{\sigma_n} = 1. \tag{6.4}$$

As a consequence of the above assumption, $\tilde{p}_0$ is a meromorphic function over the whole complex plane and on the basis of relation (4.13), it is easy to see that $\tilde{\theta}_m$ is also a meromorphic function, which must have simple poles.

The zeros in z of $Q_m(-z)$ are negative and it follows in view of relation (4.13) that these zeros may potentially be singularities for the Laplace transform $\tilde{\theta}_m(z)$. However, it can be shown that they are removable singularities. Indeed, consider z_0 a (simple) zero of polynomial $Q_{m-1}(-z)$. Using [9, eq. (12.1-23], we can write

$$\tilde{p}_0(z) = \sum_{m=0}^{\infty} \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} \frac{1}{Q_{m-1}(-z) Q_m(-z)}$$

and then, since the polynomials $Q_m(z)$ have no common zeros,

$$\lim_{z \to z_0} Q_{m-1}(-z)\tilde{p}_0(z) = \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} \frac{1}{Q_m(-z_0)} + \frac{\mu_1 \dots \mu_{m-2}}{\lambda_0 \dots \lambda_{m-2}} \frac{1}{Q_{m-2}(-z_0)} = 0,$$

where we have used the recurrence relations (1.4) in the last step and the fact that $Q_m(-z_0) = 0$. As a consequence, since $Q_{m-1}(z)$ and $Q_{m-2}(z;1)$ have no common zeros,

$$\lim_{z \to z_0} Q_{m-1}(-z)\tilde{\theta}_m(z) = Q_m(-z_0) + \frac{\mu_1 \dots \mu_{m-1}}{\lambda_1 \dots \lambda_{m-1} Q_{m-2}(-z_0;1)}.$$

It follows that $\lim_{z \to z_0} Q_{m-1}(-z)\tilde{\theta}_m(z)$ exists and that z_0 is a removable singularity for $\tilde{\theta}_m(z)$. The actual singularities of $\tilde{\theta}_m(z)$ are in fact located in $(-\infty, 0]$ and correspond to the negative roots of the equation

$$\tilde{p}_0(z) = [\tilde{p}_0(z)]_{m-1}.$$

This allows us to state the following result.

Theorem 2. *The poles of the Laplace transform $\tilde{\theta}_m(z)$ of the random variable θ_m are located at the negative roots of the equation*

$$\tilde{p}_0(z) = [\tilde{p}_0(z)]_{m-1}, \tag{6.5}$$

where $[\tilde{p}_0(z)]_{m-1}$ is the $(m-1)$th approximant of the continued fraction $\tilde{p}_0(z)$.

Equation (6.5) gives a means of numerically computing the singularities and the corresponding residues of the Laplace transform $\tilde{\theta}_m(z)$. Note in addition that by using the relation [9, p. 601]

$$\tilde{p}_0(z) - [\tilde{p}_0(z)]_{m-1} = \int_0^\infty \left[\frac{P^-_{m-1}(-\tau)}{P^-_{m-1}(z)}\right]^2 \frac{d\mu(\tau)}{z+\tau},$$

we deduce that

$$\tilde{p}_0(z) - [\tilde{p}_0(z)]_{m-1} = \frac{1}{Q^2_{m-1}(-z)} \sum_{n=1}^\infty \frac{r_n Q^2_{m-1}(\sigma_n)}{\sigma_n} \frac{1}{z+\sigma_n}.$$

Hence, for each $m \geq 1$, $\tilde{\theta}_m$ has a unique pole on each interval $(-\sigma_{n+1}, -\sigma_n)$ for $n \geq 1$. The residue r at the pole s of $\tilde{\theta}_m$ is finally given by

$$r = \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1} \sum_{n=1}^\infty \frac{r_n Q^2_{m-1}(\sigma_n)}{\sigma_n (s+\sigma_n)^2}}. \tag{6.6}$$

The above results show how the different measures $\hat{\theta}_m(dx)$ for $m \geq 1$ depend on the continued fraction $\tilde{p}_0(z)$, the measure $\mu(dx)$ and the basic OPS $\{Q_m(x)\}$. Specifically, if we denote by $0 < s^m_1 < s^m_2 < \dots$ the atoms of $\hat{\theta}_m(dx)$, where $s^m_j \in (\sigma_j, \sigma_{j+1})$, $-s^m_j$ for $j \geq 1$ are the poles of the Laplace transform $\tilde{\theta}_m$ and we have

$$\mu_m \hat{\theta}_m(dx) = \sum_{j=1}^\infty \frac{R^m_j}{s^m_j} \delta_{s^m_j}(dx) \tag{6.7}$$

where

$$R^m_j = \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1} \sum_{n=1}^\infty \frac{r_n Q^2_{m-1}(\sigma_n)}{\sigma_n (-s^m_j+\sigma_n)^2}}. \tag{6.8}$$

To conclude this paper, let us briefly discuss a conjecture formulated in [5]. In that paper, it is claimed that for asymptotically proportional birth and death processes (i.e., $\lambda_m \sim \rho m^\alpha$ and $\mu_m \sim m^\alpha$ as $m \to \infty$ for some positive constants α and ρ), the poles of the continued fraction $\tilde{p}_0(z)$ are asymptotically proportional to m^α (i.e., $\sigma_m \sim C m^\alpha$ for some constant C, which depends only on ρ). In this paper, we assume that $\rho < 1$.

We first note that under the asymptotic proportionality assumption, the birth and death process $\{\Lambda_t\}$ satisfies Aldous' local linearization property (as discussed in Section 3) and $\tilde{\theta}_m(m^\alpha z) \to \tilde{\zeta}(z)$ as $m \to \infty$, where $\tilde{\zeta}$ is the Laplace transform of the duration of a busy period of an $M/M/1$ queue with input rate $\rho < 1$ and unit service rate. It follows that

$$\sum_{j=1}^\infty \frac{R^m_j}{m^\alpha z + s^m_j} \to \tilde{\zeta}(z) \text{ as } m \to \infty$$

and a classical theorem in Stieltjes transform theory then implies

$$\sum_{j=1}^\infty \frac{R^m_j}{s^m_j} \delta_{\frac{s^m_j}{m^\alpha}}(dx) \xrightarrow{v} \hat{\zeta}(dx) \text{ as } m \to \infty,$$

where $\hat{\zeta}(dx)$ is the measure such that its Stieltjes transform is equal to $\tilde{\zeta}$. In the Appendix, it is shown that the measure $\hat{\zeta}(dx)$ has a continuous density on the compact support $[(1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2]$.

In view of the above arguments, it seems reasonable to conjecture as in [5] that the atoms s_m^j and the residues R_j^m are of the same order as n^α. Thus, one may expect that $s_j^m = O(m^\alpha)$ for $j \sim m$ as $m \to \infty$. Since $s_j^m \in (\sigma_j, \sigma_{j+1})$, these observations heuristically support the conjecture that $\sigma_m = O(m^\alpha)$ as $m \to \infty$. It is also reasonable to conjecture that the coefficient of proportionality depends only on ρ since it is the case for $\tilde{\zeta}(z)$.

Appendix: Analysis of the duration of a busy period in an $M/M/1$ queue

Consider an $M/M/1$ queue with input rate $\rho < 1$ and unit service rate. It is well known in queueing literature that the Laplace transform of the duration ζ of a busy period is given by [14]

$$\tilde{\zeta}(z) = \frac{1 + z + \rho - \sqrt{(1+\rho+z)^2 - 4\rho}}{2\rho}$$

which is the Laplace transform of the probability density function

$$f(y) = \frac{1}{y\sqrt{\rho}} e^{-(1+\rho)y} I_1(2y\sqrt{\rho}),$$

where I_1 is the Bessel function of the first kind of order one. The functions $s \to e^{-ks}/s$ and $s \to e^{-ks} I_1(ks)$ for $k > 0$ are the Laplace transforms of the functions $\hat{\zeta}_1 : t \to u(t-k)$ and

$$\hat{\zeta}_2 : t \to \frac{k-t}{\pi k \sqrt{t(2k-t)}} \left[u(t) - u(t-2k) \right],$$

respectively [1], where $u(t)$ is the unit step function defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Noting that

$$f(y) = \frac{1}{\sqrt{\rho}} \frac{1}{y} e^{-(1+\rho-2\sqrt{\rho})y} e^{-2y\sqrt{\rho}} I_1(2y\sqrt{\rho}),$$

the function f is the Laplace transform of the convolution $\hat{\zeta}_d = \hat{\zeta}_1 * \hat{\zeta}_2$. Straightforward manipulations then yield

$$\hat{\zeta}_d(t) = \frac{1}{2\pi\rho} \sqrt{4\rho - (t-1-\rho)^2}\, \mathbf{1}_{[(1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2]}(t),$$

where $\mathbf{1}_{[(1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2]}$ is the indicator function of the interval $[(1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2]$.

The above analysis shows that the Laplace transform $\tilde{\zeta}$ is the Stieltjes transform (i.e., the iterated Laplace transform) of the measure $\hat{\zeta}$ with compact support and continuous density $\hat{\zeta}_d$, that is,

$$\hat{\zeta}(dx) = \hat{\zeta}_d(x)dx = \frac{1}{2\pi\rho} \sqrt{4\rho - (x-1-\rho)^2}\, \mathbf{1}_{[(1-\sqrt{\rho})^2, (1+\sqrt{\rho})^2]}(x) dx.$$

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