

NEW TECHNIQUES FOR BOUNDS ON THE TOTAL NUMBER OF PRIME FACTORS OF AN ODD PERFECT NUMBER

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ABSTRACT. Let $\sigma(n)$ denote the sum of the positive divisors of n . We say that n is perfect if $\sigma(n) = 2n$. Currently there are no known odd perfect numbers. It is known that if an odd perfect number exists, then it must be of the form $N = p^\alpha \prod_{j=1}^k q_j^{2\beta_j}$, where p, q_1, \dots, q_k are distinct primes and $p \equiv \alpha \equiv 1 \pmod{4}$. Define the total number of prime factors of N as $\Omega(N) := \alpha + 2 \sum_{j=1}^k \beta_j$. Sayers showed that $\Omega(N) \geq 29$. This was later extended by Iannucci and Sorli to show that $\Omega(N) \geq 37$. This was extended by the author to show that $\Omega(N) \geq 47$. Using an idea of Carl Pomerance this paper extends these results. The current new bound is $\Omega(N) \geq 75$.

1. INTRODUCTION

Here and throughout, n is any natural number, and N is a hypothetical odd perfect number. Let $\sigma(n)$ denote the sum of the positive divisors of n . We say that n is perfect if $\sigma(n) = 2n$. It is known that if $\sigma(n) = 2n$ and n is even, then $n = 2^{k-1}(2^k - 1)$ where $2^k - 1$ is a Mersenne prime. Currently there are no known odd perfect numbers. First shown by Euler, it is well known that if an odd perfect number exists, then it must be of the form

$$(1) \quad N = p^\alpha \prod_{j=1}^k q_j^{2\beta_j},$$

where p, q_1, \dots, q_k are distinct primes and $p \equiv \alpha \equiv 1 \pmod{4}$.

Based on (1) we define the total number of prime factors of an odd perfect number as

$$(2) \quad \Omega(N) := \alpha + 2 \sum_{j=1}^k \beta_j,$$

and we define the total number of distinct prime factors of N as

$$(3) \quad \omega(N) := 1 + k.$$

A number of bounds have been derived for $\Omega(N)$. Cohen showed that $\Omega(N) \geq 23$ [3]. Sayers showed that $\Omega(N) \geq 29$ [12]. Iannucci and Sorli showed that $\Omega(N) \geq 37$ [8]. The author extended this to give $\Omega(N) \geq 47$ [7]. This paper extends this result to give

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TABLE 1. Timing results for algorithm

Result	Time to Prove result	Lines to Prove result
$\Omega(N) \geq 69$	7 days, 17 hours, 23 minutes, 40 seconds	25,234,126 lines
$\Omega(N) \geq 71$	10 days, 22 hours, 17 minutes, 58 seconds	37,752,001 lines
$\Omega(N) \geq 73$	17 days, 1 hour, 28 minutes, 28 seconds	56,352,180 lines
$\Omega(N) \geq 75$	25 days, 3 hours, 17 minutes, 27 seconds	83,900,721 lines

Theorem 1.1. *If N is an odd perfect number, then $\Omega(N) \geq 75$.*

In proving these results, the methods of [7] were modified and applied. For an introduction and explanation of the algorithm we refer the interested reader there.

To some extent, with this modification of the algorithm given here, the calculation becomes a matter of book-keeping. In [7] there was a specific road-block which would have prevented any further calculation. This is no longer the case. With enough computational power, we could get to any number, (although this amount appears to double every time we increase the bound by 4), so it is just a matter of how much computer power to dedicate towards the problem.

The choice of 75 in the theorem above was motivated by the fact that 75 is an “esthetically nice” number. (Based on the completely arbitrary and non-scientific reasoning that we work in base 10.) The next number that the author would consider nice, namely 101, is beyond reach, using the current techniques and computational power. (Some timing data is given in Table 1.) That being said, although 75 could doubtless be extended to 77 or possibly even 79, no purpose was seen in doing this, as the numbers themselves do not provide any greater insight, and the bound of 75 was sufficient to demonstrate the improvement of the current technique over that found in [7].

Between 69 and 71, there is an increase by a factor of about 1.414. Between 71 and 73, there is an increase by a factor of about 1.561. Between 73 and 75, there is an increase by a factor of about 1.473.

Based on these numbers, we could estimate that the amount of time to compute up to 101 would be over 11 years of CPU time.

2. DEFINITIONS AND NOTATION

By $p^a || N$ we mean $p^a | N$ and $p^{a+1} \nmid N$. We define the function $\sigma_{-1}(n)$ as

$$(4) \quad \sigma_{-1}(n) := \sum_{d|n} d^{-1} = \frac{\sigma(n)}{n}.$$

A number of simple results concerning $\sigma_{-1}(n)$ are summarized below.

Lemma 2.1. *Let n be any natural number. Then*

- $\sigma_{-1}(n)$ is a multiplicative function, i.e. if $(n, m) = 1$ then $\sigma_{-1}(n \cdot m) = \sigma_{-1}(n)\sigma_{-1}(m)$,
- $\sigma_{-1}(n) > 1$ for all $n > 1$,
- $\sigma_{-1}(n) = 2$ if and only if n is perfect,
- $\frac{p+1}{p} \leq \sigma_{-1}(p^a) < \sigma_{-1}(p^{a+1}) < \frac{p}{p-1}$ for all primes p and integers $a > 1$.

There are a number of useful results concerning $\omega(N)$, the total number of distinct prime factors.

Lemma 2.2. *Let N be an odd perfect number. Then*

- $\omega(N) \geq 8$ [2, 5].
- If $3 \nmid N$ then $\omega(N) \geq 11$ [6, 9].
- If $3 \nmid N$ and $5 \nmid N$ then $\omega(N) \geq 15$ [10].
- If $3 \nmid N$, $5 \nmid N$ and $7 \nmid N$ then $\omega(N) \geq 27$ [10].

This last result is well known in the literature. We use the version given in [4].

Lemma 2.3 (Cohen, Sorli [4]). *Let*

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_1} \cdots q_r^{\beta_1}$$

be an odd perfect number, where the p_i and the α_i are known. Define

$$S := \sigma_{-1}(p_1^{\alpha_1} \cdots p_k^{\alpha_k})$$

Then we have,

$$\min q_i < \frac{2 + S(r-1)}{2 - S}$$

moreover, if $r \geq 3$ we have the tighter bound

$$\min q_i < \frac{8 + 2S^2(r-1)}{4 - S^2}.$$

Furthermore we have the lower bound

$$\min q_i > \max\left(3, \frac{S}{2 - S}\right)$$

Proof. Taking $h = \sigma_{-1}$, $\mu = 1$, $h(\lambda) = S$ and $w = r$, this follows directly from equations (2.3), (4.1) and (2.4) of [4]. \square

Example 2.4. *Assume that $N = 3^2 \cdot 13^4 \cdot 30941 \cdot q_1^{\beta_1} \cdot q_2^{\beta_2}$ is an odd perfect number. Then by Lemma 2.3 we can assume that $3.5962 < q_1 < 8.1924$. So in, particular we can assume that $q_1 = 5$ or 7 .*

3. THE ALGORITHM AND PROOF OF THEOREM 1.1

Suppose $N = p^\alpha \prod q_i^{2\beta_i}$, as before. To prove that $\Omega(N) \geq K$, we assume that $\Omega(N) = \alpha + \sum 2\beta_i \leq K - 2$ and obtain a contradiction for every combination of α and β_i .

There are two main modifications to the algorithm in [7]. First, as opposed to doing every individual case of $[\alpha, \beta_1, \dots, \beta_k]$ where $\Omega(N) = \alpha + 2 \sum \beta_i$, we combine them into one test. For example, if we want to prove $3 \nmid N$, where $\Omega(N) = 57$, we look at the possibilities $3^2 \parallel N, 3^4 \parallel N, \dots, 3^{56} \parallel N$, and recurse. (Actually, for any power $n > 44$, we have that 3^n will give rise to a contradiction as $\omega(N) \geq 8$.) This was done because the original method of looking at every possible partition of N caused a large amount of duplicated effort in the automated proofs. By combing these tests together we speed up the calculation, and significantly reduce the storage space requirements.

It should be pointed out that this modification introduced a new contradiction, which in [7] was taken care of in the choice of partitions. This is listed as contradiction (5) below.

The second modification to the algorithm is the use of Lemma 2.3. If the algorithm finds itself in a situation where previously it could not continue, because it did not know the factorization of some very large number, then we compute X from

Lemma 2.3. If X is reasonably small, then we run through all of the possibilities of q_i prime, and $q_i \leq X$, as a means of continuing the calculation. Here “small” was defined as anything less than 100000. This was sufficient for these calculations. It should also be noted that the primes are checked in order. After proving that a particular prime (say p) causes a contradiction, it is assumed that it cannot occur as a factor for the next cases being checked, within the same sub-branch. (For example, with an abuse of notation, we assume $p^0 \parallel N$, and would arrive at contradiction (1) if a factor of p occurs.)

There are five contradictions that we test for. The first four are from the original algorithm, or the equivalent variation needed to combine all tests into one test. The last is commented on above.

- (1) Excess of a given prime:

By assuming $p^k \parallel N$ we derive the contradiction that $p^{k+1} \mid N$. This is denoted in the output by “xs= p ” where p is the prime in question.

- (2) Excess of the number of primes:

We have more primes than we are allowed, given the restrictions on $\Omega(N)$ and the fact that only one prime can have an exponent of 1. This is denoted in the output by “xs=prime”. Incompletely factored numbers are counted as contributing two primes, even though this may be too low. Incompletely factored numbers are known not to be perfect powers. Furthermore, incompletely factored numbers are checked to ensure that they are co-prime with each other, as well as other primes within the relevant branch.

For example, if at some point in the calculation, we check the assumption that $26881^{14} \parallel N$, we would get as a result that 3, 5, 31, 43, 3368729516337631, 6717545999551, 5601667, 8265157321, 18691, 145861 and 1801 must all divide N . If we only had 6 primes unaccounted for, and none of primes listed were potential factors already, we would get this contradiction.

- (3) Partition cannot be satisfied:

The factors that must divide N , along with their powers, cannot satisfy the partition. In the original algorithm there were a number of different ways that it could occur. Given that in this implementation, there are not specific exponent bounds assigned before starting to recurse, this can only occur in one way. This is if of the remaining primes, one of them has to be the special prime (with an exponent $a \equiv 1 \pmod{4}$), and of all of the remaining primes, all of them are such that $p \not\equiv 1 \pmod{4}$. This is denoted in the output by “exponent bounds exceeded”.

- (4) Excess of σ_{-1} :

A lower bound for $\sigma_{-1}(N)$ using known factors gives $\sigma_{-1}(N) > 2$. This is denoted in the code by “S= number”, giving a floating point approximation for a lower bound of $\sigma_{-1}(N)$. (The code uses exact rational arithmetic to check the inequality.)

- (5) ω -bound exceeded.

Given the choices of primes and exponents made so far, it is not possible to satisfy Lemma 2.2. For example, if $N = 57$, and we knew that we have the factor 3^{46} , we would then have at most 6 other factors, which contradicts $\omega(N) \geq 8$ from Lemma 2.2. This is denoted as “violate omega bound”.

In Table 2 an example of running this code is given, and the five possible exceptions. These exceptions are indicated by the numbers (1) through (5) on the left hand side of the page. The use of Lemma 2.3 is indicated with an star, *, on the left hand side of the page. Minor formatting has been done to the output, to avoid lines with more than 80 characters. (The output of the actual code would have put the line starting with “It would be nice to know” all on one line.)

As was done in [7], numbers were factored using the ifactor command in MAPLE, with the easy option specified. If easy factors were not found, then number was then checked in a hints database (which currently contains over 700 completely, or partially factored numbers).

4. COMMENTS AND ACKNOWLEDGMENTS

In [7] it was said that there were three numbers that needed to be factored, without which the algorithm in [7] could not be continued. Using the methods of Carl Pomerance, this problem was avoided. At this point, the calculations was terminated not by a particular obstruction, but instead because 75 was esthetically pleasing, and the next obvious goal of 101 is beyond the reach of the current techniques and computer power.

The purpose of this paper was to demonstrate a new technique to extend this bound, and not necessarily to extend this bound to the farthest extend possible.

I would like to thank Phil Carmody, Christophe Clavier, Don Leclair Paul Leyland, Tom Womack, Paul Zimmermann, and the people at <http://mersenneforum.org> who helped provide factorizations for the numbers

- $\sigma(\sigma(547^{18})^{16})$, a 789-digit number, with a factor of:
1520135498523547561326750429418247.
- $\sigma(\sigma(3221^{12})^{22})$, a 927-digit number, with a factor of:
46973400441039677515399714233826061.

which I had listed in [7] as obstructions to my calculations. This information was included in the hints database used by the code.

I am also indebted to Carl Pomerance for bringing a weaker form of Lemma 2.3 to my attention [11], and more importantly how this could be applied to improve the existing bound. I would also like to thank the unknown referee for the reference [4], which replaced Lemma 2.3, as well as many other useful suggestions and comments.

Also, I would like to acknowledge William Lipp, the creator of www.oddperfect.org, for providing some additional factorizations. (The website www.oddperfect.org is a website, which when complete, will co-ordinate an attack to improve the lower bound on odd perfect numbers, of $N \geq 10^{300}$, given by [1].)

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TABLE 2. Parts of the proof that $\Omega(N) \geq 75$

$3^6 \Rightarrow 1093$
 $1093 \Rightarrow 2 \ 547$
 $547^2 \Rightarrow 3 \ 163 \ 613$
 $163^2 \Rightarrow 3 \ 7 \ 19 \ 67$
 $7^2 \Rightarrow 3 \ 19$
 $19^2 \Rightarrow 3 \ 127$
(1) $67^2 \Rightarrow 3 \ 7^2 \ 31 \quad xs=7$
....
 $547^{18} \Rightarrow 19256021298645399074821884828797791764310604858317$
 $19256021298645399074821884828797791764310604858317^2 \Rightarrow 3$
 $3081128010533825683 \ 143739375561423904832226409$
 $279078490021828554466220167808251320188667868911072727$
....
 $19256021298645399074821884828797791764310604858317^{36} \Rightarrow c_{1775}$
It would be nice to know
 $19256021298645399074821884828797791764310604858317^{36}$
(*) By Cohen/Sorli's argument, it has a factor between 3 and 18
Trying each one in turn
Next prime to try is 5
 $5^2 \Rightarrow 31$
....
(5) $17^8 \Rightarrow 19 \ 307 \ 1270657$ violate omega bound
Finished Cohen/Sorli's argument
....
 $1093^2 \Rightarrow 3 \ 398581$
 $398581 \Rightarrow 2 \ 17 \ 19 \ 617$
....
 $398581^4 \Rightarrow 5 \ 1866871 \ 2703853428809791$
 $5 \Rightarrow 2 \ 3$
....
 $5^{10} \Rightarrow 12207031$
 $1866871^2 \Rightarrow 3 \ 19 \ 331 \ 184725139$
(4) $19^2 \Rightarrow 3 \ 127 \ S=2.001549342$
....
 $19^{16} \Rightarrow 3044803 \ 99995282631947$
 $331^2 \Rightarrow 3 \ 7 \ 5233 \ S=2.310779791$
....
 $331^{18} \Rightarrow 282349518620419 \ 8080363351283173500917653281271$
(3) $3044803^2 \Rightarrow 3 \ 3090276117871$ exponent bounds exceeded
(2) $3044803^4 \Rightarrow 11 \ 631 \ 12382686952067349629581 \quad xs=prime$
....

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